Information Aggregation and Investment Cycles with Strategic Complementarity

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Abstract

This paper studies the interaction between information aggregation and investment cycles with investments exhibiting strategic complementarity. The composition of information aggregation varies across different phases of cycles, which in turn affects the course of investment cycles. The phases of cycles are history dependent for informational reasons, and changes in phases depend on the growth rate of aggregate investment: a slowdown in growth is interpreted as bad news and a slowdown in downturn is considered as good news. A small structural change in low cost investments can have a large effect on the pattern of cycles. Investment cycles might be characterized by sudden crashes and slow recoveries.

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1 Introduction

A same level of aggregate investment may generate different sentiments across different phases of business cycle. A moderate investment level may be interpreted as good news during a slump as much lower level was originally expected, and thus triggers a recovery. On the other hand, the same moderate level of investment during a boom may be interpreted as bad news as much higher level was originally expected, and thus triggers an economic downturn.

To explain the possibility of the above mentioned phenomenon, we study the interaction between information aggregation and investment cycles in a dynamic coordination setting, and explore its implications about investment cycles. We emphasize that the *amount and composition* of aggregated information about the fundamental of the economy not only fluctuates with investment cycle, but also changes its course.

Specifically, we develop a simple overlapping generations (OLG) model with the main modeling elements as follows. (1) In each period there are two active generations of investors,

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and their investments exhibit strategic complementarities. (2) Each generation has three types of investors with different costs of investment: the low cost types who always invest, the medium cost types and the high cost types whose investment decisions depend on the expected total investment in the same period. The mass of each type in a generation is uncertain. Moreover, the existence of an investor and his cost type is his own private information. This captures the fact that the relevant information is dispersed throughout the economy. (3) The history of aggregate investment is observable.

In this economy, the fundamental in a particular period consists of the masses of three types of agents. Due to heterogeneity among investors, different types might make different investment decisions. In any period, the economy can be in one of the three regimes (equilibria): the H-regime (boom) in which all types invest, the M-regime in which only the low and medium cost types invest, and the L-regime (slump) in which only the low cost types invest.

Since information is aggregated by only investing, the compositions of information aggregation are different in different regimes. In a period of boom each type invests, thus by observing the aggregate investment players learn the total measure of all investors, but not the composition of investors' population. If there is a slump today, so that only the low cost types invest, then the aggregate investment reveals the population size of the low cost types but nothing about those of the other two cost types. Now the interactions between information aggregation and investment cycles naturally emerge: the previous period's regime determines the composition of information aggregation in that period, which in turn, through the OLG structure, affects the current period's regime and composition of information aggregation, and so on. This adds an informational aspect to the dynamics of cycles: the composition of information aggregation not only varies with investment cycle, but also feedbacks to and changes its course.

Several interesting features emerge from these interactions. First, the equilibrium evolution of regimes (cycles) is history dependent for informational reasons. In particular, the interpretation of an observed aggregate investment depends on the phase of investment cycles. A same moderate level of investment may be interpreted as bad news during a boom but good news during a slump, since more investment is anticipated during a boom but less is expected in a slump.

Second, a change of the distribution of the shocks to lower cost types have a greater impact on regime switches (cycles) than a corresponding change of the distribution of the shocks to higher cost types.[1] This property arises because the information about shocks to the low cost

[1] The changes in the distribution of shocks are observed by all investors. A shock to a specific cost type refers to the realized mass of that type.

types is always aggregated (incorporated in total investments), while information about shocks to the higher cost types might not be aggregated. This informational aspect leads to another interesting feature: once the economy is in a slump whether the economy can be pulled out of it only depends on the shock to the low cost type, no matter how favorable the shocks to the higher cost types are.

Third, regime switches possibly exhibit asymmetry: while it is always possible for the economy to switch directly downward from a boom to a slump, upward regime switch from a slump directly to a boom might not be possible, which must go through some recovering phase (regime M). This asymmetry is due to the following informational reasons. When the least favorable shock occurs in a boom, it is fully revealed. On the other hand, when the most favorable shock occurs in a slump, it is only partially revealed, since the information about the shocks to higher cost types is not aggregated.

In an extension we study a general model with N types of investors. A new feature emerges in the N-type model: regime switches in investment cycles depend on the *growth rate* of aggregate investment. In particular, during an upward trend of regime switches (recovery) if the growth of aggregate investment slows down, then it is interpreted as bad news and might trigger an economic downturn. On the other hand, during a downward trend of regime switches (economic downturn) if the decease of aggregate investment slows down, then it is interpreted as good news and might trigger a recovery. This feature arises because higher aggregate investments are expected in higher regimes, thus an upward (a downward) regime switch without a corresponding increase (decrease) in aggregate investment is interpreted as bad (good) news.

1.1 Related Literature

Our model is a dynamic coordination game with incomplete information, thus it is related to the literature on coordination games. Cooper and John (1988) study static coordination games, and they use the feature of multiple equilibria to explain economic fluctuations.[2] Adopting the global game approach, Morris and Shin (1998) introduce heterogenous beliefs into static coordination games and equilibrium becomes unique. Angeletos and Werning (2006) extend the model of Morris and Shin by adding a stage of information aggregation in financial markets before a coordination game is played. Multiple equilibria reemerge in their setting.

Several recent papers study cycles in dynamic global games. Oyama (2004) considers an OLG model with the actions of two successive generations having strategic complementarity.

[2] For dynamic coordination games with complete information, see Gale (1995).

The equilibrium cycle exhibits hysteresis. With two thresholds $\theta^* < \theta^{**}$, a boom switches to a slump only if the fundamental falls below θ^*, but a slump switches to a boom only if the fundamental rises above θ^{**}. Steiner (2008) assumes a player's current action affects not only her current payoff but also future payoffs. Specifically, higher expected payoff tomorrow makes today's investment more risky and therefore harder to coordinate on. This dynamic link leads to endogenous cycles. Giannitsarou and Toxvaerd (2007) study cycles when players' actions not only have contemporaneous complementarities but also exhibit a positive intertemporal link: a high action today increases the payoff of a high action tomorrow. A common feature of these models is that the intertemporal link results from intertemporal payoff externalities, and there is no social learning. In contrast, in our model the intertemporal link is purely informational.

Angeletos et. al (2007) study a dynamic global game of currency attacks. Players update their beliefs by receiving new private signals and observing the outcomes of previous attacks. The failure of a more aggressive attack yesterday leads to stronger beliefs about the economic fundamental, which decrease players' incentive to attack today. This effect generates equilibrium fluctuations in the intensity of attacks. The observability of the outcomes of previous attacks adds an aspect of social learning in their model. Other than that their paper has a different focus from ours, in their model the total number of agents who attacked in each previous round is never observed, while in our model the aggregate investment in each previous period is observable.[3]

Chamley (1999) studies a dynamic coordination game with cycles. Agents' investment costs follow a known distribution but with an unknown mean, which fluctuates over time. The history of aggregate investment is observable. He shows that there is a unique equilibrium and the economy fluctuates symmetrically between booms and recessions. However, in his model information aggregation is just a by-product of investment cycles and has no feedback effect: information (about the mean of cost distribution) is either fully revealed or not revealed at all in each period. In contrast, in our model the compositions of information aggregation are richer since the fundamental in a period consists of the masses of three type agents, which is multi-dimensional. Moreover, in our model information aggregation affects the course of investment cycle.[4]

[3] Chamley (2003) studies currency attacks in a dynamic coordination model. The number of speculators is fixed but uncertain in the beginning. His main result is that multi-period interactions facilitate coordination, and thus increases the probability of successful attack. However, there is no cycle in his model.

[4] Jeitschko and Taylor (2001) consider a dynamic coordination game with local learning. They show that the fear about coordination breakdown can quickly break down coordination. There is neither cycle nor social learning in their model. Caplin and Leahy (1994) present a dynamic model in the first stage of which routine behavior keeps information trapped in private hands. When private information reaches some threshold, some

The informational aspect of our model is related to the literature on social learning / information cascades. Banerjee (1992) and Bikhchandani et. al (1992) show that when information is dispersed and agents act sequentially, learning from others may lead to "follow-the-leader" or herding behavior. Chamley and Gale (1994) endogenize the timing of actions and show that herding is still a possibility.[5] In all these models the fundamental is fixed, hence there is no cycle. In an endogenous timing model, Peck and Yang (2007) study information cascades with the state of the world changing over time. Their main focus is on how the possibility of waiting affects information cascades and investment cycles. In their model no information is revealed either in booms or slumps, which is very different from the features about information aggregation in this model.

Several related papers study the informational aspects of business cycle. Veldkamp (2005) and Nieuwerburgh and Veldkamp (2006) develop models in which agents learn the return of their investments, which is uncertain and changing over time. Each investment is like an experiment, with its outcome revealing some information about the common investment return. Naturally, agents learn the common investment return more accurately during booms than during recessions, as the signal-to-noise ratio increases with the total number of investments (experiments). Based on this learning asymmetry, they show that business cycle exhibits slow boom and sudden crash. Our model is different in that agents not necessary learn more about the fundamental during booms. Instead, agents learn different aspects of the fundamental in different phases of cycles, which leads to dynamics different from those in their models. Zeira (1994) shows that investment cycles can be generated by learning about the changing market size. The market size can only be learned when total investment overshoots it. Therefore, learning only occurs at the end of booms, which is quite different from the information dynamics in our model.

The rest of the paper is structured as follows. Section 2 presents a basic model with three types of investors. Section 3 characterizes the equilibrium regime switches, and their properties are examined in Section 4. In Section 5, we generalize the model to N types of investors. Section 6 discusses some assumptions of the model, and Section 7 concludes.

agents act, which perfectly reveals the state of the world. In their model there is no cycle either and in the end information is always fully aggregated.

[5] For endogenous timing herding model with cost heterogeneity, see Levin and Peck (2009). For the application of exogenous timing herding model in financial markets, see Avery and Zemsky (1998). For a detailed survey of dynamic informational externalities, see Chapter 14 of Vives (2008).

2 A Three Type Model

Consider an OLG model with each generation of agents living for two periods. Thus each period consists of two generations: old and new. In each generation there are three types of investors who have different investment costs. Type 1 agents (low cost type) have investment cost c_1, type 2 agents (medium cost type) c_2, and type 3 agents (high cost type) c_3, with $c_1 < c_2 < c_3$. These costs are common knowledge. For each generation, however, the mass of each type is uncertain. Let m_i be the mass of type i agents in a generation. We assume that m_i, $i = 1, 2, 3$, is drawn from the same distribution $F(m)$ with support $[\underline{m}, \overline{m}]$ and mean m^* and that $F(m)$ is common knowledge.[6] Moreover, all the m_i are independent across types and generations. Once a new agent draws a cost type, he remains that type for his lifetime (two periods).

