Targeted Advertising on Competing Platforms

Siqi Pan and Huanxing Yang*
Department of Economics, Ohio State University

March 5, 2014

Abstract

This paper studies targeted advertising in two sided markets. Two platforms, with different targeting abilities, compete for single-homing consumers, while advertising firms are multi-homing. We show that the platform with a higher targeting ability will attract more consumers and have more advertising firms. When the targeting ability of either platform increases, all consumers benefit as they will incur lower nuisance costs from advertising. Compared to the equilibrium outcome, monopoly ownership always leads to more skewed consumer allocation between two platforms. We also compare the advertising levels and consumer allocation under the social optimum and those under equilibrium. We find that in most cases, platforms underinvest in targeting abilities in equilibrium.

Key Words: Targeting; Advertising; Two sided markets
JEL Classification: D43, L13, L15

1 Introduction

The Internet has revolutionized the advertising industry. One distinguishing feature of online advertising is that online platforms are able to provide customized advertisements to relevant consumers. In other words, online platforms have high targeting abilities. This is achieved mainly because online platforms are able to track consumers' web browsing activities. For instance, social network websites, such as Facebook, can roughly know the current interests of a consumer by tracing his activities on the network, and provide the ads of relevant products that might interest him. The high targeting ability of online advertising brings two potential benefits. First, for advertising firms, fewer advertising messages get lost, as ads are sent to more relevant consumers on average. Second, for consumers, who usually do not like irrelevant ads, on average they are less likely to encounter irrelevant ads. Probably for these reasons, the aggregate spending of advertising on the Internet has increased dramatically in recent years, while that of traditional media (TV, newspaper, radio) has declined steadily.

Two recent papers studied targeted advertising. Bergemann and Bonatti (2011) model advertising markets as perfectly competitive markets, focusing mainly on the impact of changes in targeting ability on equilibrium ad prices. In their model, consumers are passive and do not incur any nuisance cost by viewing irrelevant ads. Moreover, platforms do not play any active

*pan.229@osu.edu and yang.1041@osu.edu.

role as the advertising markets are assumed to be perfectly competitive. In Johnson (2013), consumers incur nuisance costs by viewing irrelevant ads, and they might not participate if there are too many irrelevant ads. In this setting, he investigates the impacts of increasing targeting ability on market outcomes. However, in his model there is no advertising price and platforms play no active role. In the real world, platforms, such as Google and Facebook, play an active role in the advertising market. They do not only bring together consumers and advertising firms for potential match, but also actively set ad prices and actively develop new technologies and methods to improve targeting ability.

The goal of this paper is to study targeted advertising in two sided markets, with platforms playing an active role in identifying relevant consumers and setting prices. Moreover, platforms are competing with each other. In particular, we ask the following questions. How do changes in targeting ability affect the prices of ads, the number of advertising firms, and the total volume of ads? Will consumers always benefit from increases in targeting ability? Will increases in targeting ability affect different types of firms differently? How does the equilibrium allocation of consumers between platforms compare to the socially optimal allocation? Will platforms invest too little or too much in targeting ability relative to the socially optimal level?

Specifically, based on Anderson and Coate (2005) we develop a model with two competing platforms acting as bridges between consumers and advertising firms. Consumers' tastes about the two platforms' contents are horizontally differentiated a la Hotelling. Advertising firms (simply firms sometimes) are heterogeneous in terms of the profitability of each product sold. While firms can be multi-homing, that is, they can participate on both platforms, consumers are single-homing, which means each consumer only participates on one platform. With some probability, a consumer is interested in (or relevant for) a firm's product, or, in other words, there is a potential match within the consumer-firm pair. The role of advertising is to turn potential matches into actual purchases: a sale is realized if and only if there is a potential match within the consumer-firm pair and the consumer receives an advertisement from the firm. For each consumer-firm pair on a platform, the platform generates a binary and informative signal regarding whether there is a potential match. The accuracy of the signals indicates the targeting ability of a platform, and two platforms have different targeting abilities.[1] Consumers are neutral about relevant ads, but incur nuisance costs in viewing irrelevant ads. In terms of timing, first the two platforms simultaneously set ad prices per impression. Then firms decide whether to participate on each platform, and at the same time consumers decide which platform to join.

Naturally, only more profitable firms will advertise on platforms, and the cutoff firms are different for the two platforms. Compared to the platform with a lower targeting ability, in equilibrium the platform with a higher targeting ability always has more advertising firms, attracts more consumers, has more relevant ads in total, and is more profitable. Intuitively, from the platforms' point of view, the number of participating firms and the number of participating consumers are complements. However, given the negative externality imposed by advertising firms on consumers, there is a tradeoff between the number of participating firms and the number of participating consumers. Since there is a smaller proportion of irrelevant ads on the platform with a higher targeting ability, this platform will accommodate more participating firms, but at

[1] One can consider the one with a higher targeting ability as an online platform and the other one as an offline platform.

the same time restrict the total number of ads per consumer such that the number of irrelevant ads per consumer is smaller, so that it can attract more consumers. This implies that consumers joining the platform with a higher targeting ability encounter less irrelevant ads and thus incur a lower nuisance cost. However, the ad price for the platform with a higher targeting ability could be relatively higher or lower. This is because more participating firms, other things equal, imply lower prices. Similarly, the platform with a higher targeting ability could have relatively more or less total number of ads. The reason is that, while more participating firms tend to increase the volume of ads, the higher targeting ability tend to reduce the number of irrelevant ads and hence the total ads volume.

We then investigate how the equilibrium changes as the targeting ability of the advantaged platform increases. It turns out that the disadvantaged platform will have fewer advertising firms, lose market share on the consumers' side (simply market share sometimes), have fewer total ads, and charge a higher ad price than before. The advantaged platform will have more advertising firms and gain market share, but the changes in the total number of ads and the advertising price are ambiguous. Intuitively, as the advantaged platform becomes even more advantaged, in order to protect its market share the disadvantaged platform accommodates fewer firms. This means the marginal firm becomes more profitable, which implies that the platform charges a higher price. Naturally, the advantaged firm "spends" its additional advantage on both accommodating more firms and attracting more consumers, since they are complements. Interestingly, all consumers, regardless of the platform they participate in, are always better off. This is because the consumers on the disadvantaged platform benefit from fewer advertising firms, while the consumers on the advantaged platform benefit even more since otherwise the advantaged platform would have lost its market share. As to firms, less profitable firms (those initially participating only on the advantaged platform and those newly participating on the advantaged platform after the change) are better off.

When the targeting ability of the disadvantaged platform increases, the changes in the number of participating firms and market shares for individual platforms are reversed compared to the previous case. However, all consumers are still better off: they encounter less relevant ads and incur lower nuisance costs. Combining this with the previous result, all consumers are better off when the targeting ability of either platform increases. This result is different from Johnson (2013), in which consumers might be worse off as the targeting ability of ads increases. As to firms, less profitable firms (those who participated initially on the advantaged platform but no longer do after the change) are worse off.

When the two platforms are owned by a monopoly, the disadvantaged platform always accommodates more firms and charges a lower ad price than the advantaged platform does. Compared to the equilibrium outcome, under monopoly ownership both platforms always accommodate more firms (thus having more ads) and the market share always skews more toward the advantaged platform. The underlying reason for these results is that monopoly ownership internalizes the competition between two platforms, hence the business stealing effect under equilibrium no longer exists.

We then study socially optimal advertising levels and compare them to those in equilibrium. In the social optimum, the business stealing effect is absent as the planner internalizes the competition, but there is a distribution effect as the planner cares about how to allocate consumers efficiently between the two platforms. Additionally, the planner cares about the total social surplus generated by each platform while platforms only care about their own revenues or profits. Due to these differences, the equilibrium levels of ads in general are different from the socially optimal ones. Similar to Anderson and Coate (2005), in equilibrium the platforms could under-provide or over-provide ads. While usually both platforms under-provide ads or over-provide ads at the same time, sometimes it could be the case that one platform over-provides ads while the other one under-provides ads. Regarding the allocation of consumers between two platforms, again in equilibrium the advantaged platform could have more consumers or fewer consumers relative to the social optimum. We identify conditions under which the advantaged platform under-accommodates and over-accommodates consumers in equilibrium, respectively. Interestingly, under some conditions, the socially optimal market share of the disadvantaged platform could be bigger than that of the advantaged platform.

Finally, we compare social incentives and private incentives to invest in targeting abilities. We consider two settings. In the first setting, two platforms are symmetric and they both make investment decisions in the first stage. In the second setting, only the advantaged platform makes investment decisions. In both settings, platforms could underinvest in targeting ability as well as overinvest in targeting ability in equilibrium. Relative to the social optimum, the fact that platforms do not care about consumers' nuisance costs per se and cannot fully appropriate firm surplus imply that platforms tend to underinvest in targeting ability. On the other hand, the business stealing effect under private incentives implies that platforms tend to overinvest in targeting ability, since a higher targeting ability means a bigger market share and a bigger profit at the expense of the other platform. Quantitatively, underinvestment in targeting ability is much more likely to occur, while overinvestment occurs only under very special conditions.

This paper is related to the literature on informative advertising (Butters, 1977; Grossman and Shapiro, 1984). In this literature, there has been an increasing interest in the role of targeting ability. Esteban et al. (2001) and Iyer et al. (2005) focus on how targeted advertising affects the competition and equilibrium prices in product markets, instead of analyzing the level and price of advertising.[2] Athey and Gans (2010) identify a supply-side impact of targeting: it allows more efficient allocation of scarce advertising space, and the resulting increase in the supply of ads space might push down the price of advertising.[3] Taylor (2011) mainly addresses the effect of targeting accuracy on competition in product markets and medium's choices regarding content differentiation. In two empirical studies, Chandra (2009) on newspapers and Chandra and Kaiser (2010) on magazines, the ad prices are found to be higher in markets with more homogeneous subscribers (more segmented or high targeting ability).

As mentioned earlier, in terms of studying targeted advertising, this paper is closely related to Bergemann and Bonatti (2011) and Johnson (2013), and we already pointed out earlier the main differences between our paper and their papers. In the second part of Bergemann and Bonatti (2011) they study competing platforms. But in their model consumers are multi-homing, so ads on two platforms are substitutes. Moreover, platforms do not play any active role as the equilibrium prices on both platforms are determined by demand and supply.[4]

[2] See also Galeotti and Moraga-Gonzlez (2008), and Gal-Or et al. (2010).

[3] In a consumer search model, de Corniere (2010) studies how a search engine's targeting ability of keywords search affects the fee of advertising. Yang (2012) develops a model of targeted search, analyzing how quality of search affects the variety of goods offered, prices, and consumer welfare.

[4] More differences in results between our paper and these two papers are discussed in the text later.

To the best of our knowledge, this paper provides the first analysis of targeted advertising under the framework of competing two-sided platforms. Hence, it also contributes to the rapidly expanding literature on competition in two-sided markets (see Rochet and Tirole, 2003, 2006; and Armstrong, 2006). The most relevant model to our paper is the model of "competitive bottlenecks" named by Armstrong (2006), in which one side of the market is single-homing, while the other is multi-homing. The major insight of Armstrong (2006) is that, due to the competitive pressure on the single-homing side, platforms might be forced to transfer some of the surplus towards this side, which leads to too few multi-homing agents in equilibrium. In terms of modeling, our paper is closely related to Anderson and Coate (2005), who study advertising on competing platforms. The difference is that in their model two platforms are symmetric and advertising cannot be targeted. In our model, platforms are able to target advertising to relevant consumers, and the two platforms are asymmetric in that they have different targeting abilities.

Related to Anderson and Coate (2005), Peitz and Valletti (2008) show that advertising intensity is higher in free-to-air television than in pay-tv stations. Ambrus and Reisinger (2006) assume that consumers/viewers can be multi-homing, and compare the equilibrium advertising levels to those in the case that consumers are single-homing. Athey et al. (2013) also assume that consumers can be multi-homing, and study how the tracking technologies of platforms affect the equilibrium outcomes in the advertising market.

The rest of the paper is organized as follows. Section 2 sets up the model. In Section 3 we characterize the equilibrium outcome and conduct comparative statics regarding changes in targeting ability. Section 4 studies monopoly ownership. In Section 5 we investigate the social optimum and compare it to the equilibrium outcome. In Section 6 we endogenize the targeting abilities of platforms. Section 7 concludes.

2 Model

There are two platforms, A and B, who bring together consumers and advertising firms (simply firms sometimes). Consumers consume the content of the platform, and firms participate on platforms in order to send ads to consumers. Both consumers and firms are of unit mass.[5] Two platforms' contents are horizontally differentiated. Using Hotelling's location model, we assume that platforms A and B are located at 0 and 1, respectively. Consumers' tastes about the platforms' contents are also differentiated. Specifically, consumers are uniformly distributed on $[0, 1]$. Let d be the location of a consumer. Consumers are single homing, meaning that a consumer will join only one platform. Firms are potentially multi-homing: each firm can join neither, one, or both platforms.

For any given consumer-firm pair, with probability $q \in (0, 1/2)$ there is a match. That is, the consumer is interested in the firm's product, and we call such a consumer a relevant consumer, and an ad between such a pair a relevant ad. The probability that a match exists is i.i.d across all consumer-firm pairs. A consumer will buy one unit of product from a firm if and only if the consumer is relevant to the firm and he receives an ad from that firm. So in our model advertising is purely informative. Denote $S \in \{0, 1\}$ as the state indicating whether

[5] It is not essential that consumers and firms are of the same measure.

a consumer is relevant to a firm, with 1 (0) denoting that the consumer is relevant (irrelevant). For any consumer-firm pair on platform i, the platform generates a signal $s \in \{0, 1\}$ regarding the possible match. The signals are also conditionally independent across consumer-firm pairs. The information structure is as follows:

$$\Pr\{s = 1|S = 1\} = \alpha_i; \;\; \Pr\{s = 0|S = 0\} = \alpha_i.$$

The parameter $\alpha_i \in (1/2, 1]$, which measures the accuracy of the signals, captures the targeting ability of platform i. We assume that α_A and α_B are common knowledge. The posteriors that a consumer is relevant to a firm can be calculated as follows:

$$\begin{aligned}\Pr\{S &= 1|s = 1\} = \frac{\alpha_i q}{\alpha_i q + (1 - \alpha_i)(1 - q)}; \\ \Pr\{S &= 1|s = 0\} = \frac{(1 - \alpha_i)q}{(1 - \alpha_i)q + \alpha_i(1 - q)}.\end{aligned}$$

We assume $q < 1/2$ is small enough such that $\Pr\{S = 1|s = 0\}$ is also small enough, so that it never pays for any firm to send ads to consumers when the signal is 0. For any firm, the set of consumers with signal 1 can be considered as that firm's targeted set of consumers, to which the firm might send ads. Note that the size of the targeted set of consumers is $\alpha_i q + (1 - \alpha_i)(1 - q)$, which is increasing in q, and is decreasing in α_i as $q < 1/2$. Also note that, for each firm on the same platform, the targeted set of consumers is of the same size since the probability of being relevant is i.i.d. across consumer-firm pairs and the signals are conditionally independent, although the targeted sets of consumers are different for different firms. The Two platforms are different in targeting abilities. Specifically, $\alpha_B > \alpha_A$. One can consider platform A as a traditional offline media and platform B as an online media. Sometimes we consider symmetric platforms: $\alpha_B = \alpha_A$. In that case both platforms can be viewed as online media.

Each consumer incurs a nuisance cost γ by viewing an irrelevant ad. A consumer neither incurs any cost nor reaps any benefit from viewing a relevant ad.[6] A location d consumer gets a utility of $\overline{\beta} - td$ minus the nuisance cost of ads if participating on platform A, and she gets a utility of $\overline{\beta} - t(1 - d)$ minus the nuisance cost of ads if participating on platform B. The parameter $\overline{\beta}$ captures the gross utility of a consumer joining either platform by consuming the content provided by that platform. We assume $\overline{\beta}$ is high enough so that all consumers participate. The parameter t is the transportation cost in standard Hotelling models, which indicates the degree of horizontal differentiation between two platforms' contents.

Firms are heterogeneous in terms of profitability. Denote v as a firm's profit per sale. Firms' types v are distributed on $[0, \overline{v}]$ with cumulative distribution function $F(v)$ and density function $f(v)$, with $f(\cdot) > 0$ everywhere in the support, differentiable, and strictly logconcave.

The timing is as follows. In the first stage, the two platforms set advertising prices (per impression), p_A and p_B, simultaneously. In the second stage, observing p_A and p_B, consumers simultaneously decide which platform to join, and firms, at the same time, decide whether to participate on each platform simultaneously. All agents have rational expectations.

[6] This is a simplifying assumption to reduce the number of parameters. Alternatively, we can assume that each consumer incurs a nuisance cost γ from viewing each ad. And each consumer gets a gross payoff λ, $0 < \lambda \neq \gamma$, if she buys a relevant product. In this alternative setting, the main results of this paper will not change qualitatively.

The model is based on Anderson and Coate (2005). The difference is that we add the aspect of advertisement targeting, and two platforms are asymmetric in that they have different targeting abilities. The model also resembles Armstrong's (2006) model of "competitive bottlenecks," in which platforms are actively competing for consumers who are single-homing, while there is no competition for firms as they are multi-homing. The two-sided market only has an one-way externality: firms exert a negative externality on consumers by posting ads.

