Investment Cycles, Strategic Delay, and Self-Reversing Cascades
Web Appendix

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1. The Issue of Multiple Equilibria

Characterization of multiple equilibria

In general, the WG will have multiple equilibria, as in Chamley (2004). There is now the possibility of a new regime in which all type-1 agents invest in round 1 and type-0 agents mix in round 1. We call this Regime 2M, which is a sort of "reverse" to Regime M. Let p be the probability that a type-0 agent invests in round 1, and $V_0(\mu_0, p)$ be the expected payoff of a type-0 agent who waits in round 1. Similar to the proof of Lemma (3.1), it can be shown that $V_0(\mu_0, p)$ is decreasing in p, and the advantage of investing right away to waiting in round 1, $\mu_0 - c - V_0(\mu_0, p)$ is increasing in μ_0. This is because as type-0 agents invest with a higher probability (type-1 agents invest for sure) in round 1, less information is revealed by round 1 investment, since the two types of agents' behavior become closer to each other. An equilibrium of Regime 2M is characterized by a p such that $\mu_0 - c - V_0(\mu_0, p) = 0$.

A complete characterization of equilibria in the WG is as follows. (1) If $\mu < \underline{\mu}^W$, there is a unique equilibrium of Regime 0. (2) If $\underline{\mu}^W \leq \mu < \min\{\widehat{\mu}^W, \overline{\mu}^{NW}\}$, there is a unique equilibrium of Regime M. (3) If $\mu > \overline{\mu}^W$, there is a unique equilibrium of Regime 2. Suppose $\overline{\mu}^{NW} > \widehat{\mu}^W$, then we have two more cases. (4) If $\widehat{\mu}^W < \mu < \overline{\mu}^{NW}$, there is a unique equilibrium of Regime 1. To see this, note that $\mu < \overline{\mu}^{NW}$ implies that type-0 agents have no incentive to invest in round 1, thus Regime 2 and 2M do not exist. On the other hand, $\widehat{\mu}^W < \mu$ ensures that all type-1 agents invest in round 1, so Regime 1 exists. (5) If $\overline{\mu}^{NW} < \mu < \overline{\mu}^W$, then there are three equilibria: Regime 1, Regime 2M, and Regime 2. Regime 2 exists because if all other agents invest in round 1, then nothing is learned by waiting and it is optimal for an agent to invest in round 1. To see that Regime 2M exists, note that by $\overline{\mu}^{NW} < \mu$ we have $\mu_0 - c - V_0(\mu_0, 1) = (1-\delta)(\mu_0 - c) > 0$. By $\mu < \overline{\mu}^W$, we have $\mu_0 - c - V_0(\mu_0, 0) < 0$. Since $V_0(\mu_0, p)$ is decreasing in p, we must have a unique $p \in (0, 1)$ such that $\mu_0 - c - V_0(\mu_0, p) = 0$. Suppose $\overline{\mu}^{NW} < \widehat{\mu}^W$, then case (4) and (5) will change. (4') If $\overline{\mu}^{NW} < \mu < \widehat{\mu}^W$, there are three equilibria: Regime M, Regime 2M, and Regime 2. (5') If $\widehat{\mu}^W < \mu < \overline{\mu}^W$, there are three equilibria: Regime 1, Regime 2M and Regime 2. The reason for the multiplicity of equilibria is that if type-0 agents invest with a higher probability in round 1, then less information is revealed in round 1 and agents have less incentive to wait.

We now discuss the implications of the equilibrium refinement, payoff dominance. In case (5) Regime 1 is payoff dominant. In case (4') Regime M is payoff dominant and in case (5') Regime 1 is payoff dominant. Since the arguments are similar for all the cases, here we just demonstrate it for case (4'). First compare Regime M and Regime 2. Type-1 agents get the same payoff in both Regimes, since in Regime M type-1 agents are indifferent between waiting and investing in round 1. But type-0 agents get a higher payoff in Regime M. This is because they have no incentive to invest in round 1 in regime M, which means that waiting yields a higher payoff than investing right away. Now compare Regime M and Regime 2M. Type-1 agents get the same payoff in both Regimes. However, type-0 agents get a higher payoff in Regime M. The reason is that they have no incentive to invest in round 1 in Regime M, implying waiting yields a higher payoff than investing right away, while in regime 2M, they are indifferent between waiting and investing.