In each period, each agent simultaneously makes a zero-or-one decision regarding whether to invest. Investment is reversible across periods: an agent can invest when he is young and not invest when he is old. All the investments in one period have a common return. Moreover, in each period the investment return exhibits strategic complementarity. Denote v as the investment return and y as the total mass of agents who invested in that period. For simplicity, we assume $v(y) = y$.[7] If a type i agent invests in a period, his payoff in that period is $y - c_i$. If an agent does not invest in a period, his payoff is 0. Each agent is risk neutral and has a discount factor δ.

Let m^o_{it} and m^n_{it} be the mass of old agents and new agents of type i respectively in period t. Thus $\{m^o_{it}, m^n_{it}\}_{i=1,2,3}$ defines the underlying state of the world in period t. We assume that the investment cost of each agent is his own private information, and neither $\{m^o_{it}\}_{i=1,2,3}$ nor $\{m^n_{it}\}_{i=1,2,3}$ is observable. This captures the fact that the information about the underlying state is dispersed among active agents: aggregating each agent's private information would perfectly reveal the state of the world.

Given the structure of the OLG model, $\{m^o_{it}\}_{i=1,2,3}$ might be inferred from the history of previous investments. Denote y^o_t and y^n_t as the old generation's and the new generation's total investment in period t respectively. Of course, $y_t = y^o_t + y^n_t$. We assume that not only the investment history $\{y_j\}^t_{j=1}$ is observable, but also $\{y^o_j\}^t_{j=1}$ and $\{y^n_j\}^t_{j=1}$ are distinctively observable. The observability of $\{y^n_j\}^t_{j=1}$ is a simplifying assumption, which will be discussed in Section 6.

[6] The masses of different types can be drawn from different distributions, in which case the main results of the paper continue to hold qualitatively.

[7] Using a more general function $v(y)$ with $v'(y) > 0$ only complicates the computations, but does not change the qualitative results of the model.

The timing in a typical period t is as follows. First a new generation is born, and the mass of each cost type is realized. Then, observing the history of investments $\{y_j^o\}_{j=1}^{t-1}$ and $\{y_j^n\}_{j=1}^{t-1}$, all the active agents simultaneously decide whether to invest. Finally, the old generation exits the economy.

As in standard coordination games (Cooper and John, 1998), there are possibly multiple equilibria in stage games due to the strategic complementarity among investments. The resolution of multiple equilibria is not the focus of this paper. Instead, we assume that agents coordinate on the most (Pareto) efficient equilibrium given the available information if there are multiple equilibria. We sidestep the issue of multiple equilibria in order to focus on the informational aspect of the dynamics.

To make the model interesting, we assume that

$$c_1 < \underline{m} + m^*; \; c_2 > \underline{m} + 3m^*; \text{ and } c_3 > \underline{m} + 5m^*. \tag{1}$$

The condition $c_1 < \underline{m} + m^*$ implies that type 1 agents always invest in each period. To see this, note that the worst shock for the old generation of type 1 is $\underline{m}$. Even under this worst shock, the total expected number of type 1 investors in a period, $\underline{m} + m^*$, is greater than c_1. This means that type 1 agents will always invest in the most efficient equilibrium. Similarly, the other two conditions in Assumption (1) imply that type 2 and type 3 agents do not always invest; they will invest if and only if the expected y is high enough, which depends on the distribution of active agents in the current period.

Interpretation The composition of each generation can be understood alternatively in the following way. The total population in each generation is always the same, say m. However, there are other agents who do not have investment opportunities, the mass of which is $m - \sum_{i=1}^{3} m_i$.

Although our model is highly stylized, it does capture some features in the real world of production or investment. First, investments or production exhibits strategic complementarity (at the industry level). Second, different firms usually have different costs of investment or production. In our model, the low cost investments and the higher cost investments can be interpreted as fundamental investments and induced investments respectively. This interpretation makes sense because low cost type agents always invest regardless of the state of the world, while higher cost agents invest only if the state of the world is favorable enough. In some sense, higher cost investments are induced by good macroeconomic conditions. Third, the information regarding firms' cost distribution is dispersed among the economy, which can only be (partially) aggregated by the observed aggregate investment.

Given the OLG structure, y_{t-1}^n contains information about $\{m_{it}^o\}_{i=1,2,3}$. Thus agents in period t will utilize the information contained in y_{t-1}^n when making investment decisions. Before studying the private information setting, we first consider a benchmark case with complete information.

A Benchmark Suppose at the beginning of any period t (before investment decisions are made), the information about the old generation, $\{m_{it}^o\}_{i=1,2,3}$, is perfectly observable, while $\{m_{it}^n\}_{i=1,2,3}$ is not observable. Since $\{m_{it}^o\}_{i=1,2,3}$ is observable, the previous investment history $\{y_j^n\}_{j=1}^{t-1}$ is not relevant. This assumption cuts off the link between investment history and information aggregation. Depending on $\{m_{it}^o\}_{i=1,2,3}$, the economy in period t will be in one of the three possible regimes. In the H-regime, all types of agents invest. In the M-regime, only type 1 and type 2 agents invest. In the L-regime, only type 1 agents invest.

The economy in period t is in the H-regime if and only if it is profitable for type 3 agents to invest:

$$\sum_{i=1}^{3}[m_{it}^o + E(m_{it}^n)] \geq c_3 \Leftrightarrow \sum_{i=1}^{3} m_{it}^o \geq c_3 - 3m^*. \tag{2}$$

Note that due to the strategic complementarity among investments, under condition (2) the other two equilibria (M-regime and L-regime) might exist. However, the equilibrium of H-regime is efficient. This is because compared to L-regime or M-regime, under H-regime type 1 and type 2 agents are always better off, while type 3 agents are at least weakly better off. By our equilibrium selection criterion, H-regime is the equilibrium regime under condition (2).

Similarly, in period t the economy is in M-regime if and only if it is profitable for type 2 to invest but not profitable for type 3 to invest. That is,

$$\sum_{i=1}^{2}[m_{it}^o + E(m_{it}^n)] \geq c_2 \Leftrightarrow \sum_{i=1}^{2} m_{it}^o \geq c_2 - 2m^*. \tag{3}$$

holds and (2) is violated. Note that for M-regime to exist, $c_3 - c_2 > m^*$ must hold.

Finally, in period t the economy is in L-regime if and only if both (2) and (3) are violated. Note that in this complete information benchmark, conditional on $\{m_{it}^o\}_{i=1,2,3}$, in period t the equilibrium regime and aggregate investment are independent of the previous history of investments and regimes.

3 Equilibrium Regime Switches

In the private information setting, the relevant information $\{m_{it}^o\}_{i=1,2,3}$ can only be inferred from previous investment history. Formally, a (behavioral) strategy of a type i agent in period

t, s_i^t, is: $\{y_j^n\}_{j=1}^{t-1} \times c_i \rightarrow \{0,1\}$. That is, it is a mapping from investment history and its cost type to whether to invest. Let $R_t \in \{H, M, L\}$ be the regime of the economy in period t. Conditional on y_{t-1}^n and R_{t-1}, the previous investment history contains no relevant information about $\{m_{it}^o\}_{i=1,2,3}$. This is due to the fact that in the OLG model agents live for two periods and all the shocks are independent across periods. Although R_{t-1} is not directly observable, it can be inferred from investment history $\{y_j^n\}_{j=1}^{t-1}$. To see this, first note that agents can correctly infer R_1 since there is no investment history in period 1. Now suppose agents correctly inferred R_{t-2}. Since y_{t-2}^n is observable, agents can correctly infer R_{t-1} by combining R_{t-2} and y_{t-2}^n. Therefore, the observability of $\{y_j^n\}_{j=1}^{t-1}$ and the knowledge of R_1 implies that agents can correctly infer R_{t-1} recursively (a more rigorous argument can be found in the proof of Proposition 1).

Given that R_{t-1} can be correctly inferred, s_i^t becomes: $y_{t-1}^n \times R_{t-1} \times c_i \rightarrow \{0,1\}$. In essence, in deciding whether to invest agents only care about $\widetilde{m}_t$, the mass of agents that will invest in the current period t. Thus s_i^t can be written more clearly as: $E(\widetilde{m}_t) \times c_i \rightarrow \{0,1\}$, where $E(\widetilde{m}_t) = f(y_{t-1}^n, R_{t-1})$ is the expected total investment in period t. Note that players' strategies (and the underlying belief updating) are Markovian.

Although $\widetilde{m}_t$ is a one-dimensional variable, to infer $\widetilde{m}_t$ from $y_{t-1}^n \times R_{t-1}$ agents need to know the distribution of investors, $\{m_{it}^o\}_{i=1,2,3}$, which is multi-dimensional. In this sense, the state of the world in our model is multi-dimensional. Ultimately, agents care about the following conditional expectations: $E[\sum_{i=1}^{j} m_{it}^o | R_{t-1}, y_{t-1}^n]$, $j = 1, 2, 3$. If $E[\sum_{i=1}^{3} m_{it}^o | R_{t-1}, y_{t-1}^n]$ (the mass of all types of agents) is big enough then all the types will invest. If $E[\sum_{i=1}^{3} m_{it}^o | R_{t-1}, y_{t-1}^n]$ is not big enough but $E[\sum_{i=1}^{2} m_{it}^o | R_{t-1}, y_{t-1}^n]$ (the mass of type 1 and type 2 agents) is big enough, then only type 1 and type 2 agents will invest. If both $E[\sum_{i=1}^{3} m_{it}^o | R_{t-1}, y_{t-1}^n]$ and $E[\sum_{i=1}^{2} m_{it}^o | R_{t-1}, y_{t-1}^n]$ are not big enough, then only type 1 agents will invest.

To derive agents' conditional expectations, we first prove a very useful lemma.

Lemma 1 *Suppose there are N random variables m_1, m_2, ..., m_N. Each random variable is an independent draw from the same distribution function F. Then with any $n_1 < n_2 \leq N$,*

$$E[\sum_{i=1}^{n_1} m_i | \sum_{i=1}^{n_2} m_i = y] = \frac{n_1}{n_2} y. \tag{4}$$

Proof. See the appendix. ■

Note that the last period's regime R_{t-1} affects the aggregate investment y^n_{t-1}. In particular,

$$y^n_{t-1} = \begin{cases} m^o_{1t} + m^o_{2t} + m^o_{3t} & \text{if } R_{t-1} = H \\ m^o_{1t} + m^o_{2t} & \text{if } R_{t-1} = M \\ m^o_{1t} & \text{if } R_{t-1} = L \end{cases} . \tag{5}$$

This shows that the compositions of information aggregation are different in different regimes. Applying Bayes' rule to (5) and using Lemma 1, we have the following lemma about how agents form conditional expectations.