3 Equilibrium with Competing Platforms

Given an ad price p_i, $i = A, B$, a firm of type v will get the following profit per ad if advertising on platform i: $\alpha_i q v - [\alpha_i q + (1-\alpha_i)(1-q)]p_i$. Thus a firm of type v will advertise on platform i if and only if

$$v \geq \frac{\alpha_i q + (1-\alpha_i)(1-q)}{\alpha_i q} p_i \equiv \widehat{v}_i.$$

The term $\widehat{v}_i$ is the cutoff or marginal type of firms for platform i: firms with types above the cutoff type will advertise on the platform and those with types below the cutoff will not. Given that firms are multi-homing, a firm's decision to join platform i is independent of its decision to join platform j. Moreover, since firms pay prices per impression, a firm's decision as to whether to join either platform does not depend on consumers' platform choices. Let μ_i be the fraction of firms that advertise on platform i. Setting the ad price p_i is equivalent to choosing the cutoff firm type $\widehat{v}_i$ or the fraction of participating firms μ_i. In particular, $\mu_i = 1 - F(\widehat{v}_i)$ and $\widehat{v}_i = F^{-1}(1-\mu_i)$. The relationship between p_i and μ_i can be expressed as follows:

$$p_i(\mu_i) = \frac{\alpha_i q F^{-1}(1-\mu_i)}{\alpha_i q + (1-\alpha_i)(1-q)}. \tag{1}$$

Note that a higher price p_i leads to a higher cutoff type $\widehat{v}_i$ and a smaller fraction of participating firms μ_i.

Let m_i be the fraction of consumers joining platform i. Sometimes we call m_i the (consumer) market share of platform i. Recall that consumers are single-homing. Because potential matches are i.i.d. and signals are conditionally i.i.d. across all consumer-firm pairs, all consumers joining platform i will receive the same number of ads. If a consumer joins platform i, he will incur the following total nuisance cost: $\gamma(1-q)(1-\alpha_i)\mu_i$.[7] Define $\widehat{d}$ as the location of the consumer who is indifferent between joining the two platforms. In particular,

$$\widehat{d} = \frac{1}{2} + \frac{\gamma(1-q)}{2t}[(1-\alpha_B)\mu_B - (1-\alpha_A)\mu_A].$$

Consumers with $d \leq \widehat{d}$ will join platform A and those with $d \geq \widehat{d}$ will join platform B.[8] Thus,

[7] For each firm that advertises on platform i, with probability $(1-q)(1-\alpha_i)$ the consumer will be in that firm's targeted set (signal is 1 for the consumer-firm pair) but the consumer is actually irrelevant for the firm's product.

[8] Recall that firms and consumers move simultaneously. But this does not matter as consumers have rational expectations: given the ad prices, consumers can figure out how many firms will advertise on each platform and how many ads will be sent on each platform.

the market shares can be written as

$$m_A(\mu_A,\mu_B) = \frac{1}{2}+\frac{\gamma(1-q)}{2t}[(1-\alpha_B)\mu_B-(1-\alpha_A)\mu_A], \tag{2}$$

$$m_B(\mu_A,\mu_B) = \frac{1}{2}+\frac{\gamma(1-q)}{2t}[(1-\alpha_A)\mu_A-(1-\alpha_B)\mu_B]. \tag{3}$$

Let $\frac{\gamma(1-q)}{2t}\equiv y$ and $y(1-\alpha_i)\equiv x_i$. Then the market shares can be more compactly written as

$$m_i(\mu_i,\mu_j)=\frac{1}{2}+x_j\mu_j-x_i\mu_i.$$

Since consumers do not pay any participation fee to either platform, their choice of platform depends solely on the nuisance costs of ads. The term $\gamma(1-q)(1-\alpha_i)$ captures the sensitivity of the nuisance cost incurred on platform i to μ_i, and the term x_i captures how sensitive the market share is to changes in μ_i. A decrease in the nuisance cost γ, an increase in the transportation cost t, or an increase in targeting ability α_i would make market shares less sensitive to the volume of ads.

Platform i's profit, $\Pi_i(\mu_i,\mu_j)$, can be computed as

$$\Pi_i(\mu_i,\mu_j)=[\alpha_i q+(1-\alpha_i)(1-q)]p_i(\mu_i)m_i(\mu_i,\mu_j)\mu_i.$$

The term $m_i\mu_i$ is the total number of consumer-firm pairs on platform i. Among those, a $\alpha_i q+(1-\alpha_i)(1-q)$ fraction of consumer-firm pairs have signal 1 and ads will be sent. Therefore, the total number of impressions is $\alpha_i q+(1-\alpha_i)(1-q)$ times $m_i\mu_i$. Multiplying the ad price per impression p_i, we get the above expression of Π_i. Using (1) to get rid of p_i, the profit function can be written as

$$\Pi_i(\mu_i,\mu_j)=\alpha_i qF^{-1}(1-\mu_i)m_i(\mu_i,\mu_j)\mu_i=\alpha_i q\mu_i\widehat{v}_i m_i(\mu_i,\mu_j). \tag{4}$$

In (4), the term $\alpha_i q\mu_i\widehat{v}_i\equiv\alpha_i qR(\mu_i)$ is platform i's revenue per consumer. From (4), we can see that an increase in α_i will directly increase platform i's profit. Intuitively, an increase in targeting ability will enable platform i to identify more relevant consumers and rule out more irrelevant consumers, which enables the platform to charge a higher price without affecting the fraction of participating firms.

Denote an equilibrium as (μ_A^*,μ_B^*). Differentiating $\Pi_i(\mu_i,\mu_j)$ with respect to μ_i, we get

$$\frac{\partial\Pi_i}{\partial\mu_i}=\alpha_i q\{[\widehat{v}_i-\frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)}]m_i(\mu_i,\mu_j)-x_i\widehat{v}_i[1-F(\widehat{v}_i)]\}. \tag{5}$$

According to (5), an increase in μ_i affects Π_i through two channels. First, it affects the marginal revenue per consumer, which is captured by the term $R'(\mu_i)=\widehat{v}_i-\frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)}$. Second, it will change platform i's market share (the business stealing effect), which is captured by the second term in (5). Let $H(v)\equiv R'(\mu)$ be the marginal revenue (per consumer) function. Define $\widetilde{v}$ as follows: $H(\widetilde{v})=0$. That is, $\widetilde{v}$ is the marginal type of firms such that the marginal revenue of a platform is zero. Let $\widetilde{\mu}=1-F(\widetilde{v})$ be the corresponding measure of participating firms. Since f is logconcave, $\frac{1-F(v)}{f(v)}$ is strictly decreasing in v.[9] Therefore, the marginal revenue $H(v)$ is

[9] See Bagnoli and Bergstrom (2005) for details.

strictly increasing in v, and hence strictly decreasing in μ. Moreover, $H(\overline{v}) > 0$ and $H(0) < 0$. Thus there is a unique $\widetilde{v} \in (0, \overline{v})$, which also implies that there is a unique $\widetilde{\mu} \in (0, 1)$. Note that $\widetilde{\mu}$ or $\widetilde{v}$ uniquely maximizes revenue per consumer $R(\mu)$.

Lemma 1 *(i)* $\frac{\partial \Pi_i}{\partial \mu_i} < 0$ *if* $\mu \in (\widetilde{\mu}, 1]$. *(ii)* $\mu_i^* \in [0, \widetilde{\mu}]$. *(iii) For* $\mu_i \in [0, \widetilde{\mu})$, $\frac{\partial^2 \Pi_i}{\partial^2 \mu_i} < 0$ *and* $\frac{\partial^2 \Pi_i}{\partial \mu_i \partial \mu_j} > 0$.

By part (ii) of Lemma 1, in search for equilibrium we can restrict our attention to the domain $[0, \widetilde{\mu}]$, which we will do in the subsequent analysis. The underlying reason is that the business stealing effect is always negative (business losing): an increase in μ_i reduces the number of participating consumers of platform i. This means that to satisfy the first order condition, the marginal revenue per consumer has to be positive. Part (iii) shows that, in the relevant domain the second order condition is satisfied, which implies that the first order condition is sufficient for the best response function $\mu_i^*(\mu_j)$. The fact that $\frac{\partial^2 \Pi_i}{\partial \mu_i \partial \mu_j} > 0$ implies that μ_A and μ_B are strategic complements.

Following previous analysis, an equilibrium (μ_A^*, μ_B^*) is characterized by the following first order conditions:

$$x_B \mu_B^* = x_A \mu_A^* + \frac{x_A \mu_A^* \widehat{v}_A^*}{\widehat{v}_A^* - \frac{1 - F(\widehat{v}_A^*)}{f(\widehat{v}_A^*)}} - \frac{1}{2}; \tag{6}$$

$$x_A \mu_A^* = x_B \mu_B^* + \frac{x_B \mu_B^* \widehat{v}_B^*}{\widehat{v}_B^* - \frac{1 - F(\widehat{v}_B^*)}{f(\widehat{v}_B^*)}} - \frac{1}{2}. \tag{7}$$

In equilibrium, the business stealing effect and the marginal revenue effect exactly cancel out each other for both platforms.

Lemma 2 *There is a unique equilibrium.*

After establishing the existence of a unique equilibrium, we compare the two platforms' equilibrium behavior in the following proposition.

Proposition 1 *In equilibrium, the following properties hold. (i) Platform B has more participating firms:* $\mu_B^* > \mu_A^*$ *and* $\widehat{v}_A^* > \widehat{v}_B^*$. *(ii) Platform B has a bigger market share:* $x_A \mu_A^* > x_B \mu_B^*$ *and* $m_A(\mu_A^*, \mu_B^*) < 1/2 < m_B(\mu_A^*, \mu_B^*)$. *(iii) Platform B has a higher equilibrium profit:* $\Pi_A^* < \Pi_B^*$. *(iv) Platform B charges a higher ad price* ($p_A^* < p_B^*$) *if* $\frac{(1-\alpha_A)(1-\alpha_B)}{\alpha_A \alpha_B} \frac{1-q}{q} \geq 1$.

Proposition 1 implies that, compared to the platform with a lower targeting ability, the platform with a higher targeting ability has more participating firms, a bigger consumer base, and a higher profit. The intuition for these results is as follows. Since platform B has a higher targeting ability, its market share is less sensitive to μ_B, or the business stealing (losing) effect of platform B is smaller. Recall that for both platforms, the business stealing effect and the marginal revenue effect exactly cancel out each other in equilibrium. Therefore, in equilibrium platform B should have a smaller marginal revenue, which implies more participating firms.[10]

[10] Recall that the marginal revenue is decreasing in μ for $\mu \in [0, \widetilde{\mu}]$.

However, in equilibrium, platform B, who has a natural advantage in attracting more consumers, will not accommodate too many firms resulting in its market share falling below 1/2. This is because, from the platforms' point of view, the number of participating firms and the number of participating consumers are complements. Thus, in equilibrium platform B has a bigger market share. The reason that platform B has a bigger profit is intuitive: by mimicking platform A's strategy, platform B can always guarantee a higher profit than platform A's profit since it will have a bigger market share.

However, part (iv) of Proposition 1 indicates that it is not clear whether the platform with a higher targeting ability will charge a higher ad price. On the one hand, a higher targeting ability means that platform B can charge a higher ad price for the same cutoff type of firms. On the other hand, platform B has a lower cutoff type of firms ($\widehat{v}_A^* > \widehat{v}_B^*$), which tends to make the price charged by platform B relatively lower. The overall effect can go either way. One sufficient condition for the first effect to dominate is that the probability of potential match, q, is small enough. This is because when q becomes smaller the first effect is magnified: for the same cutoff type of firms, any given difference in targeting ability leads to a bigger price difference. Here we provide an example in which $p_A^* > p_B^*$. Suppose v is uniformly distributed on $[0, 1]$, $q = 0.3$, $\gamma = 3$, $t = 0.1$, $\alpha_A = 0.88$, and $\alpha_B = 0.93$. Then $0.5745 = p_A^* > p_B^* = 0.5543$.

Corollary 1 *In the unique equilibrium, the following properties hold. (i) Compared to consumers on platform A, each consumer on platform B sees less irrelevant ads and incurs a lower nuisance cost:* $(1 - \alpha_A)(1 - q)\mu_A^* > (1 - \alpha_B)(1 - q)\mu_B^*$. *(ii) Platform B has more relevant ads per consumer and more total relevant ads. (iii) For any firm that advertises on both platforms, the firm earns more profit per consumer and more total profit on platform B. (iv) Platform B has less ads per consumer and less ads in total if* γ *is small enough or* t *is big enough.*

Part (i) of Corollary 1 is directly implied by the fact that platform B has a bigger equilibrium market share, which means that consumers on platform B must incur a lower nuisance cost than those on platform A. This also implies that on platform B there are less irrelevant ads per consumer. Since platform B has more participating firms and a lower marginal firm, each firm on platform B must earn more profit per consumer than it can earn on platform A. Each participating firm's total profit should also be higher on platform B as it has a bigger market share. Platform B has more relevant ads because it has more participating firms and a higher targeting ability, which means more relevant consumers are identified. Since it also has a bigger market share, Platform B has more relevant ads in total than platform A does. But it is not clear whether platform B has more total ads (including both relevant and irrelevant ads). This is because platform A might have more total irrelevant ads, which might lead to more total ads on platform A. Part (iv) of Corollary 1 identifies conditions under which platform A has more ads in total.[11]

The following proposition shows how the equilibrium will change as the nuisance cost or the transportation cost changes.

[11] In the limit, when γ goes to zero or t goes to infinity, both platforms act like local monopolists and both accommodate the same number of firms, $\widetilde{\mu}$. In this limiting case, platform A has more ads in total as it has a bigger or more noisy targeted set of consumers.

Proposition 2 *(i) Both* μ_A^* *and* μ_B^* *are increasing in* t *and decreasing in* γ*; both* p_A^* *and* p_B^* *are decreasing in* t *and increasing in* γ*. (ii)* Π_A^* *is increasing in* t *and decreasing in* γ*.*

The results of Proposition 2 are intuitive. Since firms' ads impose a negative externality on consumers, an increase in the nuisance cost γ or a decrease in the transportation cost t means consumers become more sensitive to ads volume. In other words, competition for consumers becomes fiercer. As a result, both platforms reduce ads volume per consumer by accommodating fewer firms. The profit of platform A decreases because it does not only suffer from intensified competition, but also becomes more disadvantaged since an increase in γ or a decrease in t amplifies platform B's advantage in attracting consumers.

However, as γ increases or t decreases, the profit of platform B could either decrease or increase. This is because, although it suffers from intensified competition, an increase in γ or a decrease in t, in the mean time, amplifies platform B's advantage in attracting consumers. Similarly, as either γ increases or t decreases, in general it is not clear whether the market share of platform A will decrease or increase. Fixing the fractions of participating firms, an increase in γ or a decrease in t tends to increase the difference between the market shares of the two platforms. However, the fractions of participating firms of both platforms will decrease, which might reduce the difference between the market shares of two platforms. The overall effect is ambiguous. The above patterns are illustrated in the following example. Suppose v is uniformly distributed on $[0,1]$, $q = 0.3$, $t = 0.3$, $\alpha_A = 0.6$, and $\alpha_B = 0.8$. As γ increases from 0.3 to 1, m_B^* increases from 0.05310 to 0.5693, and Π_B^* increases from 0.0318 to 0.0338. However, as γ increases from 2 to 3, m_B^* decreases from 0.5733 to 0.5622, and Π_B^* decreases from 0.0331 to 0.0310.

In the following comparative statics exercise, we study how the equilibrium will change as platform B's targeting ability increases.

Proposition 3 *Suppose* $\alpha_B' > \alpha_B$ *and* α_A *remains the same. Let the superscript* $'$ *denote the endogenous variables in the equilibrium under* (α_A, α_B')*. Then, (i)* $\mu_A^{*\prime} < \mu_A^*$ *and* $\mu_B^{*\prime} > \mu_B^*$*; (ii)* $m_A^{*\prime} < m_A^*$ *and* $m_B^{*\prime} > m_B^*$*; (iii)* $p_A^{*\prime} > p_A^*$*, and* $p_B^{*\prime} > p_B^*$ *if* $\frac{(1-\alpha_B)(1-\alpha_B')}{\alpha_B \alpha_B'} \frac{1-q}{q} \geq 1$*; (iv)* $\Pi_A^{*\prime} < \Pi_A^*$ *and* $\Pi_B^{*\prime} > \Pi_B^*$*.*

As the targeting ability of the advantaged platform (B) increases, Proposition 3 shows that the disadvantaged platform (A) reduces the fraction of participating firms, charges a higher ad price, but ends up with a lower market share and a lower profit. The reason that platform A accommodates fewer firms is that it tries to protect its market share as platform B becomes more advantaged. And this directly implies that platform A will charge a higher ad price as the type of the marginal firms shifts upward. For platform B, Proposition 3 shows that it will increase the fraction of participating firms, but not significantly enough relative to the increase in targeting ability such that its market share still increases (again because revenue per consumer and market share are complements), and will earn a higher profit.

However, for the same reason as mentioned earlier, whether the price charged by platform B will increase, as α_B increases, is ambiguous, as the type of the marginal firms shifts downward. As indicated in part (iii), when q, the probability of potential match, is small enough, the effect that a higher targeting ability enables platform B to charge a higher price dominates, and platform B's ad price will increase in α_B. The condition in part (iii) also indicates that p_B^* is increasing in α_B if α_B is small enough, and only when α_B is close enough to 1 can p_B^* be decreasing in α_B.[12]

Corollary 2 *As α_B increases, in equilibrium: (i) both platforms will have less irrelevant ads per consumer, each consumer incurs a lower nuisance cost, and consumers on platform B gain more than consumers on platform A do; (ii) the number of relevant ads per consumer and the total number of relevant ads on platform B increase, while those on platform A decrease; the combined total number of relevant ads on the two platforms increases if $\alpha_A \leq 2/3$; (iii) the total number of ads on platform A decreases; if y is small enough, then the total number of ads on platform B decreases; (iv) firms with $v \in (\widehat{v}_B^{*\prime}, \widehat{v}_A^*]$ are strictly better off; for firms participating on both platforms ($v \geq \widehat{v}_A^{*\prime}$), every firm sends more relevant ads in total if $\alpha_A \leq 2/3$, and more profitable firms gain relatively more from the increase in α_B.*

Part (i) of Corollary 2 indicates that all consumers benefit from an increase in α_B. Intuitively, as α_B increases, platform A reduces ad volume per consumer, and thus consumers on platform A are better off. Since platform B gains more consumers, consumers remaining with platform B must gain more than those on platform A, because otherwise platform B would have lost consumers to platform A. As α_B increases, since platform A reduces the number of participating firms while its market share decreases, both the total number of relevant ads and the total number of ads on platform A decrease. Similarly, since platform B accommodates more firms and its market share increases, the total number of relevant ads on platform B increases. However, the total number of ads on platform B could decrease, as the total number of irrelevant ads might decrease. When the targeting ability of platform A is low ($\alpha_A \leq 2/3$), the combined total number of relevant consumers on the two platforms increases as α_B increases. This means that the increase in relevant ads on platform B is bigger than the corresponding decrease on platform A. This implies that when $\alpha_A \leq 2/3$, an increase in α_B would lead to more matches identified and more realized sales.

On the firms' side, an increase in α_B induces some new firms, who initially did not participate on either platform, to participate on platform B. Those firms are clearly better off as now they have access to relevant consumers. As to firms who initially participated only on platform B, they are also better off with a bigger α_B, since platform B is now appealing to even lower types. Note that both types of firms are the relatively low-profitable ones. This result can be considered as one manifestation of the "long tail of advertising"[13]: the increase in the targeting ability of the advantaged platform enables less profitable firms, who were previously excluded, to have access to the advertising market and hence consumers.

Now we study how the equilibrium will change as platform A's targeting ability increases.

Proposition 4 *Suppose $\alpha_A' > \alpha_A$, α_B remains the same, and $\alpha_A' < \alpha_B$. Let the superscript $'$ denote the endogenous variables in the equilibrium under (α_A', α_B). Then, (i) $\mu_A^{*\prime} > \mu_A^*$ and $\mu_B^{*\prime} < \mu_B^*$; (ii) $m_A^{*\prime} > m_A^*$ and $m_B^{*\prime} < m_B^*$; (iii) both platforms will have less irrelevant ads*

[12] In our numerical examples with v being uniformly distributed on $[0, 1]$ and γ being relatively big, p_B^* is increasing in α_B when α_B is relatively small and p_B^* is decreasing in α_B when α_B is relatively close to 1.

[13] Anderson (2006).

per consumer, each consumer incurs a lower nuisance cost, and consumers on platform A gain more than consumers on platform B; (iv) the number of relevant ads per consumer and the total number of relevant ads on platform A increase, while those on platform B decrease; (v) The total number of ads on platform B decreases; if y is small enough, then the total number of ads on platform A decreases; (vi) firms with $v \in (\widehat{v}_B^*, \widehat{v}_A^{*\prime})$ *are strictly worse off.*

As the targeting ability of the disadvantaged platform (A) increases, compared to the case that α_B increases, Proposition 4 shows that in most aspects the directions of changes in the equilibrium variables of the two individual platforms are reversed. However, all consumers are still better off since they end up with lower nuisance costs. This is because as α_A increases, platform B reduces the number of participating firms, which means that consumers on platform B are better off. The fact that platform A gains more market share means that consumers on platform A must gain more than those on platform B do. Combining with previous results, we reach the general conclusion that consumers always benefit from increases in targeting abilities, regardless of on which platform the increases occur.

This result is different from Johnson (2013), in which consumers might incur a higher total nuisance cost when the targeting ability of an implicit monopolist platform increases. In his model, an increase in targeting ability has two effects. First, there is a mix effect as the fraction of irrelevant ads decreases. Second, there is a volume effect as firms will send more ads. The overall effect is ambiguous. In our model, these two effects are also present. In particular, for the platform whose targeting ability increases, the fraction of irrelevant ads decreases but the number of participating firms increases. However, in our model the mix effect always dominates the volume effect so that overall consumers benefit from a higher targeting ability. The underlying reason for the difference is that in our model the two platforms are competing with each other, while in Johnson (2013) there is just a single platform (hence no competition).[14] With competition, each platform has to worry about its market share. As a result, when a platform's targeting ability increases, in order to gain more market share, it will restrict the increase in the number of participating firms so that the volume effect is smaller than the mix effect.

In some respects, an increase in α_A has qualitatively different effects from an increase in α_B. The first difference is regarding the impacts on firms. While an increase in α_B benefits less profitable firms, an increase in α_A actually makes less profitable firms worse off, since the advantaged platform (B) will accommodate fewer firms and charge a higher price. Thus, an increase in the targeting ability of the disadvantaged platform is "anti long tail." The second difference is regarding the efficiency of the allocation of consumers. When α_B increases, platform B gains market share. Since platform B is the more efficient platform, more consumers joining the more efficient platform means that the allocation of consumers becomes more efficient. In contrast, when α_A increases, the allocation of consumers becomes less efficient as more consumers join the less efficient platform.

[14] Another difference is that in Johnson (2013), the platform is passive in the sense that the per-impression ad price is exogenously given.

4 Monopoly Ownership

In this section we study the situation in which two platforms are owned by a single monopoly. The monopolist, by choosing μ_A and μ_B, tries to maximize the joint profit of the two platforms, $\Pi(\mu_A, \mu_B)$. In particular,

$$\Pi(\mu_A, \mu_B) = \alpha_A q \widehat{v}_A m_A[1 - F(\widehat{v}_A)] + \alpha_B q \widehat{v}_B m_B[1 - F(\widehat{v}_B)].$$

Let the superscript M denote the optimal solution under monopoly ownership. The monopoly solution is characterized by the following first order conditions:[15]

$$\frac{x_A\left(\alpha_A \widehat{v}_A^M \mu_A^M - \alpha_B \widehat{v}_B^M \mu_B^M\right)}{\alpha_A\left(\widehat{v}_A^M - \frac{1-F(\widehat{v}_A^M)}{f(\widehat{v}_A^M)}\right)} = \frac{1}{2} + x_B \mu_B^M - x_A \mu_A^M = m_A^M, \tag{8}$$

$$\frac{x_B\left(\alpha_B \widehat{v}_B^M \mu_B^M - \alpha_A \widehat{v}_A^M \mu_A^M\right)}{\alpha_B\left(\widehat{v}_B^M - \frac{1-F(\widehat{v}_B^M)}{f(\widehat{v}_B^M)}\right)} = \frac{1}{2} + x_A \mu_A^M - x_B \mu_B^M = m_B^M. \tag{9}$$

Proposition 5 *When two platforms are owned by a single monopoly, we have (i) $m_A^M < \frac{1}{2} < m_B^M$; (ii) $\mu_A^M > \widetilde{\mu} > \mu_B^M > \mu_B^* > \mu_A^*$; (iii) $m_A^* > m_A^M$, and $m_B^* < m_B^M$; (iv) $p_A^M < p_A^*$, $p_B^M < p_B^*$, and $p_A^M < p_B^M$.*

Proposition 5 indicates that, compared to the equilibrium of competing platforms, under monopoly ownership each platform has more participating firms, charges lower prices, and the advantaged (disadvantaged) platform has a bigger (smaller) market share. Moreover, the disadvantaged platform accommodates more firms, and charges a lower price than the advantaged firm does. Intuitively, the monopoly owner internalizes the competition between the two platforms. The removal of the business stealing effect naturally leads to more advertising firms on each platform, which in turn implies lower ad prices. For the same reason, the monopoly owner tries to steer more consumers to the advantaged platform, as each consumer on that platform generates more revenue. In order to achieve that, it intentionally increases the ads per consumer on the disadvantaged platform by much more. Actually, under the monopoly solution platform A's marginal revenue is negative, which balances with the positive distribution effect (the difference of revenues per consumer between the two platforms).

Compared to the case of competing platforms, under monopoly ownership each platform has more ads per consumer. Since platform B has a bigger market share, it also has more ads in total under monopoly. However, platform A could have fewer ads in total, as its market share decreases under monopoly. Previous analysis also suggests that consumer allocation under monopoly is more skewed toward the more efficient platform. This implies that monopoly ownership could lead to a higher social welfare than competition does.

[15] Notice that we no longer need $\mu \in [0, \widetilde{\mu})$ for the second order condition to be satisfied.

$$\frac{\partial^2 \Pi}{\partial \mu_i^2} = \alpha_i q \left\{ -2x_i \left(\widehat{v}_i - \frac{1 - F(\widehat{v}_i)}{f(\widehat{v}_i)} \right) - m_i \left(1 - \left(\frac{1 - F(\widehat{v}_i)}{f(\widehat{v}_i)} \right)' \right) \frac{1}{f(\widehat{v}_i)} \right\}.$$

From the above expression, $\left(\widehat{v}_i - \frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)}\right)$ could be negative but $\frac{\partial^2 \Pi}{\partial \mu_i^2} < 0$.

5 Socially Optimal Allocation

In this section, we study the socially optimal advertising levels and the corresponding allocation of consumers between the two platforms. For that purpose, consider a social planner who chooses μ_A and μ_B, given α_A and α_B, to maximize social surplus. Denote the social surplus as $SS(\mu_A, \mu_B)$. In particular,

$$\begin{aligned} SS(\mu_A, \mu_B) &= \bar{\beta} - t(\frac{1}{2} - m_A + m_A^2) - \gamma(1-q)[(1-\alpha_A)\mu_A m_A + (1-\alpha_B)\mu_B m_B] \\ &\quad +[\alpha_A q m_A S_A + \alpha_B q m_B S_B], \end{aligned}$$

where $S_i = \int_{\hat{v}_i}^{\overline{v}} vf(v)dv$ denotes the (total) firm surplus generated by platform i per consumer. According to the above expression, the social surplus has four terms. The first term $\bar{\beta}$ is consumers' basic utility of joining platforms. The second term is the total transportation cost incurred by consumers. The third term is the total nuisance cost suffered by consumers. The last term includes the firm surplus generated on platform A and platform B.

The first order conditions characterizing the socially optimal advertising levels, μ_A^o and μ_B^o, are given by

$$(\alpha_A S_A^o - \alpha_B S_B^o) + m_A^o[-\frac{1}{y}\frac{\alpha_A}{1-\alpha_A}\hat{v}_A^o + \frac{2t}{q}] = 0, \tag{10}$$

$$(\alpha_A S_A^o - \alpha_B S_B^o) + m_B^o[\frac{1}{y}\frac{\alpha_B}{1-\alpha_B}\hat{v}_B^o - \frac{2t}{q}] = 0. \tag{11}$$

According to (10) and (11), the socially optimal allocation is determined by two effects. The first effect is the distribution effect, which is captured by the term $\alpha_A S_A^o - \alpha_B S_B^o$. Roughly speaking, this measures the difference between firm surplus per consumer on platform A and that on platform B. The second effect is the surplus effect, which measures the (normalized) marginal social surplus of a platform. In particular, the terms in the brackets of (10) and (11) represent the surplus effect of platform A and platform B, respectively (the first term is the marginal firms' surplus and the second term is consumers' nuisance cost). The following Lemma shows some useful properties of the socially optimal allocation.

Lemma 3 *Suppose the solution is interior. Then (i)* $\alpha_A S_A^o - \alpha_B S_B^o < 0$*; (ii)* $\frac{\alpha_B}{1-\alpha_B}\hat{v}_B^o > \gamma\frac{1-q}{q} > \frac{\alpha_A}{1-\alpha_A}\hat{v}_A^o$*; (iii) given parameter values, the socially optimal* (μ_A^o, μ_B^o) *or* $(\hat{v}_A^o, \hat{v}_B^o)$ *is unique.*

Lemma 3 tells us that, under the socially optimal allocation, the distribution effect is negative: more firm surplus will be generated by a consumer on platform B because it has a higher targeting ability. Moreover, platform A's marginal social surplus is negative while platform B's marginal social surplus is positive. This is because increasing μ_A and decreasing μ_B can induce more consumers to join the more efficient platform B, which is embodied in the distribution effect.

We will compare ad volumes and the allocation of consumers, between the social optimum and the equilibrium outcome. Although we can conceptually separate ad volume from the allocation of consumers, they are intimately related. Anderson and Coate (2005) also compare ad volume between the social optimum and equilibrium. The difference is that in their model, two platforms are symmetric, so the allocation of consumers is not a concern.

Proposition 6 *(i) If γ is small enough such that y is small enough, then $\mu_A^* < \mu_A^o$ and $\mu_B^* < \mu_B^o$; if y is bounded and $\gamma > \frac{q}{1-q}\frac{\alpha_B}{1-\alpha_B}\overline{v}$, then $\mu_A^* > \mu_A^o$ and $\mu_B^* > \mu_B^o$. (ii) Suppose t is big enough. If $\gamma > \frac{q}{1-q}\frac{\alpha_B}{1-\alpha_B}\widetilde{v}$, then $\mu_A^* > \mu_A^o$ and $\mu_B^* > \mu_B^o$; if $\gamma < \frac{q}{1-q}\frac{\alpha_A}{1-\alpha_A}\widetilde{v}$, then $\mu_A^* < \mu_A^o$ and $\mu_B^* < \mu_B^o$; if $\frac{q}{1-q}\frac{\alpha_A}{1-\alpha_A}\widetilde{v} < \gamma < \frac{q}{1-q}\frac{\alpha_B}{1-\alpha_B}\widetilde{v}$, then $\mu_A^* > \mu_A^o$ and $\mu_B^* < \mu_B^o$. (iii) Suppose t is small enough. If $\gamma < \frac{q}{1-q}\frac{\alpha_B}{1-\alpha_B}\overline{v}$, then $\mu_A^* < \mu_A^o$ and $\mu_B^* < \mu_B^o$. (iv) If α_B is close enough to 1, then $\mu_B^* < \mu_B^o$.*

As pointed out earlier, the fractions of participating firms under the social optimum and those under equilibrium are determined by different forces. So generically they will be different. Part (i) of Proposition 6 is intuitive. When the nuisance cost is very small, in equilibrium there is essentially no competition, so each platform will act like a local monopolist. Under social optimum, the distribution effect is zero and each platform should accommodate all firms. Since platforms cannot appropriate all the surplus generated by advertising firms, in equilibrium platforms naturally under-provide ads by accommodating fewer firms. When the nuisance cost is very high (higher than the profit per sale of the most profitable firm), under the social optimum no ads should be provided. But in equilibrium, both platforms will still accommodate positive fractions of firms since they do not care about consumers' nuisance costs per se. When platform B has perfect targeting ability (part iv), under the social optimum platform B should accommodate all firms. But in equilibrium platform B will only accommodate a fraction of firms because it cannot appropriate all the firm surplus generated, which leads to the under-provision of ads.

In part (ii) of Proposition 6, a big transportation cost implies that in equilibrium the business stealing effect is absent, and each firm will choose the same $\widetilde{\mu}$ (zero marginal revenue). Under the social optimum a big transportation cost implies the distribution effect is zero, so for each platform the marginal social surplus must be zero. Depending on the magnitude of the nuisance cost, in equilibrium both platforms could over-provide or under provide ads. Interestingly, when the nuisance cost is intermediate, the following scenario could occur: platform A over-provides ads while platform B under-provides ads. Intuitively, platform B should provide more ads than platform A as it is more efficient. But a big transportation cost gets rid of the competition between the two platforms and they choose the same fraction of participating firms in equilibrium. On the other hand, when the transportation cost goes to zero (part iii), the competition for consumers is so strong that in equilibrium both platforms provide no ads. However, in the social optimum a zero transportation cost just means that all consumers should join the more efficient platform B, and platform B should provide ads if the nuisance cost is not too big. To ensure all consumers join platform B, platform A should also accommodate a big enough fraction of firms. Thus both platforms under-provide ads in equilibrium.

Under the social optimum, it is possible that platform A accommodates more firms than platform B does: $\mu_A^o > \mu_B^o$. Moreover, it is also possible that in equilibrium platform A under-provides ads while platform B over-provides ads: $\mu_A^* < \mu_A^o$ and $\mu_B^* > \mu_B^o$. These possibilities are illustrated in the following example. Suppose v is uniformly distributed on $[0, 1]$, $q = 0.1$, $\gamma = 1$, $t = 0.1$, $\alpha_A = 0.8$, and $\alpha_B = 0.93$. Then $\mu_A^o \geq 0.7439$, $\mu_B^o = 0.3553$, $\mu_A^* = 0.2725$, and $\mu_B^* = 0.4363$. Since the nuisance cost γ is high and the transportation cost t is low, it is socially optimal to only advertise on platform B. To ensure all consumers go to platform B, platform A accommodates a large fraction of firms. This explains why $\mu_A^o > \mu_B^o$. Since γ is big, $\mu_B^o < \mu_B^*$ as the social planner cares about consumer surplus while firms do not.

Now we investigate the difference between the equilibrium allocation of consumers and the socially optimal allocation of consumers between the two platforms. This is related to the comparison between the equilibrium difference of ad levels and the socially optimal difference of ad levels between the two platforms. In particular, if the equilibrium market share of platform A is bigger than the socially optimal market share of platform A, then it indicates that, relative to the social optimum, the equilibrium ad level of platform A is too low relative to that of platform B.