Lemma 2 *Given R_{t-1} and y^n_{t-1}, agents form conditional expectations as follows:*

$$E[m^o_{1t}|R_{t-1} = H, y^n_{t-1}] = \frac{1}{3}y^n_{t-1}; \;\; E[\sum_{i=1}^{2} m^o_{it}|R_{t-1} = H, y^n_{t-1}] = \frac{2}{3}y^n_{t-1};$$

$$E[\sum_{i=1}^{3} m^o_{it}|R_{t-1} = H, y^n_{t-1}] = y^n_{t-1}; E[m^o_{1t}|R_{t-1} = M, y^n_{t-1}] = \frac{1}{2}y^n_{t-1};$$

$$E[\sum_{i=1}^{2} m^o_{it}|R_{t-1} = M, y^n_{t-1}] = y^n_{t-1}; \;\; E[\sum_{i=1}^{3} m^o_{it}|R_{t-1} = M, y^n_{t-1}] = y^n_{t-1} + m^*;$$

$$E[m^o_1|R_{t-1} = L, y^n_{t-1}] = y^n_{t-1}; \;\; E[\sum_{i=1}^{2} m^o_{it}|R_{t-1} = L, y^n_{t-1}] = y^n_{t-1} + m^*;$$

$$E[\sum_{i=1}^{3} m^o_{it}|R_{t-1} = L, y^n_{t-1}] = y^n_{t-1} + 2m^*.$$

Lemma 2 shows that R_{t-1} affects the expectation formation or signal extraction from y^n_{t-1}. This is because the composition of information aggregation depends on R_{t-1}. In particular, information aggregation exhibits two interesting features. First, in any period some information is aggregated, as y^n_{t-1} always contains information about $\{m^o_{it}\}_{i=1,2,3}$.[8] Second, in any period information is never fully revealed: y^n_{t-1}, a single dimensional variable, cannot perfectly reveal m^o_{1t}, $\sum_{i=1}^2 m^o_{it}$ and $\sum_{i=1}^3 m^o_{it}$ at the same time.[9] Specifically, in L-regime the information about m^o_{1t} is perfectly revealed, but that of m^o_{2t} and m^o_{3t} is not aggregated at all. On the other hand, though $\sum_{i=1}^3 m^o_{it}$ is aggregated in H-regime, the exact composition of $\{m^o_{it}\}_{i=1,2,3}$ is obscured. Unlike Veldkamp (2005) where agents have more accurate information regarding the state of the economy during times of high activities, in our model this kind of informational

[8] This of course depends on our assumption that type 1 investors always investment. If c_1 is high, there might be a regime in which nobody invests and no information is aggregated.

[9] This result is related to Jehiel and Moldovanu (2001) and Mikoucheva and Sonin (2004), who in auction settings show the impossibility of efficient aggregation of multidimensional information in a one-dimensional bid.

monotonicity does not hold. This feature is due to the fact that in our model the state of the world is multi-dimensional.

Since each agent lives for two periods, his objective is to maximize his discounted lifetime payoff. It turns out that this is equivalent to maximizing current period payoff. To see this, consider an arbitrary period t. For old agents, maximizing the current period payoff is clearly optimal. For new agents, maximizing the current period payoff is also optimal for two reasons. First, a new agent's action today does not reduce the flexibility of his action in the next period (investment is reversible). Second, a single deviation would not change the composition of information aggregation in the current period, as there is a continuum of new agents. Therefore, old agents and new agents have the same equilibrium strategy, and we do not need to distinguish them when we derive equilibria.

An equilibrium in the dynamic game is characterized by a sequence of equilibrium regimes for each period. The equilibrium conditions of regime switches are summarized in the following table:

Table 1: Equilibrium Conditions of Regime Switches

	H	M	L
H	$y_{t-1}^n \geq c_3 - 3m^*$	$\frac{3}{2}c_2 - 3m^* \leq y_{t-1}^n < c_3 - 3m^*$	$y_{t-1}^n < \frac{3}{2}c_2 - 3m^*$
M	$y_{t-1}^n \geq c_3 - 4m^*$	$c_2 - 2m^* \leq y_{t-1}^n < c_3 - 4m^*$	$y_{t-1}^n < c_2 - 2m^*$
L	$y_{t-1}^n \geq c_3 - 5m^*$	$c_2 - 3m^* \leq y_{t-1}^n < c_3 - 5m^*$	$y_{t-1}^n < c_2 - 3m^*$

(The rows denote initial regimes, and the columns denote new regimes)

Proposition 1 *The conditions that govern equilibrium regime switches are described by Table 1. Agents can correctly infer regime R_{t-1} from previous investment history $\{y_j^n\}_{j=1}^{t-1}$.*

Proof. In the appendix, we derive the conditions in Table 1. From Table 1, we can see that for each row the three inequalities are mutually exclusive. Thus, given R_{t-1} and y_{t-1}^n, R_t is uniquely determined. This feature is due to our equilibrium selection criterion: the most efficient equilibrium is selected in every period. What remains to be shown is that agents are able to infer R_{t-1} from $\{y_j^n\}_{j=1}^{t-1}$. To see this, first note that agents can correctly infer R_1, since no information is available in period 1. Now suppose agents know R_{t-2}. Since y_{t-2}^n is observable, agents can correctly infer R_{t-1} by applying the equilibrium regime switching conditions. Therefore, the observability of $\{y_j^n\}_{j=1}^{t-1}$ and the knowledge of R_1 implies that agents can correctly infer R_{t-1} recursively, which justifies our presumption. ■

According to Proposition 1, the equilibrium regime switches form a Markov chain. Table 1 implicitly defines the probabilities of equilibrium regime switches P_{ij}, $i, j = L, M, H$, where

i denotes the initial regime and j the new regime. For example, $P_{HH} = \Pr(y^n_{t-1} \geq c_3 - 3m^*)$. Without imposing further restrictions on the parameters, some of the transition probabilities P_{ij} might be zero. To make regime switches meaningful, we impose two sets of conditions. First, $P_{ii} > 0$, $i = L, M, H$. This ensures that each regime has the potential to be persistent. Second, each regime is able to transit to at least one of the other regimes, so that there is no absorbing regime. That is, for any regime i there is a regime $j \neq i$ such that $P_{ji} > 0$. These two sets of conditions boil down to the following conditions on parameter values:

$$\underline{m} + 3m^* < c_2 < \overline{m} + 3m^*; \; c_2 + 2m^* < c_3 < 2\overline{m} + 4m^* \tag{6}$$

Under assumption (6), $P_{ii} > 0$, $i = L, M, H$, $P_{ML} > 0$, $P_{HL} > 0$, $P_{LM} > 0$, and $P_{MH} > 0$. Hereafter we assume (6) holds.

Since the equilibrium regime switching is a Markov chain, the steady-state probability distribution of regimes exists. Let π_i, $i = L, M, H$, be the steady-state probability of regime i, and $\pi = [\pi_L, \pi_M, \pi_H]$. Denote P as the matrix of transition probabilities

$$P = \begin{bmatrix} P_{LL} & P_{LM} & P_{LH} \\ P_{ML} & P_{MM} & P_{MH} \\ P_{HL} & P_{HM} & P_{HH} \end{bmatrix}.$$

Then π can be computed from condition $\pi P = \pi$. Let $\widetilde{y}$ be the expected aggregate investment per period in the steady state, and $\widetilde{y}_i$, $i = L, M, H$, be the expected aggregate investment in regime i. In particular,

$$\widetilde{y} = \pi_H \widetilde{y}_H + \pi_M \widetilde{y}_M + \pi_L \widetilde{y}_L. \tag{7}$$

4 Equilibrium Properties

In this section we investigate the properties of equilibrium regime switches. We say that R_t increases if R_t changes in the direction of L, M, H. If R_t changes in the opposite direction, we say that R_t decreases.

History Dependence

Proposition 2 *(i) Conditional on $\{m^o_{it}\}_{i=1,2,3}$, R_t depends on R_{t-1}; (ii) conditional on R_{t-1}, the bigger the y^n_{t-1}, the (weakly) higher the R_t; (iii) conditional on y^n_{t-1}, the higher the R_{t-1}, the (weakly) lower the R_t; (iv) for some realizations of $\{m^o_{it}\}_{i=1,2,3}$, the higher the R_{t-1}, the lower the R_t.*

Proof. Part (i) is straightforward, since R_{t-1} not only affects the realized y_{t-1}^n but also affects the interpretation of y_{t-1}^n. Part (ii) also can be easily shown. Conditional on R_{t-1}, by Table 1 R_t is weakly increasing in y_{t-1}^n. Part (iii) is evident from Table 1 as well. For example, conditional on y_{t-1}^n, the higher the R_{t-1}, the more stringent the condition is for $R_t = H$. To show part (iv), consider the following realizations of $\{m_{it}^o\}_{i=1,2,3}$: $\sum_{i=1}^3 m_{it}^o < c_3 - 3m^*$, but $\sum_{i=1}^2 m_{it}^o \geq c_3 - 4m^*$ (this means that $m_{3t}^o < m^*$). Then by Table 1, $R_{t-1} = M$ means that $R_t = H$, but R_t will be strictly lower than H if $R_{t-1} = H$. ■

Proposition 2 shows two interesting properties of equilibrium regime switches. First, regime switches are history dependent: the current period's regime depends on the last period's regime even conditional on the realized shocks to the old agents. Note that in the full information benchmark, regime switches are not history dependent: R_t does not depends on R_{t-1}. The key difference is that in the private information setting, the last period's regime affects the composition of information aggregation, which in turn affects the regime in the current period. On the other hand, this informational link is missing in the full information benchmark as $\{m_{it}^o\}_{i=1,2,3}$ is perfectly observed. Thus the history dependence arises for informational reasons. The history dependence also implies that shocks have persistent effects on the economy. In the private information setting, shocks in period $t-1$, $\{m_{it-1}^n\}_{i=1,2,3}$, not only affect R_t, but also affect R_{t+1} since it is affected by R_t. By similar logic, the regimes after period $t+1$ are also affected by shocks in period $t-1$, thus shocks have persistent effects. In contrast, under the full information benchmark shocks in period $t-1$ only affect the regime in period t, thus the effect of shocks is temporary.

This shows that the dynamics of endogenously determined information aggregation are an integral part of investment cycles. The composition of information aggregation is not only a by-product of business cycle, but also affects the its course.

The second property is that the history-dependence is not monotonic: for some realizations of shocks, a higher R_{t-1} leads to a lower R_t. The reason for this non-monotonicity is as follows. Agents' belief updating is based on two pieces of information: last period's aggregate investment y_{t-1}^n and last period's regime R_{t-1}. Conditional on R_{t-1}, an increase in y_{t-1}^n is always good news since it implies more active agents. However, conditional on y_{t-1}^n, the higher the R_{t-1}, the more pessimistic the agents are.[10] This is because under a higher R_{t-1} more agents are expected to invest, thus a higher y_{t-1}^n is expected. Therefore, an increase in R_{t-1} without a corresponding increase in y_{t-1}^n actually is interpreted as bad news, which leads to a

[10]This feature is not possible if there were only one type of investors, say type 1, since there is only one regime, L.

lower regime in period t.[11]

The Impact of Shocks to Different Cost Types Another interesting property is that shocks to different cost types have different impacts on regime switches. Specifically, shocks to type 1 agents affect all possible regime switches, since m^o_{1t} is always a part of y^n_{t-1}. On the other hand, shocks to type 2 (3) agents affect regime switches only if the initial regime is weakly higher than M (H), since m^o_{2t} (m^o_{3t}) is aggregated under y^n_{t-1} only if $R_{t-1} \geq M$ ($R_{t-1} = H$). Therefore, shocks to the low cost type have the biggest impact on the economy, whereas shocks to the high cost type have the smallest impact. This property is again due to informational reasons: while the information about m^o_{1t} is always aggregated into y^n_{t-1}, the information about m^o_{2t} and m^o_{3t} might not always be.

Since only m^o_{1t} is revealed in L-regime, once the economy is in L-regime, whether the economy can be pulled out of it only depends on the shocks to the low cost type m^o_{1t}, regardless of the shocks to higher cost types. This result can be interpreted as follows. Once the economy is in recession, lacking enough fundamental investment will still trap the economy in recession, no matter how favorable the shocks to induced investments are. Note that this feature does not arise in the full information setting, under which favorable enough shocks to induced investment alone can pull the economy out of regime L.

To examine the impacts of the shocks to the low cost type on the economy, we conduct the following comparative statics. Consider two system of shocks. The first system is the same as the basic model. In the second system, $m'_1 - \varepsilon$ with $\varepsilon > 0$ is drawn from $F(m)$. Thus the support of m'_1 is $[\underline{m} + \varepsilon, \overline{m} + \varepsilon]$ and the mean of m'_1 is $m^* + \varepsilon$. The other two shocks are the same as those in the first system. The distribution of shocks in each system is common knowledge. Compared to system 1, system 2 has higher expected fundamental investments.

Denote P^k_{ij}, $k = 1, 2$, as the transition probabilities under system k, and π^k_i as the steady state probability of regime i under system k. Let $\widetilde{y}^k_i$, $i = H, M, L$, $k = 1, 2$, be the expected investment in regime i under system k, and $\widetilde{y}^k$, $k = 1, 2$, be the expected investment under system k. Specifically,

$$\begin{aligned}
\widetilde{y}^1_H &= 6m^*;\ \widetilde{y}^1_M = 4m^*;\ \widetilde{y}^1_L = 2m^*; \\
\widetilde{y}^2_H &= 6m^* + 2\varepsilon;\ \widetilde{y}^2_M = 4m^* + 2\varepsilon;\ \widetilde{y}^2_L = 2m^* + 2\varepsilon.
\end{aligned}$$

[11] Another example for part (iv) of Proposition 2 is as follows:. $c_3 - 4m^* \leq \sum_{i=1}^2 m^o_{it} < c_3 - 3m^* - \underline{m}$, and $m^o_{3t} = \underline{m}$. For this realization of shocks, if $R_{t-1} = M$ then $R_t = M$, but if $R_{t-1} = H$ then $R_t = L$. This pattern arises because when the shock to high cost agents is very unfavorable, under regime H agents will suspect that a relatively low aggregate investment is caused partially by unfavorable shocks to lower cost agents, as they cannot identify the composition of shocks in regime H.

By (7), we compute $\widetilde{y}^2 - \widetilde{y}^1$.

$$\begin{aligned} \widetilde{y}^2 - \widetilde{y}^1 &= 2\varepsilon + 6m^*(\pi_H^2 - \pi_H^1) + 4m^*(\pi_L^1 + \pi_L^2 - \pi_H^2 - \pi_H^1) + 2m^*(\pi_L^2 - \pi_L^1) \\ &= 2\varepsilon + 2m^*[(\pi_H^2 - \pi_L^2) - (\pi_H^1 - \pi_L^1)]. \end{aligned} \tag{8}$$

Proposition 3 *Compare two systems with system 2 having a higher expected fundamental investment. (i) The equilibrium probabilities of regime switches exhibit* $P_{iH}^1 < P_{iH}^2$ *and* $P_{iL}^1 > P_{iL}^2$, $i = L, M, H$. *(ii) The steady-state probabilities exhibit* $\pi_H^2 - \pi_L^2 > \pi_H^1 - \pi_L^1$. *(iii) The expected investment in steady state is higher under system 2:* $\widetilde{y}^2 > \widetilde{y}^1$.

Proof. See the Appendix. ■

Intuitively, due to the strategic complementarity an upward shift of the distribution of the fundamental investments makes the induced investment more profitable, so the investment opportunities of the induced investment are more likely to be materialized. Part (i) of Proposition 3 implies that compared to system 1, recessions are (on average) shorter and booms are longer under system 2. To see this, the expected (or average) length of a recession is $1/(1 - P_{LL}^i)$, and the expected length of a boom is $1/(1 - P_{HH}^i)$. By the fact that $P_{LL}^1 > P_{LL}^2$ and $P_{HH}^1 < P_{HH}^2$, we get the desired result. Part (ii) of Proposition 3 implies that relative to system 1, the economy under system 2 is (on average) more likely in booms (regime H) and less likely in recessions (regime L). This naturally leads to part (iii): the expected investment is higher under system 3.

From (8), the increase in expected investment can be decomposed into two effects. The first effect is the direct effect: an increase in fundamental investment directly increases the investments in each regime. This effect is captured by the term 2ε. The second effect is an indirect effect, which is captured by the second term in (8). An increase in fundamental investments increases the probability of regime H and reduces the probability of regime L, leading to higher expected investment.

To evaluate these two effects, consider a specific example. Let $c_1 = 0$, $c_2 = 2.2$, and $c_3 = 3.6$. In the first system, the distribution of each shock is uniform on $[0, 1]$. The second system is the same as the first one except that m_1 is distributed uniformly on $[0.1, 1.1]$. Note that the expected increase in fundamental investments (ε) is 0.1. The transition probability matrix under system k, P^k, is as follows:

$$P^1 = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.68 & 0.24 & 0.08 \\ 0.716 & 0.1625 & 0.1215 \end{bmatrix}; \quad P^2 = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.5 & 0.32 & 0.18 \\ 0.5 & 0.342 & 0.158 \end{bmatrix}.$$

Clearly, part (i) of Proposition 3 holds in this example. The steady state probability distributions are

$$\begin{aligned}(\pi_L^1, \pi_M^1, \pi_H^1) &= (0.7113, 0.2657, 0.0242);\\(\pi_L^2, \pi_M^2, \pi_H^2) &= (0.5, 0.363, 0.137).\end{aligned}$$

We can see that π_L^2 is significantly less than π_L^1, and π_H^2 is significantly higher than π_H^1. For expected investments,

$$\widetilde{y}^1 = 1.315;\ \widetilde{y}^2 = 1.837;\ \widetilde{y}^2 - \widetilde{y}^1 = 0.522.$$

The increase in the expected investment due to the direct effect is $2\varepsilon = 0.2$, and that due to the indirect effect is 0.322, which is bigger than the direct effect. In percentage terms, a 20% (ε/m^*) increase in the fundamental investment leads to a 40% increase in expected total investment. This shows that a small change in the shocks to fundamental investment can have a big impact on the economy.

This result fits naturally into the "long wave" theory. According to Kondratiev (1926), the patterns of business cycles are different in different phases of long wave cycles. He concludes that "during the period of a rising wave in the long cycles, the intermediate business cycles are characterized by the brevity of depressions and the intensity of upswings. During the period of a downward wave in the long cycles, the picture is the opposite." Schumpeter (1939) identifies the driving force of long wave cycles as technological innovations: the upward wave in the long cycles is characterized by rapid technological innovations and fast expansion of new industries, while the downward wave has the opposite features.

The investments due to technological innovations and expansion of new industries correspond to fundamental investments (the low-cost investments) in our model.[12] Thus in the upward wave of long cycles there are more fundamental investments (corresponding to system 2), while in the downward wave there are less fundamental investments (corresponding to system 1). Note that the pattern of business cycles is consistent with the implications of our model. In particular, when there is more fundamental investment, the recessions are short, booms are longer, and average economic activity is higher. The opposite pattern holds when there is less fundamental investment.

More importantly, our result implies that the pattern of long wave cycles can be generated by small variations of the fundamental investments. This is because a small difference in the shocks to fundamental investment can generate a significant impact on the pattern of business cycles. The expansion of a new industry may only account for a small percentage of total

[12] This is because those investments are profitable regardless of the macroeconomic conditions.

investment. However, the knowledge that such an industry exists increases the profitability of higher cost investments, leading to more induced investment opportunities being materialized. As our example indicates, this indirect effect can significantly amplify the impact of fundamental investment on the economy.[13]

Next we show that shocks to induced investments have less impact on the economy than shocks to fundamental investment do. For this purpose, we introduce another system of shocks. System 3 is the same as system 1 except that $m_3' - \varepsilon$ with $\varepsilon > 0$ is drawn from $F(m)$. Compared to system 1, there is (on average) more high cost investment in system 3.