Proposition 7 *(i) Suppose* $\gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A} < \widetilde{v}$. *If* α_B *is big enough, then we have* $m_A^o < m_A^*$. *(ii) Suppose* $\widetilde{v} \ll \gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A} \leq \overline{v}$, *and* t *is big such that* $y = \gamma(1-q)/t$ *is close to zero, then* $m_A^o > m_A^*$ *if* α_B *is big enough. (iii) Suppose* γ *is big enough such that* $\gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A} < \overline{v}$ *but is very close to* $\overline{v}$. *Then for* α_B *close enough to* α_A, $m_A^o > 1/2 > m_A^*$.

Generally speaking, the distribution effect under the social optimum tends to make the optimal market share of platform A smaller than the equilibrium market share. This is because platform B is more efficient, thus steering some marginal consumers from platform A to platform B will increase social welfare. Moreover, this effect is absent in equilibrium, as both platforms do not internalize competition and try to protect their market shares. On the other hand, the distribution effect is constrained by the surplus effect. This is because to steer more consumers to platform B, the marginal surplus of platform A and that of platform B will be further away from zero. Recall that under equilibrium the marginal revenue effect does not take into account consumer surplus. Therefore, the surplus effect is always stronger than the marginal revenue effect, which means that the surplus effect tends to make the optimal market share of platform A bigger than the equilibrium market share. Overall, whether the equilibrium market share of platform A is bigger or smaller than the socially optimal one depends on which effect dominates.

Parts (i) and (ii) of Proposition 7 consider the limiting situation in which platform B almost has perfect targeting ability. In that case, if the nuisance cost γ is relatively small and/or the targeting ability of platform A, α_A, is relatively high, then the socially optimal market share of platform A is smaller than the equilibrium one. On the other hand, if the nuisance cost γ is relatively big, the targeting ability of platform A, α_A, is relatively low, and the two platforms are more horizontally differentiated (t is relatively big), then the socially optimal market share of platform A is smaller than the equilibrium one. These results can be understood in terms of the intuition mentioned earlier. Platform B having perfect targeting ability means that in either equilibrium or the social optimum the market shares only depend on platform A's level of ads. When the nuisance cost γ is small and/or the targeting ability of platform A, α_A, is relatively high, the surplus effect is rather small relative to the distribution effect, as increasing μ_A will have a rather small negative impact on the social surplus of platform A. But the dominance of the distribution effect implies that the equilibrium market share of platform A is bigger than the socially optimal one. On the other hand, when the nuisance cost γ is relatively big, and the targeting ability of platform A, α_A, is relatively low, the surplus effect becomes more important. The fact that transportation cost t is high reduces the importance of the distribution effect. Thus, overall the surplus effect becomes dominant. But this implies that the equilibrium market share of platform A is smaller than the socially optimal one.

Part (iii) of Proposition 7 considers another limiting situation in which platform B almost has the same targeting ability as platform A. It shows that, when the nuisance cost γ is relatively big, and/or the targeting ability of platform A, α_A, is relatively low, the socially optimal market share of platform A is not only bigger than the equilibrium market share of platform A, but also bigger than the socially optimal market share of platform B. The first result is easy to understand. A big γ and a low α_A imply that the surplus effect is relatively important, and the fact that α_A and α_B are very close makes the distribution effect negligible.

It is surprising that the socially optimal market share of platform A could be bigger than $1/2$. That is, the less efficient platform has more consumers. Recall that in equilibrium this will never occur. To understand the underlying intuition, consider a special case where $\alpha_B = \alpha_A$ and γ is big such that the nuisance cost incurred per ad is the same as the profit per sale of the most profitable firm. In this scenario, the socially optimal allocation is symmetric and both platforms accommodate no firms. Now suppose α_B increases slightly. Now it is socially desirable for platform B to accommodate some firms. Since the nuisance cost γ is big, the surplus effect is big, and the fact that α_B is close to α_A means that the distribution effect is negligible. Thus in the social optimum, platform A will still almost accommodate no firms, and platform B will accommodate a positive fraction of firms. The net result is that platform A gets more than half of consumers.

6 Investment in Targeting Ability

In this section, we investigate the platforms' incentives as well as the social planner's incentives to invest in targeting ability. For that purpose, we add an investment stage to the basic model, in which the targeting abilities of platforms are chosen. We study two cases. In the first case, two platforms are symmetric as they both choose targeting abilities. In the second case, only platform B chooses targeting ability in the investment stage. While the second case is relevant when an online platform competes with an offline one, the first case applies to the situation where two online platforms compete with each other.

6.1 Symmetric case

Suppose in the investment stage, each platform i, $i = A, B$, simultaneously chooses its own targeting ability α_i. Two platforms have the same cost function $C(\alpha_i)$, where $C'(\cdot) \geq 0$ and $C''(\cdot) > 0$. We further assume that $C'(\frac{1}{2}) = 0$ and $C'(1) = \infty$. These conditions ensure that the resulting targeting ability for both platforms will be interior: $\alpha_i \in (1/2, 1)$. At the end of the investment stage, the chosen α_A and α_B become publicly observable. Then the regular stage of the basic model begins.

Given α_A and α_B, the equilibrium in the regular stage has been characterized in Section 3. Denote $\mu_A^\star(\alpha_A, \alpha_B)$ and $\mu_B^\star(\alpha_A, \alpha_B)$ as the equilibrium fractions of advertising firms. Now consider the choice of targeting ability in the investment stage. We will focus on symmetric equilibrium in which the two platforms choose the same targeting ability. Denote the (symmetric) equilibrium level of targeting ability as α^*. Suppose platform j chooses the equilibrium targeting ability: $\alpha_j = \alpha^*$. Then platform i's profit evaluated at the investment stage is given by

$$\Pi_i(\alpha_i,\alpha^\star)=\alpha_i q\widehat{v}_i^*(\alpha_i,\alpha^\star)\mu_i^\star(\alpha_i,\alpha^\star)\left\{\frac{1}{2}+y\left[(1-\alpha^\star)\mu_j^\star(\alpha_i,\alpha^\star)-(1-\alpha_i)\mu_i^\star(\alpha_i,\alpha^\star)\right]\right\}-C(\alpha_i).$$

Taking the derivative with respect to α_i, using the Envelope Theorem, and imposing symmetry $\alpha_i^\star=\alpha^*$, we get the following first order condition

$$q\mu^\star\widehat{v}^*\times\left\{\frac{1}{2}+y(1-\alpha^\star)\alpha^\star\frac{\partial\mu_j^\star(\alpha^\star,\alpha^\star)}{\partial\alpha_i}+y\alpha^\star\mu^\star\right\}=C'(\alpha^*), \tag{12}$$

where in the symmetric equilibrium $\mu^\star\equiv\mu^\star(\alpha^\star,\alpha^\star)$ is given by

$$Z^*\equiv\frac{\mu^\star\widehat{v}^*}{\widehat{v}^*-\frac{1-F(\widehat{v}^*)}{f(\widehat{v}^*)}}=\frac{1}{2y(1-\alpha^\star)}. \tag{13}$$

The term $\frac{\partial\mu_j^\star(\alpha^\star,\alpha^\star)}{\partial\alpha_i}$ can be calculated as follows:

$$\begin{aligned}\frac{\partial\mu_j^\star(\alpha^\star,\alpha^\star)}{\partial\alpha_i} &= \frac{1}{(1-\alpha^*)}\frac{\frac{dZ^*}{d\mu^*}\mu^*-Z^*}{1-(1+\frac{dZ^*}{d\mu^*})^2},\\ \frac{dZ^*}{d\mu^*} &= \frac{\frac{(\mu^*)^2}{[f(\widehat{v}^*)]^2}+(\widehat{v}^*)^2+\frac{(\mu^*)^2\widehat{v}^*f'(\widehat{v}^*)}{[f(\widehat{v}^*)]^3}}{[\widehat{v}^*-\frac{1-F(\widehat{v}^*)}{f(\widehat{v}^*)}]^2}.\end{aligned}$$

From previous results, $\frac{\partial\mu_j^\star(\alpha^\star,\alpha^\star)}{\partial\alpha_i}<0$.

An individual platform's incentive to invest in targeting ability, represented by the LHS of (12), can be separated into two effects. The first effect is the profit margin effect, captured by the first term in the bracket. An increase in α means a platform can charge a higher price per impression without reducing the number of participating firms. The second effect is the business stealing effect, captured by the last two terms in the bracket. An increase in α_i would increase platform i's equilibrium market share in the regular stage. While the third term captures the direct impact of changes in α on the market share, the second term is the indirect impact through the changes in platform j's ad volume.

Now consider the socially optimal level of targeting ability. Suppose a social planner chooses the targeting abilities for both platforms in the investment stage, and then lets the two platforms compete by choosing levels of ads.[16] The socially optimal solution must be symmetric: the two platforms have the same targeting ability. Given the equilibrium in the later stage $\mu^\star(\alpha,\alpha)$ (or $\widehat{v}^*(\alpha,\alpha)$), the social surplus generated by both platforms evaluated at the investment stage is given by

$$SS(\alpha)=\bar{\beta}-\frac{1}{4}t-\gamma(1-\alpha)(1-q)\mu^\star(\alpha,\alpha)+\alpha q\int_{\widehat{v}^*(\alpha,\alpha)}^{\overline{v}}vf(v)dv-2C(\alpha).$$

[16] In an alternative setting, the social planner could choose both the targeting abilities in the first stage and the ad levels in the second stage. In this setting, the social and equilibrium incentive to invest are further away from each other than in the setting we are considering.

The socially optimal α^o solves

$$\frac{1}{2}\gamma(1-q)[\mu^* - (1-\alpha^o)\frac{d\mu^\star}{d\alpha}] + \frac{1}{2}q\int_{\widehat{v}^*}^{\overline{v}} vf(v)dv + \frac{1}{2}\alpha^o q\widehat{v}^*\frac{d\mu^\star}{d\alpha} = C'(\alpha^o), \tag{14}$$

where

$$\frac{d\mu^\star}{d\alpha} = \frac{1}{2y(1-\alpha^o)^2\frac{dZ^*}{d\mu^*}},\ Z^*(\alpha^o) \equiv \frac{\mu^\star\widehat{v}^*}{\widehat{v}^* - \frac{1-F(\widehat{v}^*)}{f(\widehat{v}^*)}} = \frac{1}{2y(1-\alpha^o)}.$$

The LHS of (14) represents the social incentive to invest in targeting ability. The first term (including the two terms in the bracket) is the consumer surplus effect, which measures how an increase in α impacts consumer surplus. The first term in the bracket is the direct effect: an increase in α reduces consumers' chance of viewing irrelevant ads. The second term in the bracket is an indirect effect: as α increases, both platforms will increase ad volume, which decreases consumer welfare. Overall, the consumer surplus effect is positive (this will be shown later). The second term is the direct effect on firm surplus: an increase in α means that existing firms can be matched with more relevant consumers. The last term is the indirect effect on firm surplus: platforms with a higher α can accommodate more firms, which also contributes to firm surplus. Note that the social incentive to invest does not have a business stealing effect. This is because the social planner internalizes the competition between the two platforms. Moreover, since the platforms are symmetric, there is no distribution effect either.

Now we compare the social incentive and the equilibrium incentive to invest in targeting ability. For that purpose, we assume that $C'(\alpha)$ increases fast enough such that both α^* and α^o are unique. Define the LHS of (12) as $H^*(\alpha)$ and the LHS of (14) as $H^o(\alpha)$. Specifically,

$$\begin{aligned} H^o(\alpha) - H^*(\alpha) &= \frac{1}{2}\gamma(1-q)[\mu^* - (1-\alpha)\frac{d\mu^\star}{d\alpha}] + \frac{1}{2}q[\int_{\widehat{v}^*}^{\overline{v}} vf(v)dv - \mu^\star\widehat{v}^*] \\ &\quad +\alpha q\widehat{v}^*[\frac{1}{4y(1-\alpha)^2\frac{dZ^*}{d\mu^*}} - y\mu^*\frac{Z^* + \mu^*[\frac{dZ^*}{d\mu^*} + (\frac{dZ^*}{d\mu^*})^2]}{2\frac{dZ^*}{d\mu^*} + (\frac{dZ^*}{d\mu^*})^2}]. \end{aligned} \tag{15}$$

According to (15), given α the difference between the social incentive and equilibrium incentive can be attributed to three terms. The first term is the consumer surplus effect, which is positive and only present under the social incentive. This term implies that in equilibrium platforms tend to underinvest in targeting ability. Intuitively, this is because platforms do not care about consumer surplus per se. The second term is the difference between the direct firm surplus effect under the social incentive and the profit margin effect under the individual incentive. It can be readily seen that this term is positive. Intuitively, as the targeting ability increases, platforms cannot appropriate the increase in firm surplus, instead they can only get the increase in the marginal firm's surplus. The third term is the difference between the indirect firm surplus effect under the social incentive and the business stealing effect under the individual incentive. The sign of this term is indeterminate. The following proposition describes situations under which platforms underinvest or overinvest.

Proposition 8 *(i) If either t is big enough or γ is small enough such that $y \to 0$, then $\alpha^\star < \alpha^o$, or in equilibrium platforms underinvest in targeting ability. (ii) Fix y, $y > 0$ and bounded, and* *let γ and t increase at the same rate. If γ is big enough, then $\alpha^\star < \alpha^o$. (iii) Suppose v is uniformly distributed on $[0,1]$. If either t is big, γ is small, or the marginal cost of investment in targeting ability is high enough such that $\mu^* \leq \widehat{\mu}$, where $\widehat{\mu} \in (0, 1/2)$ is the solution to $1 - 7\mu + 14\mu^2 - 10\mu^3 = 0$, then $\alpha^\star < \alpha^o$. Overinvestment ($\alpha^\star > \alpha^o$) occurs only if γ is close enough to 0, t is even closer to 0 such that y is quite big, and the marginal cost of investment in targeting ability is low enough.*

In part (i) of Proposition 8, since t is big enough or γ is small enough, there is essentially no competition between the two platforms. Both platforms will choose the optimal ads volume $\widetilde{\mu}$ as if they were local monopolists, and the equilibrium level of ads is independent of the targeting abilities. So there is neither the business stealing effect under competition, nor the indirect firm surplus effect under the social optimum. According to (15), the third term is zero and the equilibrium incentive to invest in targeting ability is lower than the socially optimal level. In the scenario of part (ii), the nuisance cost of ads is very high so that the consumer surplus effect dominates. Thus again in equilibrium platforms underinvest in targeting ability.[17]

In part (iii) with the firms' profits uniformly distributed, it shows that the third term of (15) is negative, or the business stealing effect under the equilibrium incentive is stronger than the indirect firm surplus effect under the social incentive, if and only if the equilibrium level of ads is high enough. The underlying reason for this property is that, while the indirect firm surplus effect is more or less the same as the ad level changes, the business stealing effect becomes stronger as the ad level becomes higher.[18] If the nuisance cost of ads is high, t is small, or the marginal cost of investing in targeting ability is high, then the equilibrium ad level is low and platforms will underinvest. Overinvestment will occur only if the business stealing effect is stronger than other effects, which can be precisely translated into the following conditions: the nuisance cost of ads is small (the consumer surplus effect is negligible), t is even smaller (y is big and the business stealing effect is strong), and the marginal cost of investing in targeting ability is low enough (α is high enough to ensure μ is big). In the numerical examples that we run, it is verified that overinvestment occurs in a very restrictive parameter space, while underinvestment occurs for most of the parameter space.

Holding α_A Constant

Suppose in the investment stage, $\alpha_A > 1/2$ is exogenously fixed and publicly known, and platform B chooses its own targeting ability α_B with the cost function $C(\alpha_B)$. At the beginning of the regular stage, α_B becomes publicly known. We impose the following conditions on $C(\cdot)$: $C'(\cdot) > 0$, $C''(\cdot) > 0$, $C'(\alpha_A) = 0$, and $C'(1) = \infty$. The last two conditions ensure that $\alpha_B \in (\alpha_A, 1)$.

In the investment stage, platform B's profit can be computed as

$$\Pi_B(\alpha_B) = \alpha_B q \widehat{v}_B^* \mu_B^\star \left[\frac{1}{2} + y(1-\alpha_A)\mu_A^\star - y(1-\alpha_B)\mu_B^\star\right] - C(\alpha_B).$$

[17] If γ is big enough or t is small enough such that $y \to \infty$, then $\mu^\star = 0$ and $\widehat{v}^* = \overline{v}$. The equilibrium incentive and socially optimal incentive to invest in targeting ability coincide: both are zero and $\alpha = 1/2$.

[18] In particular, as the ad level increases, a given increase in α_i will allow platform i to steal more consumers from platform j.