Proposition 4 *(i) The equilibrium probabilities of regime switches exhibit $P^1_{iH} < P^2_{iH} = P^3_{iH}$ and $P^2_{iL} < P^3_{iL} = P^1_{iL}$, $i = L, M, H$. (ii) The steady-state probabilities exhibit $\pi^2_H - \pi^2_L > \pi^3_H - \pi^3_L$. (iii) The expected investment in steady state is higher under system 2: $\widetilde{y}^2 > \widetilde{y}^3$.*

Proof. See the Appendix. ■

The reason that a corresponding structural change in the high cost investment has a smaller impact on the economy is that shocks to high cost investment only affect the transition probabilities of going into regime H. Intuitively, those shocks only affect whether regime H can be supported, but have no impact on whether regime M can be supported in the next period. Following the previous example, we introduce system 3, which is the same as the first system except that m_3 is distributed uniformly on $[0.1, 1.1]$. The probability transition matrix and steady state distribution under system 3 are:

$$\begin{aligned} P^3 &= \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.68 & 0.14 & 0.18 \\ 0.716 & 0.126 & 0.158 \end{bmatrix}; \\ (\pi^3_L, \pi^3_M, \pi^3_H) &= (0.6983, 0.1802, 0.1215);\ \widetilde{y}^3 = 1.4475. \end{aligned}$$

Note that π^3_L is very close to π^1_L and is significantly bigger than π^2_L. Moreover, $\widetilde{y}^3$ is substantially lower than $\widetilde{y}^2$. In percentage terms, a 20% (ε/m^*) increase in the high cost investment leads to a 10% increase in expected total investment, which is much lower than the 40% increase in expected total investment resulting from the same percentage increase in fundamental investment.

Asymmetric Regime Switches In assumption (6), the condition that ensures $P_{LL} > 0$ is $c_2 > \underline{m} + 3m^*$ (see Table 1). Under this condition,

$$\frac{3}{2}c_2 - 3m^* > \frac{3}{2}\underline{m} + \frac{3}{2}m^* > 3\underline{m}.$$

[13] In macroeconomics, the well-known multiplier effect also amplifies the impact of investment on aggregate economic activity. This paper identifies a different amplification mechanism due to informational reasons.

Thus $P_{HL} > 0$ (see Table 1). On the other hand, $P_{LH} > 0$ is not guaranteed. Specifically, if $c_3 > \overline{m} + 5m^*$, then from Table 1 we can see that $P_{LH} = 0$. In this case, regime switches exhibit asymmetry: while it is always possible for the economy to directly switch from regime H to regime L, the direct switch from regime L to regime H might be impossible. In other words, the only possible asymmetry is quick downward switch and slow upward switch. Thus we have the following proposition.

Proposition 5 *Suppose assumption (6) holds. Then, while P_{HL} is always strictly positive, $P_{HL} = 0$ if $c_3 > \overline{m} + 5m^*$.*

The underlying reason for this potential asymmetric pattern of regime switches is informational. In regime H the worst scenario (each realization of the shocks is $\underline{m}$) can be revealed, which causes the economy to plunge into regime L directly. On the other hand, in regime L the best possible scenario (each realization of the shocks is $\overline{m}$) cannot be revealed, since the information about m_{2t}^o and m_{3t}^o is not aggregated. Instead, the most optimistic belief is $\sum_{i=1}^{3} m_{it}^o = \overline{m} + 2m^*$. Thus if c_3 is high enough ($c_3 > \overline{m} + 5m^*$), $P_{LH} = 0$, and the direct transition from L to H is impossible.

The potential asymmetry in regime switches implies that once the economy is in the L-regime, it takes at least two periods with consecutive favorable shocks for the economy to switch to the H-regime. Therefore, the economy is characterized by abrupt crashes from boom to recession and slow buildups from recession to boom. This might explain the well-documented asymmetry in business cycles: the downturns at the end of booms are generally sharp, while the recovery from slump to boom is more gradual (Neftci, 1984; Hodrick and Prescott, 1997).

Veldkamp (2005) develops a model with learning asymmetry to explain sudden crash and slow recovery. In her model, agents learn the (uncertain) return of their investments. Each investment works like an experiment, the outcome of which reveals some information about the common investment return. Naturally, agents learn the investment return more accurately during booms than during recessions, since more investments (experiments) lead to a higher signal-to-noise ratio. Given that agents have more accurate information during booms than in slumps, beliefs are updated quickly if a negative shock occurs during booms and they are updated slowly if a positive shock occurs during slumps. This pattern results in sudden crashes and slow recoveries. In our model, the learning asymmetry comes from the fact the worst realization of underlying shocks is revealed in regime H, while the best realization of shocks cannot be revealed in regime L, since only the shock to the low cost type investment is revealed in regime L. And this leads to the feature that the biggest possible downward jump in beliefs is bigger than the biggest possible upward jump in beliefs.[14]

Steady State Distributions of Regimes It would be desirable to compare the steady state distribution of regimes in the private information setting to that of the full information benchmark. However, under general conditions this turns out to be a difficult task without the distribution of shocks being specified. In the following, we assume that $F(m)$ is uniform on $[0,1]$ with $m^* = 0.5$. On top of assumption (6), we make the following assumptions:

$$c_2 > 4m^* = 2 \text{ and } c_3 > 6m^* = 3. \tag{9}$$

We interpret Assumption (9) as the economy exhibiting *positive (information) revelation bias* in the following sense. If no information regarding the distribution of investors is revealed, Assumption (9) implies that the equilibrium regime will be L for sure. On the other hand, if some information is revealed, then with some positive probability the equilibrium regime will be H or M. Thus information revelation increases the probability that the economy is in higher regimes.

We will focus on the case with positive revelation bias. Denote π_i^B as the probability that the economy is in regime i under the benchmark case. In the following proposition, we show that, compared to the benchmark case, in the steady state in private information setting the economy has a higher probability of being in regime L and a lower probability of being in regime H.

Proposition 6 *Suppose Assumption (9) holds. Compared to the benchmark case, in the private information setting the economy is less likely to be in the H-regime and more likely to be in the L-regime. That is,* $\pi_H < \pi_H^B$ *and* $\pi_L > \pi_L^B$.

Proof. See the Appendix. ■

Proposition 6 implies that when the economy exhibits positive revelation bias, the expected aggregate economic activity is lower in the private information setting. This is because in the private information setting the dispersed information never gets fully revealed, while in the benchmark case all the relevant information is directly observable. When the economy exhibits positive revelation bias, the fact that less information is revealed in the private information setting leads to lower economic activity.

To illustrate Proposition 6, consider system 1 in the previous example. In the benchmark case, the probability distribution of regimes is as follows:

[14] Note that unlike in Veldkamp (2005), in our model agents do not necessarily learn more during booms than during slumps. For example, in regime H the composition of underlying shocks is unknown, while in regime L the shock to the low cost investment is fully revealed.

$$(\pi_L^B, \pi_M^B, \pi_H^B) = (0.6073, 0.1899, 0.2028).$$

Comparing these probabilities to π_i^1, we can see that $\pi_L^1 > \pi_L^B$, $\pi_H^1 < \pi_H^B$. Moreover,

$$1.315 = \widetilde{y}^1 < \widetilde{y}^B = 2.584.$$

That is, the expected aggregate investment is much lower in the private information setting than in the benchmark case.

5 N-Type Model

In this section we generalize the basic model to a N-type model. Suppose there are N types of investors, with $N \geq 3$. Type n ($n = 1, ..., N$) is characterized by investment cost c_n, with $c_{n+1} > c_n$ for any n. All the costs are common knowledge. Let m_n be the mass of the type n investor in a generation. Again, m_n is a random variable with a cumulative distribution function $F(\cdot)$ on support $[\underline{m}, \overline{m}]$ with mean m^*. All the m_n are i.i.d. across types and across generations. To make the regime switches meaningful, we make following assumptions about c_n:

$$\begin{aligned} c_1 &< \underline{m} + m^*; \\ n\underline{m} + (n+2)m^* &< c_{n+1} < n\overline{m} + (n+2)m^* \text{ for all } n > 1; \\ c_{n+1} - c_n &> 2m^* \text{ for all } n > 1. \end{aligned} \tag{10}$$

All the other assumptions are the same as those in the basic model.

Now we define regime n. The economy is in regime n if all the cost types $i \leq n$ invest, and all the cost types $i > n$ do not invest. The first inequality in assumption (10) implies that type 1 agents always invest. The second inequality in (10) ensures that one-step regime switches are possible in both directions for any regime $n \in \{2, 3, ..., N-1\}$. The last inequality guarantees that each regime $n \in \{1, 2, ..., N\}$ exists.

Note that if $R_{t-1} = n$, only $y_{t-1}^n = \sum_{i=1}^n m_{it}^o$ is revealed at the beginning of period t. Thus the equilibrium regime switch again depends on $\{y_{t-1}^n, R_{t-1}\}$. It is easy to see that in the N-type model, maximizing current period payoff is optimal for any agent. Moreover, a generalized version of Lemma 2 holds in the N-type model. The following proposition characterizes the general pattern of equilibrium regime switches.

Proposition 7 *Suppose $R_{t-1} = n$. Given y^n_{t-1}, the equilibrium regime in period t, R_t, will be:*

$$\begin{aligned}
R_t &= N \text{ if } y^n_{t-1} \geq c_N - (2N-n)m^*; \\
R_t &= j \text{ for } n \leq j < N \text{ if } c_j - (2j-n)m^* \leq y^n_{t-1} < c_{j+1} - [2(j+1)-n]m^*; \\
R_t &= j \text{ for } 1 < j < n \text{ if } y^n_{t-1} \geq \frac{n}{j}c_j - nm^*; \\
&\quad \text{and } y^n_{t-1} < \frac{n}{k}c_k - nm^* \text{ for any } k \text{ such that } j < k < n; \\
R_t &= 1 \text{ if } y^n_{t-1} < \frac{n}{k}c_k - nm^* \text{ for any } k \text{ such that } 1 < k < n.
\end{aligned}$$

Proof. See the Appendix. ■

Proposition 7 shows that the N-type model is qualitatively the same as the three-type model, except that now regime switches become more complex. The state of information aggregation depends on the regime in the last period, which affects the investment behavior and regime in the current period. This leads to history dependence of regime switches. Since the information about shocks to n-type investors is aggregated if and only if the regime is higher than n, shocks to lower cost types affect the economy more than those to higher cost types.

The possible asymmetry between downward and upward regime switches is also present. Under assumption (10), there is always a positive probability that the economy directly switches from a regime $n > 1$ to any lower regime $k < n$, while the opposite direct regime switch might not be possible. To see this, the direct downward regime switch from n to a lower regime k is possible if the following condition is satisfied:

$$\begin{aligned}
n\underline{m} &< \frac{n}{k}c_k - nm^* \text{ for any } k \text{ such that } 1 < k < n \\
&\Leftrightarrow k\underline{m} + km^* < c_k \text{ for any } k \text{ such that } 1 < k < n.
\end{aligned}$$

The inequality ensures that under the worst realization of shocks (from type 1 to type n), the economy will directly slump into regime 1. By assumption (10),

$$k\underline{m} + km^* < (k-1)\underline{m} + [(k-1)+2]m^* < c_k \text{ for any } k \text{ such that } 1 < k < n.$$

Thus the direct downward regime switch is always possible. On the other hand, for a direct upward regime switch from k to n to be possible, the following condition must be satisfied:

$$k\overline{m} + (2n-k)m^* \geq c_n.$$

This condition is not guaranteed by assumption (10). Actually, assumption (10) only ensures the above condition is satisfied for $k = n-1$; that is, a one-step upward switch is always possible. Again, this is due to the asymmetry in information aggregation between higher regimes and lower regimes: in higher regimes the worst realization of shocks is revealed, while in lower regimes the best realization of shocks cannot be revealed since the information about the shocks to higher cost types is not aggregated.

The non-monotonicity of history dependence is also present in the N-type model. Agents' belief updating depends on two pieces of information: the old generations' aggregate investment and the regime in the last period. Since the aggregate investment is supposed to be higher in higher regimes, an increase in regime without a corresponding increase in aggregate investment actually is interpreted as bad news and might cause a downturn switch. Similarly, a decrease in regime without a corresponding decrease in aggregate investment is interpreted as good news and might induce an upward switch. This non-monotonicity implies that the direction of regime switches depends on the last period's *growth rate* of aggregate investment.

Proposition 8 *Let $R_{t-1} = j$, $R_t = j'$. (i) Suppose $1 \leq j < j' < N$. Then $R_{t+1} > R_t$ only if $(y_t^n - y_{t-1}^n)/y_{t-1}^n > \widehat{g}^+(j, j') > 0$; (ii) Suppose $1 < j' < j \leq N$. Then $R_{t+1} < R_t$ only if $(y_t^n - y_{t-1}^n)/y_{t-1}^n < \widehat{g}^-(j, j') < 0$; where*

$$\widehat{g}^+(j, j') = \frac{(j' - j)m^*}{c_{j'+1} - [2(j' + 1) - j]m^*}; \ \widehat{g}^-(j, j') = \frac{j' - j}{j}. \tag{11}$$

Proof. See the Appendix. ■

To see Proposition 8, suppose the economy is initially in some intermediate regime, and the aggregate investment increases, which causes an upward regime switch. However, if the increase in aggregate activity in the next period is less than some cutoff value, the economy will switch to some lower regime in the period after. Therefore, to sustain an upward trend of regime switches over a number of consecutive periods, a mere increase in aggregate activities might not be enough. Instead, every step of the increase has to be big enough, since a small increase will be interpreted as bad news and might cause a downward switch. Similarly, in a downward trend of regime switches, small decreases in aggregate activities actually are good news and may reverse the trend of downward switches. Using an analogy, to keep a trend of regime switches going, the trend must have enough "momentum." In an upward trend of recovery, a slowing down of growth (decrease in the upward momentum) might trigger an economic downturn. On the other hand, in an economic downturn a slowing down of the downturn (decrease in the downward momentum) might trigger a recovery.

In some sense, this result provides a theoretical foundation for self-exciting threshold autoregressive (SETAR) models. Introduced initially by Tong (1978), SETAR models aim at capturing non-linearity in time series. Specifically, there are several regimes for the variable in question (usually the first difference of GNP). The variable is a linear autoregression within a regime but may move between regimes depending on the specific region that its own lagged variable is in. Thus consistent with our result, regime switches depend on the growth rate of the variable in the past. During the past decade, SETAR models have been applied to many time series. Tiao and Tsay (1994) and Potter (1995) found that US GNP can be usefully modelled as a SETAR process. According to Clements and Smith (1997), the performance of one-step ahead forecasting can be significantly improved by using SETAR models. Focusing on Canadian real GNP data, Feng and Liu (2003) found that SETAR model has better forecasting performance than other linear models. Though these empirical studies are not direct tests of our model, they suggest that our model has some relevance to real world business cycles.

6 Discussions

Our model is simple and highly stylized. In this section we discuss how our results will be affected if we incorporate more realistic assumptions.

The Observability of y_{t-1}^n and Risk Aversion. We have assumed that $\{y_j^n\}_{j=1}^{t-1}$ is observable. A more realistic assumption is that only $\{y_j\}_{j=1}^{t-1}$ is observable. Though agents in period t only care about y_{t-1}^n and R_{t-1}, now they will have to extract the information about y_{t-1}^n from $y_t = y_{t-1}^n + y_{t-1}^o$. Since y_{t-2} and R_{t-2} contain information about y_{t-1}^o, now both y_{t-2} and R_{t-2} affect the current period's expectation formation even conditional on R_{t-1}. In general, this recursive structure means that the entire history of $\{y_j\}_{j=1}^{t-1}$ and $\{R_j\}_{j=1}^{t-1}$ affects the belief updating in the current period. Therefore, the non-observability of y_{t-1}^n implies that regime switches are not Markovian, and equilibria become difficult to describe and the model becomes intractable. However, the general insights still hold in this alternative setting. In principle, agents form expectations based on the entire history of $\{y_j\}_{j=1}^{t-1}$ and $\{R_j\}_{j=1}^{t-1}$, which determine the regime and aggregate activities in the next period. The composition of information aggregation again depends on regimes, which in turn affects aggregate activities and regimes in the future.

Now we briefly consider the case when agents are risk averse. First of all, risk aversion does not affect the qualitative results of the basic model, but it will affect the cutoff values for the equilibrium regime switching conditions. Second, risk aversion would make regimes more persistent. This is because remaining in the same regime involves less uncertainty, while switching to other regimes involves more uncertainty. Third, risk aversion would make upward regime switches less likely, since they are involved with more uncertainty.

The Persistence of Shocks. We have assumed that the shocks are i.i.d. across time periods. More realistically, shocks might have some persistence. Consider the basic model with Markovian shocks. Specifically, for $i = 1, 2, 3$,

$$m_{it}^n = \begin{cases} m_{it-1}^n & \text{with probability } \rho \\ \text{a random draw from } F(m) & \text{with probability } 1-\rho \end{cases}.$$

where $\rho \in (0,1)$ is the persistence parameter. Shocks to different types of investors are again independent of each other.

In this setting, the expectation formation in the current period in general will depend on the entire history of $\{y_j^n\}_{j=1}^{t-1}$ and $\{R_j\}_{j=1}^{t-1}$. [15] This non-Markovian structure of expectation formation makes it difficult to characterize equilibrium regime switches in general. However, we believe that the persistence of shocks will not change the main insights of the basic model. The composition of information aggregation still depends on regimes, which in turn affects aggregate investment and regimes in the future. The difference is that the equilibrium regime switches now depend on the entire history of regimes and aggregate activities, though more distant history has less impact. The history dependence of regime switches is again not monotonic, as more aggregate investments are expected in higher regimes. The shocks to fundamental investment still have more impact on the economy than induced investment does, since the information about the former is always contained in aggregate investment, while that of induced investment is not always aggregated. Moreover, regime switches again might exhibit sudden crashes and slow recoveries, as the worst possible realization of shocks is revealed in regime H, and the best possible realization of shocks is not revealed in regime L.

The persistence of shocks in general makes regimes more persistent. This adds a more realistic feature to the equilibrium regime switches.[16] The persistence of shocks might make the asymmetry in regime switches more prominent. If a negative shock occurs in high regimes, it is immediately revealed and the economy switches to low regimes immediately. Given the persistence of shocks and the information about high cost investments being not aggregated in low regimes, the economy will remain in low regimes for a longer period of time. When positive shocks to higher cost investment occur in low regimes, since the information about those shocks is not aggregated, the economy still remains in low regimes.

[15] This example shows that additional relevant information might be contained in the history besides y_{t-1}^n and R_{t-1}. Suppose $R_{t-1} = M$ and $R_{t-2} = L$. Then, m_{1t-1}^o is perfectly revealed by y_{t-2}^n. However, m_{1t}^o is not perfectly revealed by y_{t-1}^n. Given that shocks are persistent, m_{1t-1}^o contains additional information about m_{1t}^o, conditional on y_{t-1}^n and R_{t-1}.

[16] In the examples of the basic model, we see that regime H and regime M have low persistence.

Correlated Shocks within a Generation Here we briefly discuss the situation in which the shocks to three types of agents within a generation are correlated. We start with positive correlations. As long as m_{1t}^o, m_{2t}^o, and m_{3t}^o are not perfectly correlated, the qualitative results of the basic model still hold. The composition of information aggregation again varies across different regimes. For example, learning m_{1t}^o in regime L and learning $\sum_{i=1}^{3} m_{it}^o$ in regime H will lead to different expectations about $\{m_{it}^o\}_{i=1,2,3}$. Higher aggregate investments are still expected in higher regimes. Shocks to low cost types again have a bigger impact on investment cycles. Given positive correlations, whether the economy will be pulled out of regime L will depend more sensitively on the shocks to low cost types, magnifying their impacts on cycles. Of course, positive correlations reduce the learning asymmetry across different regimes, which will make the asymmetry of cycles less prominent.

Negative correlations among m_{1t}^o, m_{2t}^o, and m_{3t}^o might qualitatively change some results of the basic model. For example, a low m_{1t}^o revealed in regime L might be a good news if this means m_{2t}^o and m_{3t}^o are very high, and hence increases the regime next period. But as long as the negative correlations are not strong enough, the qualitative results of the basic model still hold.

7 Conclusion

We study the interaction between information aggregation and investment cycles in a dynamic coordination setting with heterogenous agents, and explores its implications for investment cycles. We emphasize that the composition of information aggregation varies across different phases of investment cycles, which affects agents' expectation formation and the dynamics of investment cycles.

Our paper sheds some light on the observed pattern of business cycles. Specifically, regime switches are history dependent for informational reasons, and the history dependence is not monotonic. Regime switches depend on the growth rate of aggregate investment: a slowdown in growth is interpreted as bad news and a slowdown in downturn is considered as good news. A small structural change in fundamental investment can have a large effect on the pattern of cycles. Investment cycles might be characterized by sudden crashes and slow recoveries.