Using the Envelope Theorem, the optimal $\alpha_B^\star$ solves

$$q\hat{v}_B^\star \mu_B^\star \left\{ m_B^\star + y\alpha_B^\star(1-\alpha_A)\frac{\partial \mu_A}{\partial \alpha_B} + y\alpha_B^\star \mu_B^\star \right\} = C'(\alpha_B^\star). \tag{16}$$

where

$$\begin{aligned}
(1-\alpha_B)\frac{\partial \mu_B^\star}{\partial \alpha_B} &= \frac{\mu_B^* \frac{\partial Z_A^*}{\partial \mu_A} + Z_B^*(1+\frac{\partial Z_A^*}{\partial \mu_A})}{\frac{\partial Z_A^*}{\partial \mu_A} + \frac{\partial Z_B^*}{\partial \mu_B} + \frac{\partial Z_A^*}{\partial \mu_A}\frac{\partial Z_B^*}{\partial \mu_B}}, \\
(1-\alpha_A)\frac{\partial \mu_A^\star}{\partial \alpha_B} &= \frac{-\mu_B^* \frac{\partial Z_B^*}{\partial \mu_B} + Z_B^*}{\frac{\partial Z_A^*}{\partial \mu_A} + \frac{\partial Z_B^*}{\partial \mu_B} + \frac{\partial Z_A^*}{\partial \mu_A}\frac{\partial Z_B^*}{\partial \mu_B}}, \\
\frac{dZ_i^*}{d\mu_i} &= \frac{\frac{(\mu_i^*)^2}{[f(\widehat{v}_i^*)]^2} + (\widehat{v}_i^*)^2 + \frac{(\mu_i^*)^2 \widehat{v}_i^* f'(\widehat{v}_i^*)}{[f(\widehat{v}_i^*)]^3}}{[\widehat{v}_i^* - \frac{1-F(\widehat{v}_i^*)}{f(\widehat{v}_i^*)}]^2}.
\end{aligned}$$

Again, platform B's incentive to invest in targeting ability, represented by the LHS of (16), can be separated into two effects. The first effect is the profit margin effect, which is captured by the first term in the bracket. The last two terms in the bracket are the business stealing effect.

As y increases (either γ increases or t decreases), the equilibrium α_B^* could either increase or decrease. In particular, as y increases, by the profit margin effect platform B's incentive to invest in targeting ability will be dampened. This is because an increase in y means more intense competition, and hence a smaller profit margin in the second stage, which reduces the return on investing in targeting ability. However, as y increases, by the business stealing effect platform B will have a stronger incentive to invest in targeting ability. Intuitively, more intense competition in the second stage means that a given amount of increase in α_B will now steal more market shares from platform A. Overall, either effect could dominate, so α_B^* could increase or decrease as y increases.

Now consider the socially optimal level of targeting ability. Suppose a social planner chooses the targeting ability for platform B in the investment stage, and then lets the two platforms compete. Given the ensuing equilibrium in the regular stage $(\mu_A^\star, \mu_B^\star)$, for which we suppress their dependence on α_B, the total surplus evaluated at the investment stage is given by

$$\begin{aligned}
B(\alpha_B) &= \bar{\beta} - t(1-m_A^\star) + \frac{t}{2}(1-2m_A^{\star 2}) - \gamma(1-q)(1-\alpha_A)\mu_A^* m_A^\star - \gamma(1-q)(1-\alpha_B)\mu_B^* m_B^\star \\
&\quad + \alpha_A q m_A^\star \int_{\hat{v}_A}^{\bar{v}} v f(v) dv + \alpha_B q m_B^\star \int_{\hat{v}_B}^{\bar{v}} v f(v) dv - C(\alpha_B).
\end{aligned}$$

Therefore, the socially optimal α_B^o solves

$$\begin{aligned}
C'(\alpha_B^o) &= [\alpha_B^o q S_B^* - \alpha_A q S_A^*]\frac{\partial m_B^*}{\partial \alpha_B} + q m_B^* S_B^* + \gamma(1-q)[\mu_B^* m_B^* - (1-\alpha_B^o) m_B^* \frac{\partial \mu_B^*}{\partial \alpha_B} \\
&\quad -(1-\alpha_A) m_A^* \frac{\partial \mu_A^*}{\partial \alpha_B}] + \alpha_A q \hat{v}_A^* m_A^* \frac{\partial \mu_A^*}{\partial \alpha_B} + \alpha_B^o q \hat{v}_B^* m_B^* \frac{\partial \mu_B^*}{\partial \alpha_B}
\end{aligned} \tag{17}$$

The RHS of (17) represents the social planner's incentive to invest in α_B. The first term is the distribution effect: an increase in α_B leads more consumers to the more efficient platform. The second term is the direct effect on the firm surplus of platform B. The third term is the consumer surplus effect. While the first term in the bracket is the direct effect on the consumer surplus of platform B, the second and third terms are the indirect effects on the consumer surplus of platform B and platform A. The last terms are the indirect effects on firm surplus of platform A and platform B.

To compare the social incentive and the equilibrium incentive to invest in targeting ability, we again assume that $C'(\alpha_B)$ increases fast enough such that both α_B^* and α_B^o are unique. Define the LHS of (16) as $H^*(\alpha_B)$ and the RHS of (17) as $H^o(\alpha_B)$. Specifically,

$$\begin{aligned} H^o(\alpha_B) - H^*(\alpha_B) \quad = \quad & [\alpha_B q S_B^* - \alpha_A q S_A^*] \frac{\partial m_B^*}{\partial \alpha_B} + \gamma(1-q) \\ & \times \left[\mu_B^* m_B^* - m_B^*(1-\alpha_B) \frac{\partial \mu_B^*}{\partial \alpha_B} - (1-\alpha_A) m_A^* \frac{\partial \mu_A^*}{\partial \alpha_B} \right] + q m_B^* [\int_{\widehat{v}^*}^{\overline{v}} v f(v) dv - \mu_B^\star \widehat{v}^*] \\ & + \{ \alpha_A q \hat{v}_A^* m_A^* \frac{\partial \mu_A^*}{\partial \alpha_B} + \alpha_B q \hat{v}_B^* m_B^* \frac{\partial \mu_B^*}{\partial \alpha_B} - \alpha_B q \hat{v}_B^\star \mu_B^\star y [(1-\alpha_A) \frac{\partial \mu_A^*}{\partial \alpha_B} + \mu_B^\star] \}. \end{aligned} \tag{18}$$

According to (18), given α_B the difference between the social incentive and the equilibrium incentive can be attributed to four terms. The first term is the distribution effect, which is positive and only present under the social incentive. The second term is the consumer surplus effect, which is again positive (this will be shown later) and only present under the social incentive. These two terms imply that in equilibrium platform B tends to underinvest in targeting ability. The third term is the difference between the direct firm surplus effect under the social incentive and the profit margin effect under the individual incentive. It can be readily seen that this term is positive, since platform B cannot appropriate the total firm surplus. The last term is the difference between the indirect firm surplus effect under the social incentive and the business stealing effect under the individual incentive. The sign of this term is indeterminate. The following proposition describes situations under which platform B underinvests.

Proposition 9 *(i) If either t is big enough or γ is small enough such that $y \to 0$, then $\alpha_B^\star < \alpha_B^o$, that is, platform B underinvests in targeting ability in equilibrium. (ii) Fix y, $y > 0$ and bounded, and let γ and t increase at the same rate. If γ is big enough, then $\alpha_B^\star < \alpha_B^o$.*

Just like in the symmetric case, when t is big enough or γ is small enough, there is essentially no competition between the two platforms. As a result, the business stealing effect under the individual incentive vanishes. Thus the equilibrium incentive to invest in targeting ability is lower than the socially optimal level. In the scenario of part (ii), the nuisance cost of ads is very high so that the second term of (18) or the consumer surplus effect dominates. Thus again in equilibrium platform B underinvests in targeting ability.

In the numerical examples that we run, overinvestment occurs only in a very restrictive parameter space, while underinvestment occurs for most of the parameter space. This is because, for overinvestment to occur, the business stealing effect under the individual incentive has to dominate all other effects. In particular, y has to be big so that the business stealing effect is strong enough. In the mean time, γ has to be small to make the consumer surplus effect negligible. This means that t should be even smaller than γ so that y is big enough. A big y will reduce platform B's equilibrium revenue and dampen the individual incentive to invest. To prevent this from happening or to maintain platform B's equilibrium revenue at some level, the marginal cost of investing in targeting ability has to be small enough so that the resulting equilibrium α_B^* is close enough to 1. These conditions are reflected in the following example, in which overinvestment occurs. When $\alpha_A = 0.975$, $q = 0.1$, $\gamma = 0.23$, $t = 0.007$, and $C'(\alpha_B) = \max\{0, e^{5(\frac{20}{19}\alpha_B - 1)} - 1\}$, $\alpha_B^* = 0.9770 > 0.9765 = \alpha_B^o$.

7 Conclusion

This paper studies targeted advertising in two sided markets. Two platforms, with different targeting abilities, compete for single-homing consumers, while advertising firms are multi-homing. We show that the platform with a higher targeting ability will attract more consumers, have more advertising firms, and have more relevant ads in total. However, it might charge a lower advertising price and have fewer ads in total. When the targeting ability of either platform increases, all consumers benefit as they will incur lower nuisance costs from advertising. When the targeting ability of the advantaged platform increases, less profitable advertising firms are better off; while the disadvantaged platform increases its price, the price charged by the advantaged firm could increase or decrease. When the targeting ability of the disadvantaged platform increases, less profitable firms are better off. Compared to the equilibrium outcome, monopoly ownership always leads to a more skewed consumer allocation between the two platforms.

We also compare the advertising levels and consumer allocation under social optimum and those under equilibrium. Similar to Anderson and Coate (2005), in equilibrium the platforms could under-provide or over-provide ads. While usually both platforms under-provide ads or over-provide ads at the same time, interestingly it could sometimes be the case that one platform over-provides ads while the other one under-provides ads. Regarding the allocation of consumers between the two platforms, again in equilibrium the advantaged platform could have more consumers or fewer consumers relative to the social optimum. Under some conditions, the socially optimal market share of the disadvantaged platform could be bigger than that of the advantaged platform. Finally, we compare social incentives and private incentives to invest in targeting abilities. Platforms could underinvest in targeting ability as well as overinvest in targeting ability in equilibrium. Quantitatively, underinvestment in targeting ability is much more likely to occur, while overinvestment occurs only under very special conditions.

Appendix

Proof of Lemma 1.

Proof. Part (i). Recall that the logconcavity of f implies that $\widehat{v}_i - \frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)}$ is strictly increasing in $\widehat{v}_i$. By the definition of $\widetilde{v}$ and $\widetilde{\mu}$, we have $\widehat{v}_i - \frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)} \leq 0$ if $\mu \in [\widetilde{\mu}, 1]$, with the strict inequality if $\mu \neq \widetilde{\mu}$. Since $x_i\widehat{v}_i[1 - F(\widehat{v}_i)] \geq 0$ for any μ, we reach the conclusion that $\frac{\partial \Pi_i}{\partial \mu_i} < 0$ if $\mu \in (\widetilde{\mu}, 1]$.

Part (ii). A necessary condition for an equilibrium μ_i^* is $\frac{\partial \Pi_i}{\partial \mu_i}(\mu_i^*) = 0$. Following part (i) we must have $\mu_i^* \in [0, \widetilde{\mu}]$, if equilibrium exists.

Part (iii). We first show that $\frac{\partial^2 \Pi_i}{\partial \mu_i^2} < 0$ for $\mu_i \in [0, \widetilde{\mu})$. By (5),

$$\frac{\partial^2 \Pi_i}{\partial \mu_i^2} \propto \frac{\partial[\widehat{v}_i - \frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)}]}{\partial \mu_i} m_i(\mu_i, \mu_j) + [\widehat{v}_i - \frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)}] \frac{\partial m_i(\mu_i, \mu_j)}{\partial \mu_i} - x_i \frac{\partial\{\widehat{v}_i[1-F(\widehat{v}_i)]\}}{\partial \mu_i}.$$

Now inspect each term in the above expression. Recall that, for $\mu_i \in [0, \widetilde{\mu})$, $\widehat{v}_i - \frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)} > 0$ and is strictly decreasing in μ_i. The term $m_i(\mu_i, \mu_j)$ is also positive and strictly decreasing in μ_i. By previous analysis, $\widehat{v}_i[1-F(\widehat{v}_i)] = R(\mu_i)$ is strictly increasing in μ_i for $\mu_i \in [0, \widetilde{\mu})$. Therefore, $\frac{\partial^2 \Pi_i}{\partial \mu_i^2} < 0$ for $\mu_i \in [0, \widetilde{\mu})$.

Next we show that $\frac{\partial^2 \Pi_i}{\partial \mu_i \partial \mu_j} > 0$. Again by (5),

$$\frac{\partial^2 \Pi_i}{\partial \mu_i \partial \mu_j} \propto [\widehat{v}_i - \frac{1-F(\widehat{v}_i)}{f(\widehat{v}_i)}] x_j,$$

which is strictly greater than 0 for $\mu_i \in [0, \widetilde{\mu})$. ■

Claim 1 *Let $W(v) \equiv \frac{v}{v - \frac{1-F(v)}{f(v)}}$ and $Z(v) \equiv \frac{v[1-F(v)]}{v - \frac{1-F(v)}{f(v)}}$. Both $W(v)$ and $Z(v)$ are strictly decreasing in v for $v \in [\widetilde{v}, \overline{v})$, and hence are strictly increasing in μ for $\mu \in [0, \widetilde{\mu})$.*

Proof. Taking derivatives, we get

$$\frac{dW(v)}{dv} \propto -\frac{1-F(v)}{f(v)} + (\frac{1-F(v)}{f(v)})' < 0,$$

where the inequality uses the fact that $(\frac{1-F(v)}{f(v)})' < 0$ by the logconcavity of $f(\cdot)$. Similarly,

$$\frac{dZ(v)}{dv} \propto -f(v)[v - \frac{1-F(v)}{f(v)}]^2 - v[1-F(v)][v - \frac{1-F(v)}{f(v)}]' < 0,$$

where the inequality uses the fact that $[v - \frac{1-F(v)}{f(v)}]'$ is strictly increasing in v. ■

Proof of Lemma 2.

Proof. Recall that, by part (iii) of Lemma 1, μ_i and μ_j are strategic complements. And the domain of $[0, \widetilde{\mu}]$ is compact. By Theorem 4.2(ii) of Vives (1990), the set of Nash equilibria is non-empty. To show the uniqueness of Nash equilibrium, we add (6) and (7) and get

$$\frac{x_A \widehat{v}_A^*[1-F(\widehat{v}_A^*)]}{\widehat{v}_A^* - \frac{1-F(\widehat{v}_A^*)}{f(\widehat{v}_A^*)}} + \frac{x_B \widehat{v}_B^*[1-F(\widehat{v}_B^*)]}{\widehat{v}_B^* - \frac{1-F(\widehat{v}_B^*)}{f(\widehat{v}_B^*)}} = 1. \tag{19}$$

Using the definition of $Z(v)$, the above equation can be written compactly as $x_A Z(\widehat{v}_A^*) + x_B Z(\widehat{v}_B^*) = 1$. Now suppose there are two Nash equilibria: (μ_{A1}^*, μ_{B1}^*) and (μ_{A2}^*, μ_{B2}^*). By

Theorem 4.2(ii) Vives (1990), the two Nash equilibria can be ordered. Without loss of generality, suppose $(\mu^*_{A1}, \mu^*_{B1}) < (\mu^*_{A2}, \mu^*_{B2})$. But by the monotonicity of $Z(v)$, according to Claim 1, we have

$$x_A Z(\widehat{v}^*_{A1}) + x_B Z(\widehat{v}^*_{B1}) < x_A Z(\widehat{v}^*_{A2}) + x_B Z(\widehat{v}^*_{B2}),$$

which contradicts (19). Therefore, there is a unique equilibrium. ■

Proof of Proposition 1.

Proof. Part (i). Suppose, to the contrary, $\mu^*_B \leq \mu^*_A$. Since $x_A > x_B$, we have $x_A\mu^*_A > x_B\mu^*_B$. Taking the difference between (6) and (7), we have

$$x_B\mu^*_B - x_A\mu^*_A = \frac{1}{2}[x_A Z(\widehat{v}^*_A) - x_B Z(\widehat{v}^*_B)] = \frac{1}{2}[x_A\mu^*_A W(\widehat{v}^*_A) - x_B\mu^*_B W(\widehat{v}^*_B)]. \tag{20}$$

The RHS of the above equation, (20), is strictly greater than 0, which follows from the presumption that $\mu^*_B \leq \mu^*_A$, and the facts that $Z(v) > 0$ and is increasing in μ (by Claim 1) for $\mu \in [0, \widetilde{\mu})$, and $x_A > x_B$. This contradicts the result that the LHS of (20) is strictly less than 0. Thus we must have $\mu^*_B > \mu^*_A$. It immediately follows that $\widehat{v}^*_A > \widehat{v}^*_B$.

Part (ii). Suppose, to the contrary, $x_A\mu^*_A \leq x_B\mu^*_B$. This means that the LHS of (20) is greater than or equal to 0. Now consider the RHS of (20). By the fact that $\widetilde{\mu} > \mu^*_B > \mu^*_A$ and by Claim 1, we have $W(\widehat{v}^*_B) > W(\widehat{v}^*_A) > 0$. Combined with the presumption that $x_A\mu^*_A \leq x_B\mu^*_B$, we draw the conclusion that the RHS of (20) is strictly less than 0, a contradiction. Therefore, we must have $x_A\mu^*_A > x_B\mu^*_B$. It follows immediately that $m_A(\mu^*_A, \mu^*_B) < m_B(\mu^*_A, \mu^*_B)$.

Part (iii). The difference in equilibrium profits can be written as

$$\Pi^*_B - \Pi^*_A \propto \alpha_B m^*_B \widehat{v}^*_B[1 - F(\widehat{v}^*_B)] - \alpha_A m^*_A \widehat{v}^*_A[1 - F(\widehat{v}^*_A)].$$

Since $\alpha_B m^*_B > \alpha_A m^*_A$, $\widehat{v}^*_B[1-F(\widehat{v}^*_B)] - \widehat{v}^*_A[1-F(\widehat{v}^*_A)] \geq 0$ implies that $\Pi^*_B - \Pi^*_A > 0$. Recall that $v[1 - F(v)]$ is decreasing in v for $v \in [\widetilde{v}, \overline{v}]$. Combining with the result in part (i) that $\widehat{v}^*_A > \widehat{v}^*_B$, we have $\widehat{v}^*_B[1 - F(\widehat{v}^*_B)] - \widehat{v}^*_A[1 - F(\widehat{v}^*_A)] \geq 0$. This proves that $\Pi^*_A < \Pi^*_B$.