While it is reasonable for investments across different industries to exhibit strategic complementarities, within an industry the investments across firms are typically strategic substitutes. One can adopt our model to this new setting, and explore the impacts of information aggregation on industry dynamics. We left it for future research. Another possible extension of our model is to study the effect of information aggregation on political regime change. For example, citizens have different costs for rebelling; the mass of each cost type is uncertain and changing over time; the more people rebel, the more likely the success of revolution. Observing the history of the total number of rebellions, citizens form expectations about the size of rebel in the current period and decide whether to rebel.

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Appendix

Proof of Lemma 1.

Proof. First note that

$$E[m_1 + m_2 + ... + m_{n_2} | \sum_{i=1}^{n_2} m_i = y] = E[\sum_{i=1}^{n_2} m_i | \sum_{i=1}^{n_2} m_i = y] = y.$$

Since all the m_i are drawn from the same distribution function and they are independent, we have

$$E[m_1 | \sum_{i=1}^{n_2} m_i = y] = E[m_2 | \sum_{i=1}^{n_2} m_i = y] = ... = E[m_{n_2} | \sum_{i=1}^{n_2} m_i = y].$$

Now

$$E[m_1 + m_2 + ... + m_{n_2} | \sum_{i=1}^{n_2} m_i = y] = n_2 E[m_1 | \sum_{i=1}^{n_2} m_i = y] = y.$$

Hence

$$E[m_1 | \sum_{i=1}^{n_2} m_i = y] = E[m_2 | \sum_{i=1}^{n_2} m_i = y] = ... = E[m_{n_2} | \sum_{i=1}^{n_2} m_i = y] = \frac{1}{n_2} y. \tag{12}$$

It is obvious that (4) is directly implied by (12). ∎

Proof of Proposition 1.

Proof. Suppose $R_{t-1} = H$, thus $y_{t-1}^n = \sum_{i=1}^3 m_{it}^o$ is revealed in period t. For the economy to stay in H-regime in period t, it must be profitable for type 3 agents to invest. That is,

$$y_{t-1}^n + E[\sum_{i=1}^3 m_{it}^n] \geq c_3 \Leftrightarrow y_{t-1}^n \geq c_3 - 3m^*. \tag{13}$$

For the economy to switch to M regime in period t, it should be only profitable for type 1 and 2 to invest:

$$E[\sum_{i=1}^{2} m_{it}^{o}|y_{t-1}^{n}, R_{t-1} = H] + E[\sum_{i=1}^{2} m_{it}^{n}] \geq c_2 \text{ and (13) fails.}$$

By Lemma 1, the above inequality is equivalent to

$$\frac{2}{3}y_{t-1}^{n} \geq c_2 - 2m^* \text{ and (13) fails} \Leftrightarrow \frac{3}{2}c_2 - 3m^* \leq y_{t-1}^{n} < c_3 - 3m^*.$$

Finally, the economy will switch to L regime in period t if:

$$y_{t-1}^{n} < \frac{3}{2}c_2 - 3m^*.$$

Now suppose $R_{t-1} = M$, thus $y_{t-1}^{n} = \sum_{i=1}^{2} m_{it}^{o}$ is revealed in period t. For the economy to switch to H regime in period t, we must have:

$$y_{t-1}^{n} + E[m_3^o + \sum_{i=1}^{3} m_{it}^{n}] \geq c_3 \Leftrightarrow y_{t-1}^{n} \geq c_3 - 4m^*. \tag{14}$$

For the economy to stay in M regime, the following condition is sufficient and necessary:

$$y_{t-1}^{n} + E[\sum_{i=1}^{2} m_{it}^{o}] \geq c_2 \text{ and (14) fails} \Leftrightarrow c_2 - 2m^* \leq y_{t-1}^{n} < c_3 - 4m^*.$$

And the economy will switch to L regime if $y_{t-1}^{n} < c_2 - 2m^*$.

Now suppose $R_{t-1} = L$, thus only $y_{t-1}^{n} = m_{1t}^{o}$ is revealed. For the economy to switch to H regime in period t, the following condition must hold:

$$y_{t-1}^{n} + E[\sum_{i=2}^{3} m_{it}^{o} + \sum_{i=1}^{3} m_{it}^{n}] \geq c_3 \Leftrightarrow y_{t-1}^{n} \geq c_3 - 5m^*.$$

For the economy to switch to M regime, we must have:

$$c_2 - 3m^* \leq y_{t-1}^{n} < c_3 - 5m^*.$$

And the economy will remain in L regime if $y_{t-1}^{n} < c_2 - 3m^*$. ■

Proof of Proposition 3.

Proof. First note that in system 2 $m_1' - \varepsilon$ has the same distribution as m_2 and m_3. Following similar procedure in the proof of Proposition 1, we derive the equilibrium transition probabilities under system 2.

$$P_{HH}^{2} = \Pr[m_{1t}^{o\prime} + \sum_{i=2}^{3} m_{it}^{o} \geq c_3 - 3m^* - \varepsilon] = \Pr[\sum_{i=1}^{3} m_{it}^{o} \geq c_3 - 3m^* - 2\varepsilon].$$

Hence $P_{HH}^1 < P_{HH}^2$. Similarly, we can show that

$$\begin{aligned}
P_{MH}^2 &= \Pr[m_{1t}^{o\prime} + m_{2t}^o \geq c_3 - 4m^* - \varepsilon] = \Pr[\sum_{i=1}^{2} m_{it}^o \geq c_3 - 4m^* - 2\varepsilon] > P_{MH}^1; \\
P_{LH}^2 &= \Pr[m_{1t}^{o\prime} \geq c_3 - 5m^* - \varepsilon] = \Pr[m_{1t}^o \geq c_3 - 5m^* - 2\varepsilon] > P_{LH}^1.
\end{aligned}$$

The condition governs $H \to L$ in system 2 is

$$E[m_{1t}^{o\prime} + m_{2t}^o | m_{1t}^{o\prime} + m_{2t}^o + m_{3t}^o] < c_2 - 2m^* - \varepsilon. \tag{15}$$

Rearrange the left hand side,

$$\begin{aligned}
E[m_{1t}^{o\prime} + m_2^o | m_{1t}^{o\prime} + \sum_{i=2}^{3} m_{it}^o] &= \varepsilon + E[m_{1t}^{o\prime} - \varepsilon + m_{2t}^o | m_{1t}^{o\prime} - \varepsilon + \sum_{i=2}^{3} m_{it}^o] \\
&= \varepsilon + \frac{2}{3}(m_{1t}^{o\prime} - \varepsilon + \sum_{i=2}^{3} m_{it}^o) = \frac{1}{3}\varepsilon + \frac{2}{3}[m_{1t}^{o\prime} + \sum_{i=2}^{3} m_{it}^o]. \quad (16)
\end{aligned}$$

Following this (15) and (16),

$$P_{HL}^2 = \Pr[m_{1t}^{o\prime} + \sum_{i=2}^{3} m_{it}^o < \frac{3}{2}c_2 - 3m^* - 2\varepsilon] = \Pr[\sum_{i=1}^{3} m_{it}^o < \frac{3}{2}c_2 - 3m^* - 3\varepsilon].$$

Thus $P_{HL}^1 > P_{HL}^2$. Similarly,

$$\begin{aligned}
P_{ML}^2 &= \Pr[m_{1t}^{o\prime} + m_{2t}^o < c_2 - 2m^* - \varepsilon] = \Pr[\sum_{i=1}^{2} m_{it}^o < c_2 - 2m^* - 2\varepsilon] < P_{ML}^1; \\
P_{LL}^2 &= \Pr[m_{1t}^{o\prime} < c_2 - 3m^* - \varepsilon] = \Pr[m_{1t}^o < c_2 - 3m^* - 2\varepsilon] < P_{LL}^1.
\end{aligned}$$

This proves part (i).

Getting rid of π_M^i, the steady state equations of $\pi P = \pi$ can be rewritten as

$$\begin{aligned}
(1 + P_{ML} - P_{LL})\pi_L + (P_{ML} - P_{HL})\pi_H - P_{ML} &= 0; \quad (17) \\
(P_{MH} - P_{LH})\pi_L + (1 + P_{MH} - P_{HH})\pi_H - P_{MH} &= 0. \quad (18)
\end{aligned}$$

By (17) and (18), we get

$$\begin{aligned}
\pi_L &= \frac{P_{ML}(1 - P_{HH}) + P_{MH}P_{HL}}{(1 + P_{MH} - P_{HH})(1 - P_{LL} + P_{ML}) + (P_{MH} - P_{LH})(P_{HL} - P_{ML})}; \quad (19) \\
\pi_H &= \frac{P_{MH}(1 - P_{LL}) + P_{ML}P_{LH}}{(1 + P_{MH} - P_{HH})(1 - P_{LL} + P_{ML}) + (P_{MH} - P_{LH})(P_{HL} - P_{ML})}. \quad (20)
\end{aligned}$$

For the steady state distribution to be meaningful, the following conditions have to be satisfied: $\pi_L \in (0,1)$, $\pi_H \in (0,1)$, and $\pi_L + \pi_H \in (0,1)$. Those conditions boil down to the following conditions:

$$1 - P_{LL} > |P_{ML} - P_{HL}|;\ 1 - P_{HH} > |P_{MH} - P_{LH}|. \tag{21}$$

Taking derivative of π_L with respect to P_{ML},

$$\frac{\partial \pi_L}{\partial P_{ML}} = \frac{(1 + P_{MH} - P_{HH})[P_{HM}P_{LH} + P_{HL}P_{LM} + P_{HM}P_{LM}]}{[(1 + P_{MH} - P_{HH})(1 - P_{LL} + P_{ML}) + (P_{MH} - P_{LH})(P_{HL} - P_{ML})]^2} > 0.$$

Similarly, one can show that for $i = H, M, L$,

$$\frac{\partial \pi_L}{\partial P_{iL}} > 0 \text{ and } \frac{\partial \pi_H}{\partial P_{iH}} > 0.$$

Thus, π_L is strictly increasing in P_{iL} and π_H is strictly increasing in P_{iH}.

By (19) and (20), we compute

$$\frac{\partial(\pi_H - \pi_L)}{\partial P_{LL}} = \frac{-[P_{ML}(1 - P_{HH}) + P_{MH}P_{HL}][(1 + P_{MH} - P_{HH}) + (P_{MH} - P_{LH})]}{[(1 + P_{MH} - P_{HH})(1 - P_{LL} + P_{ML}) + (P_{MH} - P_{LH})(P_{HL} - P_{ML})]^2} < 0.$$

The inequality comes from condition (21). Similarly, we can show that for $i = L, M, H$,

$$\frac{\partial(\pi_H - \pi_L)}{\partial P_{iL}} < 0;\ \frac{\partial(\pi_H - \pi_L)}{\partial P_{iH}} > 0.$$

Now combining with part (i): $P^1_{iH} < P^2_{iH}$, and $P^1_{iL} > P^2_{iL}$, $i = L, M, H$, we get the desired result: $\pi^2_H - \pi^2_L > \pi^1_H - \pi^1_L$. This shows part (ii).