Part (iv). The sign of $p^*_B - p^*_A$ can be expressed as

$$p^*_B - p^*_A \propto \frac{\alpha_B q \widehat{v}^*_B}{\alpha_B q + (1 - \alpha_B)(1 - q)} - \frac{\alpha_A q \widehat{v}^*_A}{\alpha_A q + (1 - \alpha_A)(1 - q)}.$$

By the above expression, $p^*_B - p^*_A > 0$ is equivalent to

$$\frac{1 + \frac{1-\alpha_B}{\alpha_B}\frac{1-q}{q}}{1 + \frac{1-\alpha_A}{\alpha_A}\frac{1-q}{q}} < \frac{\widehat{v}^*_B}{\widehat{v}^*_A}.$$

Recall that $\mu\widehat{v}$ is strictly decreasing in $\widehat{v}$ for $\widehat{v} \in [\widetilde{v}, \overline{v})$, which means that $\mu^*_B\widehat{v}^*_B > \mu^*_A\widehat{v}^*_A$, or $\frac{\mu^*_A}{\mu^*_B} < \frac{\widehat{v}^*_B}{\widehat{v}^*_A}$. Thus, to prove $p^*_A < p^*_B$, it is enough to show that $\frac{1+\frac{1-\alpha_B}{\alpha_B}\frac{1-q}{q}}{1+\frac{1-\alpha_A}{\alpha_A}\frac{1-q}{q}} \leq \frac{\mu^*_A}{\mu^*_B}$. The fact that $x_A\mu^*_A > x_B\mu^*_B$ implies that $\frac{\mu^*_A}{\mu^*_B} > \frac{1-\alpha_B}{1-\alpha_A}$. Thus it is sufficient that $\frac{1+\frac{1-\alpha_B}{\alpha_B}\frac{1-q}{q}}{1+\frac{1-\alpha_A}{\alpha_A}\frac{1-q}{q}} \leq \frac{1-\alpha_B}{1-\alpha_A}$, which holds if $\frac{(1-\alpha_A)(1-\alpha_B)}{\alpha_A\alpha_B}\frac{1-q}{q} \geq 1$. ■

Proof of Corollary 1.

Proof. Part (i). $(1-\alpha_A)(1-q)\mu_A^* > (1-\alpha_B)(1-q)\mu_B^*$ follows naturally from the result that $x_A\mu_A^* > x_B\mu_B^*$.

Part (ii). The relevant ads per consumer and total relevant ads on platform i are $\alpha_i q\mu_i^*$ and $\alpha_i q\mu_i^* m_i^*$, respectively. Since $\alpha_A < \alpha_B$, $\mu_A^* < \mu_B^*$, and $m_A^* < m_B^*$, we have $\alpha_A q\mu_A^* < \alpha_B q\mu_B^*$ and $\alpha_A q\mu_A^* m_A^* < \alpha_B q\mu_B^* m_B^*$.

Part (iii). Consider a type v firm advertising on both platforms: $v \geq \widehat{v}_A^* > \widehat{v}_B^*$. Its profit per consumer and total profit on platform i are $\alpha_i q(v-\widehat{v}_i^*)$ and $\alpha_i q(v-\widehat{v}_i^*)m_i^*$, respectively. Since $\alpha_A < \alpha_B$, $\widehat{v}_A^* > \widehat{v}_B^*$, and $m_A^* < m_B^*$, we conclude that $\alpha_A q(v-\widehat{v}_A^*) < \alpha_B q(v-\widehat{v}_B^*)$ and $\alpha_A q(v-\widehat{v}_A^*)m_A^* < \alpha_B q(v-\widehat{v}_B^*)m_B^*$.

Part (iv). It is enough to show the claim is true when $y \to 0$. In particular, when $y \to 0$, we have $m_i^* \to \frac{1}{2}$and $\mu_i^* \to \widetilde{\mu}$. Therefore,

$$\begin{aligned} & [\alpha_A q + (1-\alpha_A)(1-q)]\,\mu_A^* m_A^* - [\alpha_B q + (1-\alpha_B)(1-q)]\,\mu_B^* m_B^* \\ \propto\ & [\alpha_A q + (1-\alpha_A)(1-q)] - [\alpha_B q + (1-\alpha_B)(1-q)] > 0, \end{aligned}$$

where the last inequality follows the fact $q < 1/2$. ■

Proof of Proposition 2.

Proof. Part (i). Define $Z_i \equiv Z(\widehat{v}_i^*)$. By the definition of y and Z_i, (6) and (7) can be rewritten as

$$\begin{aligned} (1-\alpha_B)\mu_B^* &= (1-\alpha_A)\mu_A^* + (1-\alpha_A)Z_A - \frac{1}{2y}, \\ (1-\alpha_A)\mu_A^* &= (1-\alpha_B)\mu_B^* + (1-\alpha_B)Z_B - \frac{1}{2y}. \end{aligned}$$

Differentiating the above equations with respect to y, we get

$$(1-\alpha_B)\frac{\partial\mu_B^*}{\partial y} = \frac{2+\frac{\partial Z_A}{\partial\mu_A^*}}{2y^2[1-(1+\frac{\partial Z_A}{\partial\mu_A^*})(1+\frac{\partial Z_B}{\partial\mu_B^*})]}, \tag{21}$$

$$(1-\alpha_A)\frac{\partial\mu_A^*}{\partial y} = \frac{2+\frac{\partial Z_B}{\partial\mu_B^*}}{2y^2[1-(1+\frac{\partial Z_A}{\partial\mu_A^*})(1+\frac{\partial Z_B}{\partial\mu_B^*})]}. \tag{22}$$

Since by Claim 1, $\frac{\partial Z_A}{\partial\mu_A^*} > 0$ and $\frac{\partial Z_B}{\partial\mu_B^*} > 0$, it can be readily seen from the above two expressions that $\frac{\partial\mu_A^*}{\partial y} < 0$ and $\frac{\partial\mu_B^*}{\partial y} < 0$. This implies that both μ_A^* and μ_B^* are decreasing in γ and increasing in t. Since by (1) p_i^* is decreasing in μ_i^*, we conclude that both p_A^* and p_B^* are increasing in γ and decreasing in t.

Part (ii). By (4) and the Envelope Theorem, we get

$$\frac{\partial\Pi_A^*}{\partial y} \propto \frac{\partial m_A^*}{\partial\mu_B}\frac{\partial\mu_B^*}{\partial y} + \frac{dm_A^*}{dy} < 0,$$

where the inequality follows that $\frac{\partial m_i}{\partial\mu_j} > 0$, $\frac{\partial\mu_j^*}{\partial y} < 0$, and $\frac{dm_A^*}{dy} = (1-\alpha_B)\mu_B^* - (1-\alpha_A)\mu_A^* < 0$. Therefore, Π_A^* is increasing in t but decreasing in γ. ■

Proof of Proposition 3.

Proof. Part (i). Since an increase in α_B implies a decrease in x_B, it is sufficient to show that $\frac{\partial \mu_A^*}{\partial x_B} > 0$ and $\frac{\partial \mu_B^*}{\partial x_B} < 0$. Define $W_i \equiv W(\widehat{v}_i^*)$. Differentiating (6) and (7) with respect to x_B, we get

$$\mu_B^* + x_B \frac{\partial \mu_B^*}{\partial x_B} = x_A \frac{\partial \mu_A^*}{\partial x_B}\left[1 + W_A + \mu_A^* \frac{dW_A}{d\mu_A^*}\right], \tag{23}$$

$$x_A \frac{\partial \mu_A^*}{\partial x_B} = \left(\mu_B^* + x_B \frac{\partial \mu_B^*}{\partial x_B}\right)[1 + W_B] + x_B \mu_B^* \frac{dW_B}{d\mu_B^*}\frac{\partial \mu_B^*}{\partial x_B}. \tag{24}$$

From the above two equations, we can solve for $\frac{\partial \mu_B^*}{\partial x_B}$ as follows:

$$\frac{\partial \mu_B^*}{\partial x_B} = -\frac{\left[\left(1 + W_A + \mu_A^* \frac{dW_A}{d\mu_A^*}\right)(1 + W_B) - 1\right]\mu_B^*}{\left[\left[1 + W_A + \mu_A^* \frac{dW_A}{d\mu_A^*}\right]\left(1 + W_B + \mu_B^* \frac{dW_B}{d\mu_B^*}\right) - 1\right] x_B} < 0, \tag{25}$$

where the inequality follows $\frac{dW_A}{d\mu_A^*} > 0$ and $\frac{dW_B}{d\mu_B^*} > 0$. From (23) and (25), the sign of $\frac{\partial \mu_A^*}{\partial x_B}$ can be expressed as

$$\frac{\partial \mu_A^*}{\partial x_B} \propto \mu_B^* + x_B \frac{\partial \mu_B^*}{\partial x_B} \propto \left[1 + W_A + \mu_A^* \frac{dW_A}{d\mu_A^*}\right]\left(\mu_B^* \frac{dW_B}{d\mu_B^*}\right) > 0. \tag{26}$$

Part (ii). It is sufficient to show that $\frac{\partial(x_A\mu_A^* - x_B\mu_B^*)}{\partial x_B} < 0$. More explicitly,

$$\frac{\partial(x_A\mu_A^* - x_B\mu_B^*)}{\partial x_B} = x_A \frac{\partial \mu_A^*}{\partial x_B} - \mu_B^* - x_B \frac{\partial \mu_B^*}{\partial x_B} < 0,$$

where the inequality follows (23), which implies that $\mu_B^* + x_B \frac{\partial \mu_B^*}{\partial x_B} > x_A \frac{\partial \mu_A^*}{\partial x_B}$.

Part (iii). To show $p_A^{*\prime} > p_A^*$, it is sufficient to show that $\frac{\partial p_A^*}{\partial x_B} < 0$. By (1), $\frac{\partial p_A^*}{\partial x_B} \propto \frac{d\widehat{v}_A^*}{d\mu_A^*}\frac{\partial \mu_A^*}{\partial x_B} < 0$, where the inequality follows $\frac{\partial \mu_A^*}{\partial x_B} > 0$ in part (i).

The sign of $p_B^{*\prime} - p_B^*$ can be expressed as

$$p_B^{*\prime} - p_B^* \propto -\frac{\alpha_B q \widehat{v}_B^*}{\alpha_B q + (1-\alpha_B)(1-q)} + \frac{\alpha_B' q \widehat{v}_B^{*\prime}}{\alpha_B' q + (1-\alpha_B')(1-q)}.$$

By the above expression, $p_B^{*\prime} - p_B^* > 0$ is equivalent to

$$\frac{1 + \frac{1-\alpha_B'}{\alpha_B'}\frac{1-q}{q}}{1 + \frac{1-\alpha_B}{\alpha_B}\frac{1-q}{q}} < \frac{\widehat{v}_B^{*\prime}}{\widehat{v}_B^*}.$$

Since $\widehat{v}\mu > 0$ and it is increasing in μ for $\mu \in [0, \widetilde{\mu}]$, we have $\widehat{v}_B^{*\prime}\mu_B^{*\prime} - \widehat{v}_B^*\mu_B^* > 0$, or $\frac{\mu_B^*}{\mu_B^{*\prime}} < \frac{\widehat{v}_B^{*\prime}}{\widehat{v}_B^*}$. Thus to show that $p_B^* - p_B^{*\prime} < 0$, it is enough to show that $\frac{1+\frac{1-\alpha_B'}{\alpha_B'}\frac{1-q}{q}}{1+\frac{1-\alpha_B}{\alpha_B}\frac{1-q}{q}} \leq \frac{\mu_B^*}{\mu_B^{*\prime}}$. Note that $x_B'\mu_B^{*\prime} < x_B\mu_B^*$. To see this, by (25) and (26), we have

$$\frac{\partial(x_B\mu_B^*)}{\partial x_B} = \mu_B^* + x_B \frac{\partial \mu_B^*}{\partial x_B} > 0.$$

This implies that $\frac{\partial(x_B\mu_B^*)}{\partial\alpha_B} < 0$ and $x_B'\mu_B^{*\prime} < x_B\mu_B^*$, since $\mu_B^{*\prime} > \mu_B^*$ by part (i). The fact that $x_B'\mu_B^{*\prime} < x_B\mu_B^*$ implies $\frac{1-\alpha_B'}{1-\alpha_B} < \frac{\mu_B^*}{\mu_B^{*\prime}}$. Therefore, to show $p_B^* - p_B^{*\prime} < 0$ it is sufficient that $\frac{1+\frac{1-\alpha_B'}{\alpha_B'}\frac{1-q}{q}}{1+\frac{1-\alpha_B}{\alpha_B}\frac{1-q}{q}} \leq \frac{1-\alpha_B'}{1-\alpha_B}$, which holds if $\frac{(1-\alpha_B)(1-\alpha_B')}{\alpha_B\alpha_B'}\frac{1-q}{q} \geq 1$.

Part (iv). Recall that $\widehat{v}\mu > 0$ and it is increasing in μ for $\mu \in [0, \widetilde{\mu}]$. Thus by part (i), we have $\widehat{v}_A^{*\prime}\mu_A^{*\prime} < \widehat{v}_A^*\mu_A^*$ and $\widehat{v}_B^{*\prime}\mu_B^{*\prime} > \widehat{v}_B^*\mu_B^*$. By (4), $\Pi_i^* \propto \alpha_i m_i^* \widehat{v}_i^* \mu_i^*$. Since α_A remains the same, $m_A^{*\prime} < m_A^*$ by part (ii), and $\widehat{v}_A^{*\prime}\mu_A^{*\prime} < \widehat{v}_A^*\mu_A^*$, it must be the case that $\Pi_A^{*\prime} < \Pi_A^*$. Similarly, since $\alpha_B' > \alpha_B$, $m_B^{*\prime} > m_B^*$ by part (ii), and $\widehat{v}_B^{*\prime}\mu_B^{*\prime} > \widehat{v}_B^*\mu_B^*$, we must have $\Pi_B^{*\prime} > \Pi_B^*$. ■

Proof of Corollary 2.

Proof. Part (i). Following Proposition 3, we have $\mu_A^{*\prime} < \mu_A^*$. Since α_A does not change, it implies that $(1-\alpha_A)(1-q)\mu_A^{*\prime} < (1-\alpha_A)(1-q)\mu_A^*$. The fact that $m_A^{*\prime} < m_A^*$ means that $(1-\alpha_A)(1-q)\mu_A^{*\prime} - (1-\alpha_B')(1-q)\mu_B^{*\prime} > (1-\alpha_A)(1-q)\mu_A^* - (1-\alpha_B)(1-q)\mu_B^*$. The above two inequalities imply that $(1-\alpha_B')(1-q)\mu_B^{*\prime} - (1-\alpha_B)(1-q)\mu_B^* < (1-\alpha_A)(1-q)\mu_A^{*\prime} - (1-\alpha_A)(1-q)\mu_A^* < 0$.

Part (ii). The first two claims directly follow parts (i) and (ii) in Proposition 3. The total number of relevant ads combining both platforms is: $\alpha_A q\mu_A^* m_A^* + \alpha_B q\mu_B^* m_B^*$.

$$
\begin{aligned}
\frac{\partial(\alpha_A q\mu_A^* m_A^* + \alpha_B q\mu_B^* m_B^*)}{\partial\alpha_B} &\propto \alpha_A m_A^* \frac{\partial\mu_A^*}{\partial\alpha_B} + \alpha_B m_B^* \frac{\partial\mu_B^*}{\partial\alpha_B} + \mu_B^* m_B^* + \frac{\partial m_A^*}{\partial\alpha_B}(\alpha_A\mu_A^* - \alpha_B\mu_B^*) \\
&> \alpha_A m_A^* \frac{\partial\mu_A^*}{\partial\alpha_B} + \alpha_B m_B^* \frac{\partial\mu_B^*}{\partial\alpha_B} + \mu_B^* m_B^* \\
&> m_B^*[\alpha_A \frac{\partial\mu_A^*}{\partial\alpha_B} + \alpha_B \frac{\partial\mu_B^*}{\partial\alpha_B} + \mu_B^*],
\end{aligned}
$$

where the first inequality uses $\frac{\partial m_A^*}{\partial\alpha_B} < 0$ and $\alpha_A\mu_A^* < \alpha_B\mu_B^*$, and the second inequality uses $\frac{\partial\mu_A^*}{\partial\alpha_B} < 0$ and $m_A^* < m_B^*$. Now to show $\frac{\partial(\alpha_A q\mu_A^* m_A^* + \alpha_B q\mu_B^* m_B^*)}{\partial\alpha_B} > 0$, it is enough to show that $\alpha_A\frac{\partial\mu_A^*}{\partial\alpha_B} + \alpha_B\frac{\partial\mu_B^*}{\partial\alpha_B} + \mu_B^* = \frac{\partial(\alpha_A\mu_A^* + \alpha_B\mu_B^*)}{\partial\alpha_B} \geq 0$. Equation (23) can be rewritten as

$$
-\mu_B^* + (1-\alpha_B)\frac{\partial\mu_B^*}{\partial\alpha_B} = (1-\alpha_A)\frac{\partial\mu_A^*}{\partial\alpha_B}\left[1 + W_A(\widehat{v}_A^*) + \mu_A^*\frac{dW_A}{d\mu_A^*}\right].
$$

Since $\frac{\partial\mu_A^*}{\partial\alpha_B} < 0$ and $\frac{dW_A}{d\mu_A^*} > 0$, the equation implies that

$$
(1-\alpha_A)\frac{\partial\mu_A^*}{\partial\alpha_B}[1 + W_A(\widehat{v}_A^*)] + \mu_B^* - (1-\alpha_B)\frac{\partial\mu_B^*}{\partial\alpha_B} > 0.
$$

It can be verified that $W_A(\widehat{v}_A^*) \geq 1$. Thus

$$
2(1-\alpha_A)\frac{\partial\mu_A^*}{\partial\alpha_B} + \mu_B^* - (1-\alpha_B)\frac{\partial\mu_B^*}{\partial\alpha_B} > 0.
$$

Now it is sufficient to show that $\alpha_A\frac{\partial\mu_A^*}{\partial\alpha_B} + \alpha_B\frac{\partial\mu_B^*}{\partial\alpha_B} + \mu_B^* \geq 2(1-\alpha_A)\frac{\partial\mu_A^*}{\partial\alpha_B} + \mu_B^* - (1-\alpha_B)\frac{\partial\mu_B^*}{\partial\alpha_B}$. This is equivalent to

$$
(3\alpha_A - 2)\frac{\partial\mu_A^*}{\partial\alpha_B} + \frac{\partial\mu_B^*}{\partial\alpha_B} \geq 0,
$$

which holds if $\alpha_A \leq 2/3$.