By (8), part (iii) $\widetilde{y}^2 > \widetilde{y}^1$ is immediate from part (ii). ■

Proof of Proposition 4.

Proof. Following similar procedures in the proof of Proposition 3, we can derive the equilibrium transition probabilities under system 3.

$$\begin{aligned}
P^3_{HH} &= \Pr[m^o_{1t} + m^o_{2t} + m^o_{3t} \geq c_3 - 3m^* - 2\varepsilon] = P^2_{HH} > P^1_{HH}; \\
P^3_{MH} &= \Pr[m^o_{1t} + m^o_{2t} \geq c_3 - 4m^* - 2\varepsilon] = P^2_{MH} > P^1_{MH}; \\
P^3_{LH} &= \Pr[m^o_{1t} \geq c_3 - 5m^* - 2\varepsilon] = P^2_{LH} > P^1_{LH}; \\
P^3_{HL} &= \Pr[m^o_{1t} + m^o_{2t} + m^o_{3t} < \frac{3}{2}c_2 - 3m^*] = P^1_{HL} > P^2_{HL}; \\
P^3_{ML} &= \Pr[m^o_{1t} + m^o_{2t} < c_2 - 2m^*] = P^1_{ML} > P^2_{ML}; \\
P^3_{LL} &= \Pr[m^o_{1t} < c_2 - 3m^*] = P^1_{LL} > P^2_{LL}.
\end{aligned}$$

This proves part (i). The proofs for part (ii) and (iii) are similar to those of Proposition 3, thus are omitted. ■

Proof of Proposition 6.

Proof. We first show $\pi_H < \pi_H^B$. In the benchmark case,

$$\pi_H^B = \Pr[m_{1t}^o + m_{2t}^o + m_{3t}^o \geq c_3 - 1.5]$$

Note that

$$P_{HH} = \Pr[m_{1t}^o + m_{2t}^o + m_{3t}^o \geq c_3 - 1.5] = \pi_H^B$$

Now we compute the equilibrium transition probabilities.

$$\begin{aligned} P_{HH} &= 1 - [\frac{1}{6}(c_3 - 1.5)^3 - \frac{1}{2}(c_3 - 2.5)^3 + \frac{1}{2}[\max\{0, (c_3 - 3.5)\}]^3], && (22) \\ P_{MH} &= \Pr[m_{1t}^o + m_{2t}^o \geq c_3 - 2] = \frac{(4 - c_3)^2}{2}, && (23) \\ P_{LH} &= \Pr[m_{1t}^o \geq c_3 - 2.5] = \max\{0, 3.5 - c_3\}. && (24) \end{aligned}$$

Our goal is to show that $P_{LH} < P_{MH} < P_{HH}$.

First consider the case $3 < c_3 \leq 3.5$. By (23) and (24), we have $P_{MH} - P_{LH} > 0$. By (22) and (23), we see that when $c = 3$, $P_{HH} = P_{MH}$. Moreover,

$$\frac{d(P_{HH} - P_{MH})}{dc_3} = (c_3 - 3.5)^2 \geq 0.$$

Thus $P_{HH} > P_{MH} > P_{LH}$.

Next consider the case $3.5 < c_3 < 4$. By (23) and (24), $P_{MH} > P_{LH}$ obviously holds. By (22) and (23), we see that when $c = 4$, $P_{HH} > P_{MH} = 0$. Moreover,

$$\frac{d(P_{HH} - P_{MH})}{dc_3} = -\frac{1}{2}(c_3 - 3.5)^2 < 0.$$

Hence, $P_{HH} > P_{MH} > P_{LH}$.

Combining the above results, we see that for $3 < c_3 < 4$,

$$\pi_H^B = P_{HH} > P_{MH} > P_{LH}. \qquad (25)$$

From the steady state equations and (25), we have

$$\pi_H = \pi_L P_{LH} + \pi_M P_{MH} + \pi_H P_{HH} < P_{HH}(\pi_L + \pi_M + \pi_H) = P_{HH} = \pi_H^B.$$

This yields the desired result.

Next we show $\pi_L > \pi_L^B$. Note that

$$\begin{aligned} \pi_L^B &= \Pr[(m_{1t}^o + m_{2t}^o + m_{3t}^o < c_3 - 1.5)\&(m_{1t}^o + m_{2t}^o < c_2 - 1)]; \\ P_{HL} &= \Pr[m_{1t}^o + m_{2t}^o + m_{3t}^o < \frac{3}{2}c_2 - 1.5]; \\ P_{ML} &= \Pr[(m_{1t}^o + m_{2t}^o < c_2 - 1]; \\ P_{LL} &= \Pr[m_{1t}^o < c_2 - 1.5]. \end{aligned}$$

Our goal is to show that $\pi_L^B < P_{iL}$, $i = 1, 2, 3$. Apparently, $P_{ML} > \pi_L^B$. Now it is sufficient to show that $P_{LL} \geq P_{ML}$ and $P_{HL} \geq P_{ML}$.

$$P_{LL} - P_{ML} = (c_2 - 1.5) - [1 - \frac{(3-c_2)^2}{2}] = \frac{1}{2}(c_2 - 2)^2 \geq 0.$$

When $3 < \frac{3}{2}c_2 \leq 3.5$,

$$\begin{aligned} P_{HL} - P_{ML} &= \frac{1}{6}(\frac{3}{2}c_2 - 1.5)^3 - \frac{1}{2}(\frac{3}{2}c_2 - 2.5)^3 - [1 - \frac{(3-c_2)^2}{2}] \\ &= \frac{1}{8}[-9c_2^3 + 58c_2^2 - 123c_2 + 86] \geq 0. \end{aligned}$$

Similarly, one can show that $P_{HL} \geq P_{ML}$ when $3.5 < \frac{3}{2}c_2 < \frac{9}{2}$.

By the fact that $\pi_L^B < P_{iL}$, $i = 1, 2, 3$, we have

$$\pi_L = \pi_L P_{LH} + \pi_M P_{MH} + \pi_H P_{HH} > \pi_L^B(\pi_L + \pi_M + \pi_H) = \pi_L^B.$$

■

Proof of Proposition 7.

Proof. Since $R_{t-1} = n$, $y_{t-1}^n = \sum_{i=1}^n m_{it}^o$. First, consider upward regime switches. $R_t = N$ if

$$y_{t-1}^n + E[\sum_{i=1}^N m_{it}^n + \sum_{i=n+1}^N m_{it}^o] \geq c_N \Leftrightarrow y_{t-1}^n \geq c_N - (2N - n)m^*.$$

$R_t = j$, $n \leq j < N$, if

$$\begin{aligned} & y_{t-1}^n + E[\sum_{i=1}^j m_{it}^n + \sum_{i=n+1}^j m_{it}^o] \geq c_j \text{ and } y_{t-1}^n + E[\sum_{i=1}^{j+1} m_{it}^n + \sum_{i=n+1}^{j+1} m_{it}^o] < c_{j+1} \\ \Leftrightarrow \quad & c_j - (2j - n)m^* \leq y_{t-1}^n < c_{j+1} - [2(j+1) - n]m^*. \end{aligned}$$

Next consider downward regime switches. $R_t = j$, $1 < j < n$, if

$$E[\sum_{i=1}^j m_{it}^o | y_{t-1}^n, R_{t-1} = n] + E[\sum_{i=1}^j m_{it}^n] \geq c_j; \text{ and} \tag{26}$$

$$E[\sum_{i=1}^k m_{it}^o | y_{t-1}^n, R_{t-1} = n] + E[\sum_{i=1}^k m_{it}^n] < c_k \text{ for any } k \text{ such that } j < k < n. \tag{27}$$

Applying Lemma 1, we have

$$E[\sum_{i=1}^j m_{it}^o | y_{t-1}^n, R_{t-1} = n] = \frac{j}{n} y_{t-1}^n.$$

Hence conditions (26) and (27) can be rewritten as

$$y_{t-1}^n \geq \frac{n}{j}c_j - nm^* \text{ and } y_{t-1}^n < \frac{n}{k}c_k - nm^* \text{ for any } k \text{ such that } j < k < n.$$

Similarly, $R_t = 1$ if $y_{t-1}^n < \frac{n}{k}c_k - nm^*$ for any k such that $1 < k < n$. ■

Proof of Proposition 8.

Proof. Part (i). By Proposition **7,** $j' = R_t > R_{t-1} = j$ implies that $c_{j'} - (2j' - j)m^* \leq y_{t-1}^n < c_{j'+1} - [2(j'+1) - j]m^*$. Again by Proposition **7,** $R_{t+1} > R_t = j'$ implies that $c_{j'+1} - [2(j'+1) - j']m^* \leq y_t^n$. Combining these two conditions, $R_{t+1} > R_t = j'$ implies that

$$\begin{aligned} \frac{y_t^n - y_{t-1}^n}{y_{t-1}^n} &> \frac{\{c_{j'+1} - [2(j'+1) - j')]m^*\} - \{c_{j'+1} - [2(j'+1) - j]m^*\}}{c_{j'+1} - [2(j'+1) - j]m^*} \\ &= \frac{(j' - j)m^*}{c_{j'+1} - [2(j'+1) - j]m^*} \equiv \widehat{g}^{+}(j, j') > 0. \end{aligned}$$

Part (ii). By Proposition **7,** $j' = R_t < R_{t-1} = j$ implies that $y_{t-1}^n \geq \frac{j}{j'}c_{j'} - jm^*$ and $y_{t-1}^n < \frac{j}{k}c_k - jm^*$ for any k such that $j' < k < j$. Again by Proposition **7,** $R_{t+1} < R_t = j'$ implies that $y_t^n < c_{j'} - (2j' - j')m^*$. Combining these two conditions, $R_{t+1} < R_t = j'$ implies that

$$\frac{y_t^n - y_{t-1}^n}{y_{t-1}^n} < \frac{(c_{j'} - j'm^*) - (\frac{j}{j'}c_{j'} - jm^*)}{\frac{j}{j'}c_{j'} - jm^*} = \frac{(j' - j)}{j} \equiv \widehat{g}^{-}(j, j') < 0.$$

■