Part (iii). Since α_A remains the same, $m_A^{*\prime} < m_A^*$, and $\mu_A^{*\prime} < \mu_A^*$, the total number of ads on platform A decreases. Regarding platform B, when $y \to 0$, we have $m_B^* \to \frac{1}{2}$, $m_B^{*\prime} \to \frac{1}{2}$, $\mu_B^* \to \widetilde{\mu}$, and $\mu_B^{*\prime} \to \widetilde{\mu}$. Therefore,

$$\begin{aligned} & \left[\alpha_B' q + (1-\alpha_B')(1-q)\right] \mu_B^{*\prime} m_B^{*\prime} - \left[\alpha_B q + (1-\alpha_B)(1-q)\right] \mu_B^* m_B^* \\ \propto \quad & \left[\alpha_B' q + (1-\alpha_B')(1-q)\right] - \left[\alpha_B q + (1-\alpha_B)(1-q)\right] < 0, \end{aligned}$$

since $q < 1/2$. This means that for y small enough, the total number of ads on platform B decreases.

Part (iv). Firms with $v \in (\widehat{v}_B^{*\prime}, \widehat{v}_B^*)$ are clearly better off since they were not participating on any platform initially. Firms with $v \in [\widehat{v}_B^*, \widehat{v}_A^*)$ are also better off. To see this, note that before and after the change they only participate on platform B. A type v firm's profit on platform B equals to $\alpha_B q m_B^*(v - \widehat{v}_B^*)$. Since $\alpha_B' > \alpha_B$, $m_B^{*\prime} > m_B^*$, $\mu_B^{*\prime} > \mu_B^*$, we have $\alpha_B' q m_B^{*\prime}(v - \widehat{v}_B^{*\prime}) > \alpha_B q m_B^*(v - \widehat{v}_B^*)$.

For firms participating on both platforms after the change ($v \geq \widehat{v}_A^{*\prime}$), the total number of relevant ads is $(\alpha_A \mu_A^* + \alpha_B \mu_B^*)q$. By part (ii), $\frac{\partial(\alpha_A \mu_A^* + \alpha_B \mu_B^*)}{\partial \alpha_B} > 0$ if $\alpha_A \leq 2/3$. Thus each of those firms will send more relevant ads in total as α_B increases if $\alpha_A \leq 2/3$. The relative gain of a type v firm from an increase in α_B can be computed as

$$\Delta\pi(v) = \left[\alpha_B' q m_B^{*\prime}(v - \widehat{v}_B^{*\prime}) + \alpha_A q m_A^{*\prime}(v - \widehat{v}_A^{*\prime})\right] - \left[\alpha_A q m_A^*(v - \widehat{v}_A^*) + \alpha_B q m_B^*(v - \widehat{v}_B^*)\right].$$

Taking derivative with respect to v, we get

$$\frac{\partial \Delta\pi(v)}{\partial v} \propto \left[\alpha_B' m_B^{*\prime} + \alpha_A m_A^{*\prime}\right] - \left[\alpha_A m_A^* + \alpha_B m_B^*\right] > m_B^*(\alpha_B' - \alpha_B) > 0.$$

Therefore, more profitable types gain relatively more from an increase in α_B. ■

Proof of Proposition 4.

Proof. Parts (i) and (ii). These are similar to the proof in parts (i) and (ii) of Proposition 3, and thus is omitted here.

Part (iii). Since $\mu_B^{*\prime} < \mu_B^*$, and α_B does not change, it implies that $(1-\alpha_B)(1-q)\mu_B^{*\prime} < (1-\alpha_B)(1-q)\mu_B^*$. The fact that $m_B^{*\prime} < m_B^*$ means that $(1-\alpha_A')(1-q)\mu_A^{*\prime} - (1-\alpha_B)(1-q)\mu_B^{*\prime} < (1-\alpha_A)(1-q)\mu_A^* - (1-\alpha_B)(1-q)\mu_B^*$. The above two inequalities imply that $(1-\alpha_A')(1-q)\mu_A^{*\prime} - (1-\alpha_A)(1-q)\mu_A^* < (1-\alpha_B)(1-q)\mu_B^{*\prime} - (1-\alpha_B)(1-q)\mu_B^* < 0$.

Part (iv). The results directly follow parts (i) and (ii).

Part (v). Similar to the proof of part (iv) in Corollary 2**.**

Part (vi). Firms with $v \in (\widehat{v}_B^*, \widehat{v}_B^{*\prime})$ are strictly worse off since they are driven out of the market. Firms with $v \in [\widehat{v}_B^{*\prime}, \widehat{v}_A^{*\prime})$ are also worse off. To see this, note that before and after the change they only participate on platform B. Recall that a type v firm's profit on platform B equals to $\alpha_B q m_B^*(v - \widehat{v}_B^*)$. Since α_B does not change, $m_B^{*\prime} < m_B^*$, and $\mu_B^{*\prime} < \mu_B^*$, we have $\alpha_B q m_B^{*\prime}(v - \widehat{v}_B^{*\prime}) < \alpha_B q m_B^*(v - \widehat{v}_B^*)$. ■

Proof of Proposition 5.

Proof. Part (i). Recall that $R(\mu) = \widehat{v}\mu$, which is maximized at $\widetilde{\mu}$. Notice that $m_A(\widetilde{\mu}, \widetilde{\mu}) = \frac{1}{2} + (x_B - x_A)\,\widetilde{\mu} < \frac{1}{2}$. Suppose, to the contrary, $m_A^M = m_A(\mu_A^M, \mu_B^M) \geq \frac{1}{2}$. Then, we have

$$\begin{aligned}\Pi(\mu_A^M, \mu_B^M) &= \alpha_A q R(\mu_A^M) m_A^M + \alpha_B q R(\mu_B^M) m_B^M \\ &\leq \left(\alpha_A m_A^M + \alpha_B m_B^M\right) q R(\widetilde{\mu}) \\ &< [\alpha_A m_A(\widetilde{\mu}, \widetilde{\mu}) + \alpha_B m_B(\widetilde{\mu}, \widetilde{\mu})] q R(\widetilde{\mu}) = \Pi(\widetilde{\mu}, \widetilde{\mu}),\end{aligned}$$

where the first inequality uses $R(\mu_i^M) \leq R(\widetilde{\mu})$ and the second inequality uses $m_A(\widetilde{\mu}, \widetilde{\mu}) < m_A^M$. This contradicts the fact that μ_A^M and μ_B^M are the optimal solutions. Therefore, $m_A^M < \frac{1}{2} < m_B^M$.

Part (ii). We first show that $\mu_B^M < \widetilde{\mu} < \mu_A^M$. Observing (8) and (9), for both m_A^M and m_B^M to be positive we must have either $\mu_A^M < \widetilde{\mu} < \mu_B^M$, or $\mu_B^M < \widetilde{\mu} < \mu_A^M$. Suppose $\mu_A^M < \widetilde{\mu} < \mu_B^M$. Then $m_A^M = \frac{1}{2} + x_B \mu_B^M - x_A \mu_A^M > \frac{1}{2} + x_B \widetilde{\mu} - x_A \widetilde{\mu} = m_A(\widetilde{\mu}, \widetilde{\mu})$. And then by the same logic as in the proof of part (i), $\Pi(\mu_A^M, \mu_B^M) < \Pi(\widetilde{\mu}, \widetilde{\mu})$. This contradicts the fact that μ_A^M and μ_B^M are the optimal solutions. Therefore, we must have $\mu_B^M < \widetilde{\mu} < \mu_A^M$. By (8) and (9), this further implies that $\widehat{v}_B^M - \frac{1-F(\widehat{v}_B^M)}{f(\widehat{v}_B^M)} > 0$, $\widehat{v}_A^M - \frac{1-F(\widehat{v}_A^M)}{f(\widehat{v}_A^M)} < 0$, and $\alpha_A \widehat{v}_A^M \mu_A^M - \alpha_B \widehat{v}_B^M \mu_B^M < 0$.

Now it is clear that $\mu_A^M > \widetilde{\mu} > \mu_B^* > \mu_A^*$. What remains to be shown is that $\mu_B^M > \mu_B^*$. Suppose to the contrary, $\mu_B^M \leq \mu_B^* < \widetilde{\mu}$. Then we have $x_A \mu_A^M - x_B \mu_B^M > x_A \mu_A^* - x_B \mu_B^*$. By (7) and (9), $x_A \mu_A^M - x_B \mu_B^M > x_A \mu_A^* - x_B \mu_B^*$ implies that

$$\frac{x_B \mu_B^* \widehat{v}_B^*}{\widehat{v}_B^* - \frac{1-F(\widehat{v}_B^*)}{f(\widehat{v}_B^*)}} - \frac{x_B\left(\alpha_B \widehat{v}_B^M \mu_B^M - \alpha_A \widehat{v}_A^M \mu_A^M\right)}{\alpha_B\left(\widehat{v}_B^M - \frac{1-F(\widehat{v}_B^M)}{f(\widehat{v}_B^M)}\right)} < 0.$$

Since $\widehat{v}_B^M - \frac{1-F(\widehat{v}_B^M)}{f(\widehat{v}_B^M)} > 0$, the above inequality further implies that

$$\frac{\mu_B^* \widehat{v}_B^*}{\widehat{v}_B^* - \frac{1-F(\widehat{v}_B^*)}{f(\widehat{v}_B^*)}} - \frac{\widehat{v}_B^M \mu_B^M}{\left(\widehat{v}_B^M - \frac{1-F(\widehat{v}_B^M)}{f(\widehat{v}_B^M)}\right)} < 0,$$

or more compactly, $Z(\mu_B^*) < Z(\mu_B^M)$. But by Claim 1, $Z(\mu) > 0$ and is increasing in μ for $\mu \in [0, \widetilde{\mu})$, we have $Z(\mu_B^*) \geq Z(\mu_B^M)$. A contradiction. Therefore, we must have $\mu_B^M > \mu_B^*$.

Part (iii). From part (ii), $\mu_A^M > \widetilde{\mu} > \mu_B^M > \mu_B^* > \mu_A^*$, we know that $\mu_A^M - \mu_A^* > \mu_B^M - \mu_B^* > 0$. Together with $x_A > x_B$, we have

$$\begin{aligned}m_A(\mu_A^*, \mu_B^*) - m_A(\mu_A^M, \mu_B^M) &= x_A\left(\mu_A^M - \mu_A^*\right) - x_B\left(\mu_B^M - \mu_B^*\right) \\ &> x_A\left(\mu_B^M - \mu_B^*\right) - x_B\left(\mu_B^M - \mu_B^*\right) \\ &= (x_A - x_B)\left(\mu_B^M - \mu_B^*\right) > 0.\end{aligned}$$

Part (iv). The results follow the facts that $p(\alpha_i, \mu_i)$ is increasing in α_i and decreasing in μ_i, $\mu_A^M > \widetilde{\mu} > \mu_B^M > \mu_B^* > \mu_A^*$, and $\alpha_A < \alpha_B$. ■

Proof of Lemma 3.

Proof. Part (i). Suppose to the contrary, $\alpha_A S_A^o - \alpha_B S_B^o \geq 0$. Since S is decreasing in $\widehat{v}$, it implies that $\hat{v}_A^o < \hat{v}_B^o$ as $\alpha_A < \alpha_B$. By the presumption that $\alpha_A S_A^o - \alpha_B S_B^o \geq 0$, the LHS of (10) and (11) must satisfy

$$\frac{1}{y}\frac{\alpha_A}{1-\alpha_A}\hat{v}_A^o \geq \frac{2t}{q} \geq \frac{1}{y}\frac{\alpha_B}{1-\alpha_B}\hat{v}_B^o.$$

But by $\hat{v}_A^o < \hat{v}_B^o$ and $\alpha_A < \alpha_B$, $\frac{1}{y}\frac{\alpha_A}{1-\alpha_A}\hat{v}_A^o < \frac{1}{y}\frac{\alpha_B}{1-\alpha_B}\hat{v}_B^o$. A contradiction. Therefore, we must have $\alpha_A S_A^o - \alpha_B S_B^o < 0$.

Part (ii). Given part (i), we must have $\frac{\alpha_A}{1-\alpha_A}\hat{v}_A^o < \gamma\frac{1-q}{q} < \frac{\alpha_B}{1-\alpha_B}\hat{v}_B^o$.

Part (iii). Equations (10) and (11) can be rewritten as:

$$m_A^o = \frac{\alpha_B S_B^o - \alpha_A S_A^o}{-\frac{1}{y}\frac{\alpha_A}{1-\alpha_A}\hat{v}_A^o + \frac{2t}{q}}, \tag{27}$$

$$m_B^o = \frac{\alpha_B S_B^o - \alpha_A S_A^o}{\frac{1}{y}\frac{\alpha_B}{1-\alpha_B}\hat{v}_B^o - \frac{2t}{q}}. \tag{28}$$

Adding (27) and (28), we get

$$(\alpha_B S_B^o - \alpha_A S_A^o)[\frac{1}{-\frac{1}{y}\frac{\alpha_A}{1-\alpha_A}\hat{v}_A^o + \frac{2t}{q}} + \frac{1}{\frac{1}{y}\frac{\alpha_B}{1-\alpha_B}\hat{v}_B^o - \frac{2t}{q}}] = 1. \tag{29}$$

By the results in parts (i) and (ii), the LHS of (29) is decreasing in $\hat{v}_B^o$ and increasing in $\hat{v}_A^o$. This implies that if there are two different solutions to (27) and (28), then the two solutions cannot be strictly ordered.

Now suppose there are two solutions $(\hat{v}_A^{o1}, \hat{v}_B^{o1})$ and $(\hat{v}_A^{o2}, \hat{v}_B^{o2})$, with $\hat{v}_A^{o1} > \hat{v}_A^{o2}$ and $\hat{v}_B^{o1} < \hat{v}_B^{o2}$. This implies that $m_A^{o1} > m_A^{o2}$ and $m_B^{o1} < m_B^{o2}$, $\alpha_B S_B^{o1} - \alpha_A S_A^{o1} > \alpha_B S_B^{o2} - \alpha_A S_A^{o2}$, and $\frac{1}{y}\frac{\alpha_B}{1-\alpha_B}\hat{v}_B^{o1} - \frac{2t}{q} < \frac{1}{y}\frac{\alpha_B}{1-\alpha_B}\hat{v}_B^{o2} - \frac{2t}{q}$. This further implies that the LHS of (11) under $(\hat{v}_A^{o1}, \hat{v}_B^{o1})$ is strictly less than that under $(\hat{v}_A^{o2}, \hat{v}_B^{o2})$. Therefore, equation (11) cannot be satisfied with $(\hat{v}_A^{o1}, \hat{v}_B^{o1})$ and $(\hat{v}_A^{o2}, \hat{v}_B^{o2})$ at the same time, which contradicts the presumption that $(\hat{v}_A^{o1}, \hat{v}_B^{o1})$ and $(\hat{v}_A^{o2}, \hat{v}_B^{o2})$ are both solutions to (10) and (11). ■

Proof of Proposition 6.

Proof. Part (i). Suppose γ is small enough such that $y \to 0$. By (6) and (7), both μ_A^* and μ_B^* converge to $\widetilde{\mu} < 1$. By (10) and (11), both μ_A^o and μ_B^o converge to 1. Therefore, $\mu_A^* < \mu_A^o$ and $\mu_B^* < \mu_B^o$ as $\gamma \to 0$. For the next claim, by (10) and (11), $\gamma > \frac{q}{1-q}\frac{\alpha_B}{1-\alpha_B}\overline{v}$ implies that the socially optimal solution is a corner solution: $\mu_A^o = \mu_B^o = 0$. By (6) and (7), the fact that y is bounded implies that both μ_A^* and μ_B^* are strictly positive. Thus, $\mu_A^* > \mu_A^o$ and $\mu_B^* > \mu_B^o$.

Part (ii). Suppose t is big enough such that $y \to 0$. By (6) and (7), both μ_A^* and μ_B^* converge to $\widetilde{\mu} < 1$. And equations (10) and (11) become

$$\frac{\alpha_A}{1-\alpha_A}\hat{v}_A^o = \gamma\frac{1-q}{q} = \frac{\alpha_B}{1-\alpha_B}\hat{v}_B^o.$$

If $\gamma > \frac{q}{1-q}\frac{\alpha_B}{1-\alpha_B}\widetilde{v}$, then $\hat{v}_A^o > \hat{v}_B^o > \widetilde{v}$, which implies that $\mu_A^* > \mu_A^o$ and $\mu_B^* > \mu_B^o$. The other two claims follow similar arguments.

Part (iii). It is sufficient to show the claim is true when $t \to 0$. By (6) and (7), both μ_A^* and μ_B^* converge to 0 as $t \to 0$. In the social optimum, when $t \to 0$, the consumer allocation becomes the corner solution with all consumers joining platform B. Thus $\hat{v}_B^o = \gamma\frac{1-q}{q}\frac{1-\alpha_B}{\alpha_B}$, which implies that $\mu_B^o > 0$ if $\gamma < \frac{q}{1-q}\frac{\alpha_B}{1-\alpha_B}\overline{v}$. To ensure that all consumers join platform B, $\mu_A^o > 0$ should hold. Therefore, $\mu_A^* < \mu_A^o$ and $\mu_B^* < \mu_B^o$.

Part (iv). It is enough to show that the results hold when $\alpha_B \to 1$. By (7), μ_B^* converges to $\widetilde{\mu} < 1$ as $\alpha_B \to 1$. By (11), μ_B^o converges to 1 as $\alpha_B \to 1$. Therefore, $\mu_B^* < \mu_B^o$ as $\alpha_B \to 1$. ■

Proof of Proposition 7.

Proof. Part (i). We only need to show that when $\alpha_B = 1$, $m_A^o < m_A^*$. This is because if this is true, then by the continuity of both m_A^o and m_A^* in α_B, this also holds for α_B close enough to 1. Now consider the case where $\alpha_B = 1$. Given that platform B has perfect targeting ability, we must have $\mu_B^o = 1$ and $\mu_B^* = \widetilde{\mu}$. The equations that characterize μ_A^o and μ_A^* can now be written as

$$\begin{aligned} \frac{1}{2} - y(1-\alpha_A)\mu_A^* &= y(1-\alpha_A)\frac{\mu_A^* \widehat{v}_A^*}{\widehat{v}_A^* - \frac{1-F(\widehat{v}_A^*)}{f(\widehat{v}_A^*)}}, \quad (30) \\ \frac{1}{2} - y(1-\alpha_A)\mu_A^o &= \frac{y(\overline{S} - \alpha_A S_A^o)}{\frac{2ty}{q} - \frac{\alpha_A}{1-\alpha_A}\widehat{v}_A^o}, \quad (31) \end{aligned}$$

where $\overline{S} = \int_0^{\overline{v}} vf(v)dv$, is the maximum firm surplus per consumer. The LHS of (30) and (31) are m_A^* and m_A^o, respectively. To show $m_A^o < m_A^*$, it is enough to show that $\mu_A^* < \mu_A^o$. By previous result $\mu_A^* < \widetilde{\mu}$, it is sufficient to show that $\mu_A^o > \widetilde{\mu}$ or $\widehat{v}_A^o < \widetilde{v}$. By part (ii) of Lemma 3, $\frac{\alpha_A}{1-\alpha_A}\widehat{v}_A^o < \gamma\frac{1-q}{q}$, which is equivalent to $\widehat{v}_A^o < \gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A}$. Thus, if $\gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A} < \widetilde{v}$, then $\widehat{v}_A^o < \widetilde{v}$.

Part (ii). Again we only need to show that when $\alpha_B = 1$, $m_A^o > m_A^*$, or $\widehat{v}_A^o > \widehat{v}_A^*$. Since y is close to zero, by (30) $\widehat{v}_A^*$ must be very close to $\widetilde{v}$, and by (31) $\widehat{v}_A^o$ must be very close to $\gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A}$. Now the condition $\widetilde{v} \ll \gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A} \le \overline{v}$ implies that $\widehat{v}_A^o \gg \widetilde{v}$, and hence $\widehat{v}_A^o > \widehat{v}_A^*$ and $m_A^o > m_A^*$.

Part (iii). First consider the symmetric case that $\alpha_B = \alpha_A$. In this case, the socially optimal allocation must be symmetric, which implies that $m_A^o = 1/2$, and by (10) and (11), $\widehat{v}_A^o = \widehat{v}_B^o = \gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A} \equiv \widehat{v}^o$. The equilibrium allocation must be symmetric as well: $m_A^* = 1/2$. Recall that, by part (ii) of Proposition (3), $\frac{\partial m_A^*}{\partial \alpha_B} < 0$. Thus, it is enough to show that, evaluated at $\alpha_B = \alpha_A$, $\frac{\partial m_A^o}{\partial \alpha_B} > 0$. More explicitly,

$$\frac{\partial m_A^o}{\partial \alpha_B}|_{\alpha_B=\alpha_A} > 0 \Leftrightarrow \{\widehat{v}_A^{o\prime} - \widehat{v}_B^{o\prime} - \frac{1}{1-\alpha_A}\frac{1-F(\widehat{v}^o)}{f(\widehat{v}^o)}\}|_{\alpha_B=\alpha_A} > 0,$$

where $'$ denotes partial derivative with respect to α_B. Now differentiate (10) and (11) at $\alpha_B = \alpha_A$, we get

$$\begin{aligned} \alpha_A(S_A^{o\prime} - S_B^{o\prime}) - \int_{\widehat{v}^o}^{\overline{v}} vf(v)dv - \frac{1}{2y}\frac{\alpha_A}{1-\alpha_A}\widehat{v}_A^{o\prime} &= 0, \\ \alpha_A(S_A^{o\prime} - S_B^{o\prime}) - \int_{\widehat{v}^o}^{\overline{v}} vf(v)dv + \frac{1}{2y}[\frac{\alpha_A}{1-\alpha_A}\widehat{v}_B^{o\prime} + \frac{1}{(1-\alpha_A)^2}\widehat{v}^o] &= 0. \end{aligned}$$

From the above two equations, we get

$$\{\widehat{v}_A^{o\prime} - \widehat{v}_B^{o\prime} - \frac{1}{1-\alpha_A}\frac{1-F(\widehat{v}^o)}{f(\widehat{v}^o)}\}|_{\alpha_B=\alpha_A} = \frac{\widehat{v}^o - 4y(1-\alpha_A)^2\int_{\widehat{v}^o}^{\overline{v}} vf(v)dv}{\alpha_A + 4y\alpha_A(1-\alpha_A)\widehat{v}^o f(\widehat{v}^o)} - \frac{1}{1-\alpha_A}\frac{1-F(\widehat{v}^o)}{f(\widehat{v}^o)}.$$

Given the condition that $\widehat{v}^o = \gamma\frac{1-q}{q}\frac{1-\alpha_A}{\alpha_A}$ is very close to $\overline{v}$, both $\int_{\widehat{v}^o}^{\overline{v}} vf(v)dv$ and $\frac{1-F(\widehat{v}^o)}{f(\widehat{v}^o)}$ are very close to zero. Therefore, the above expression is strictly greater than 0, which implies that $\frac{\partial m_A^o}{\partial \alpha_B}|_{\alpha_B=\alpha_A} > 0$. ■

Proof of Proposition 8.

Proof. Part (i). Since $y \to 0$, by (13) $\mu^\star = \widetilde{\mu}$ and $\widehat{v}^* = \widetilde{v}$. Moreover, $\frac{\partial \mu_j^\star}{\partial \alpha_i} = 0$ and $\frac{d\mu^\star}{d\alpha} = 0$. Thus (15) becomes

$$H^o(\alpha) - H^*(\alpha) = \frac{1}{2}\gamma(1-q)\widetilde{\mu} + \frac{1}{2}q[\int_{\widetilde{v}}^{\overline{v}} vf(v)dv - \widetilde{\mu}\widetilde{v}] > 0.$$

Therefore, by the convexity of $C(\cdot)$, $\alpha^* < \alpha^o$.

Part (ii). It is enough to show that for any $\alpha < 1$, $H^*(\alpha) < H^o(\alpha)$. The facts that $y > 0$ and $\alpha < 1$ imply that $\mu^* > 0$. Since y is bounded and γ is big, the first term of (15) dominates the third term. Thus, it is sufficient that the first term is positive or $\mu^* - (1-\alpha)\frac{d\mu^\star}{d\alpha} > 0$. In particular,

$$\begin{aligned} \mu^* - (1-\alpha)\frac{d\mu^\star}{d\alpha} &\propto 2\mu^\star y \frac{dW^*}{d\mu^*} - 1 \\ &\propto \frac{\mu^{*2}}{f^2(\widehat{v}^*)} + \frac{\mu^*\widehat{v}^*}{f^3(\widehat{v}^*)}[f^2(\widehat{v}^*) + f'(\widehat{v}^*)(1-F(\widehat{v}^*))] > 0, \end{aligned}$$

where the last inequality uses the fact that the terms in the bracket is positive due to the logconcavity of $f(\cdot)$.

Part (iii). Since v is uniformly distributed, $\widetilde{\mu} = 1/2$, and

$$Z^* = \frac{\mu^*(1-\mu^*)}{1-2\mu^*}; \quad \frac{dZ^*}{d\mu^*} = 1 + \frac{2\mu^*(1-\mu^*)}{(1-2\mu^*)^2}.$$

Now (15) becomes

$$\begin{aligned} H^o(\alpha) - H^*(\alpha) &> \alpha q \widehat{v}^*[\frac{1}{4y(1-\alpha)^2\frac{dZ^*}{d\mu^*}} - y\mu^*\frac{Z^* + \mu^*[\frac{dZ^*}{d\mu^*} + (\frac{dZ^*}{d\mu^*})^2]}{2\frac{dZ^*}{d\mu^*} + (\frac{dZ^*}{d\mu^*})^2}] \\ &= \frac{\alpha q y(1-\mu^*)\mu^{*3}}{(1-2\mu^*)^4[2\frac{dZ^*}{d\mu^*} + (\frac{dZ^*}{d\mu^*})^2]}[1 - 7\mu^* + 14\mu^{*2} - 10\mu^{*3}]. \end{aligned} \tag{32}$$

It can be verified that $1 - 7\mu + 14\mu^2 - 10\mu^3$ is decreasing in μ for $\mu \in [0, 1/2]$. Therefore, if $\mu^* \le \widehat{\mu}$, then $H^o(\alpha) - H^*(\alpha) > 0$. By (13), given y and α, μ^* is uniquely determined and it is increasing in α and t, and decreasing in γ. Therefore, if either t is big, γ is small, or the marginal cost of investment in targeting ability is high enough (α is small), then $\mu^* \le \widehat{\mu}$, which implies $\alpha^\star < \alpha^o$ by (32).

Observing (15) and (32), for overinvestment to occur, γ has to be small so that the first term in (15) is negligible, y has to be big enough so that the third term in (15) will dominate the second term, and μ^* has to be bigger than $\widehat{\mu}$. A big y and a small γ imply that t should be even smaller than γ, and a big y and a big μ^* imply that α has to be close enough to 1, which means that the marginal cost of investment in targeting ability must be low enough for α close enough to 1. For a concrete example, when $q = 0.1$, $\gamma = 0.001$, $t = 0.0005$, and $C'(\alpha) = \max\{0, e^{5(\frac{20}{19}\alpha - 1)} - 1\}$, $\alpha^* = 0.9684 > 0.9637 = \alpha^o$. ■

Proof of Proposition 9.

Proof. Part (i). Since $y \to 0$, by (6) and (7), $\mu_A^\star = \mu_B^\star = \widetilde{\mu}$, $\widehat{v}_A^* = \widehat{v}_B^* = \widetilde{v}$, and $m_A^* = m_B^* = 1/2$. Moreover, $\frac{\partial \mu_A^\star}{\partial \alpha_B} = 0$ and $\frac{\partial \mu_B^\star}{\partial \alpha_B} = 0$. Thus (18) becomes

$$H^o(\alpha) - H^*(\alpha) = \frac{1}{2}\gamma(1-q)\widetilde{\mu} + \frac{1}{2}q[\int_{\widetilde{v}}^{\overline{v}} vf(v)dv - \widetilde{\mu}\widetilde{v}] > 0.$$

Therefore, by the convexity of $C(\cdot)$, $\alpha_B^* < \alpha_B^o$.

Part (ii). It is enough to show that for any $\alpha_B < 1$, $H^*(\alpha_B) < H^o(\alpha_B)$. The facts that $y > 0$ and $\alpha_B < 1$ imply that $\mu_B^* > 0$. Since y is bounded and γ is big, by (18) the second term dominates the fourth term. Thus it is enough to show that the second term is positive. In particular,

$$\begin{aligned} & \mu_B^* m_B^* - m_B^*(1-\alpha_B)\frac{\partial \mu_B^*}{\partial \alpha_B} - (1-\alpha_A)m_A^*\frac{\partial \mu_A^*}{\partial \alpha_B} \\ > \quad & m_B^*[\mu_B^* - (1-\alpha_B)\frac{\partial \mu_B^*}{\partial \alpha_B}] \propto (\mu_B^*\frac{\partial Z_B^*}{\partial \mu_B} - Z_B^*)(1 + \frac{\partial Z_A^*}{\partial \mu_A}) \\ \propto \quad & f(\hat{v}_A^*)^2 + (1 - F(\hat{v}_A^*))\, f'(\hat{v}_A^*) > 0, \end{aligned}$$

where the first inequality uses the fact that $\frac{\partial \mu_A^*}{\partial \alpha_B} < 0$, and the last inequality uses the fact that the terms in the bracket is positive due to the logconcavity of $f(\cdot)$. ■

References

[1] Ambrus, A., and M. Reisinger. "Exclusive vs Overlapping Viewers in Media Markets." (2006), mimeo.

[2] Anderson, Simon P., and Stephen Coate. "Market Provision of Broadcasting: A Welfare Analysis." *The review of Economic studies* 72, no. 4 (2005): 947-972.

[3] Armstrong, Mark. "Competition in Two-sided Markets." *The RAND Journal of Economics* 37, no. 3 (2006): 668-691.

[4] Athey, Susan, Emilio Calvano, and Joshua Gans. "The Impact of the Internet on Advertising Markets for News Media." Available at SSRN 2180851 (2012).

[5] Athey, Susan, and Joshua S. Gans. "The Impact of Targeting Technology on Advertising Markets and Media Competition." *American Economic Review*, P&P (2010).

[6] Bergemann, Dirk, and Alessandro Bonatti. "Targeting in Advertising Markets: Implications for Offline versus Online Media." *The RAND Journal of Economics* 42, no. 3 (2011): 417-443.

[7] Bergstrom, Ted, and Mark Bagnoli. "Log-concave Probability and its Applications." *Economic Theory* 26 (2005): 445-469.

[8] Butters, G. R. "Equilibrium Distribution of Sales and Advertising Prices," *Review of Economic Studies*, 44(3), 1977, 465-491.

[9] Chandra, Ambarish. "Targeted Advertising: The Role of Subscriber Characteristics in Media Markets." *The Journal of Industrial Economics* 57, no. 1 (2009): 58-84.

[10] Chandra, Ambarish, and Ulrich Kaiser. "Targeted Advertising in Magazine Markets." No. 10-063. ZEW Discussion Papers, 2010.

[11] de Corniere, Alexandre. "Targeting with Consumer Search: an Economic Analysis of Keyword Advertising." mimeo, Paris School of Economics, 2010.

[12] Esteban, Lola, Agustin Gil, and Jose M. Hernandez. "Informative Advertising and Optimal Targeting in a Monopoly." *The Journal of Industrial Economics* 49, no. 2 (2001): 161-180.

[13] Gal-Or, Esther, Mordechai Gal-Or, Jerrold H. May, and William E. Spangler. "Targeted Advertising Strategies on Television." *Management Science* 52, no. 5 (2006): 713-725.

[14] Galeotti, Andrea, and José Luis Moraga-González. "Segmentation, Advertising and Prices." *International Journal of Industrial Organization* 26, no. 5 (2008): 1106-1119.

[15] Grossman, G.M., and C. Shapiro. "Informative Advertising with Differentiated Products," *Review of Economic Studies,* 51(1), 1984, 63-81.

[16] Iyer, Ganesh, David Soberman, and J. Miguel Villas-Boas. "The Targeting of Advertising." *Marketing Science* 24, no. 3 (2005): 461-476.

[17] Johnson, Justin P. "Targeted Advertising and Advertising Avoidance." *The RAND Journal of Economics* 44, no. 1 (2013): 128-144.

[18] Peitz, Martin, and Tommaso M. Valletti. "Content and Advertising in the Media: Pay-tv versus Fee-to-air." *International Journal of Industrial Organization* 26, no. 4 (2008): 949-965.

[19] Rochet, Jean-Charles, and Jean Tirole. "Platform Competition in Two-sided Markets." *Journal of the European Economic Association* 1, no. 4 (2003): 990-1029.

[20] Rochet, Jean-Charles, and Jean Tirole. "Two-sided Markets: A Progress Report." *The RAND Journal of Economics* 37, no. 3 (2006): 645-667.

[21] Taylor, Greg. "Attention Retention: Targeted Advertising and the Provision of Media Content." Mimeo, 2011.

[22] Vives, Xavier. "Nash Equilibrium with Strategic Complementarities." *Journal of Mathematical Economics* 19, no. 3 (1990): 305-321.

[23] Yang, H. "Targeted Search and the Long Tail Effect," *Rand Journal of Economics*, forthcoming.