Thus, imposing payoff dominance, the equilibrium is characterized by Proposition 3.2.[1]

Long-run dynamics in the large, persistent economy in the "maximal-coordination-failure" equilibrium

Suppose we select Regime 2 whenever it coexists with other equilibria. Then based on this criterion, like Proposition (3.2), the equilibrium can be characterized as follows: (i) Regime 0 if $\mu \in [0, \underline{\mu}^{NW}]$; (ii) Regime 2 if $\mu \in [\overline{\mu}^{NW}, 1]$; (iii) Regime M if $\mu \in (\underline{\mu}^{NW}, \min\{\overline{\mu}^{NW}, \widehat{\mu}^{W}\})]$; (iv) Regime 1 if $\mu \in (\min\{\overline{\mu}^{NW}, \widehat{\mu}^{W}), \overline{\mu}^{NW}]$. Note that Regime 1 might not exist if $\overline{\mu}^{NW} \leq \widehat{\mu}^{W}$.

As we can see, the range of beliefs corresponding to Regime 0 in the WG is still the same as in the NWG. But now the range of beliefs corresponding to Regime 2 in the WG is the same as in the NWG. For sufficiently high persistence, when a Regime 0 cascade ends, the economy moves into Regime M with type-1 agents investing with a *low* probability (beliefs are just slightly higher than $\underline{\mu}^{NW}$). When a Regime 2 cascade ends, there are two possible cases. (Case 1: $\overline{\mu}^{NW} \geq \widehat{\mu}^{W}$) If agents are not too patient, the economy moves into Regime 1, and the transition from booms is exactly the same as in the NWG, so we have longer recessions and booms of the same length as in the NWG. As a result, the average length of recessions is longer in the WG than in the NWG, and the economy spends a greater fraction of time in recessions in the WG than in the NWG. This shows that the results in section 4.3 regarding the comparison of the long-run dynamics across two games are robust. (Case 2: $\overline{\mu}^{NW} \leq \widehat{\mu}^{W}$) If agents are sufficiently patient, the economy moves into Regime M, which could possibly delay the end of a boom. However, it is our strong conjecture that the economy spends a greater fraction of time in recessions in the WG than in the NWG, because the rate of information flow when we transition from Regime 2 to Regime M is significant, while the rate of information flow when we transition from Regime 0 to Regime M is approximately zero.

2. More Detailed Analysis of the Large, Persistent Economy (Section 4)

All of our simulations indicate that investment cycles have longer recessions and shorter booms in the WG than in the NWG. While we conjecture that this is a general result, the nonstationary nature of the dynamics makes proving this result impossible (at least, for us). However, in this section we derive analytical results for the important case of the *large, persistent*

[1] The Regime 2M equilibria can also be eliminated as not being stable. Consider perturbations around type-0 agents' equilibrium investment probability p^*. If p is slightly higher than p^*, then $\mu_0 - c - V_0(\mu_0, p) > 0$ since $\mu_0 - c - V_0(\mu_0, p^*) = 0$ and $V_0(\mu_0, p)$ is decreasing in p. Thus each type-0 agent has incentive to increase p further, which makes Regime 2M instable. Similarly, one can show that if $p < p^*$, then each type-0 agent has incentive to further decrease p. On the other hand, the Regime M equilibrium is stable, which can be shown by similar arguments.

economy. We consider the limiting case in which $n \to \infty$, $\rho \to 1$, and the order of limits is such that $n \to \infty$ for any value of ρ. We will argue that this allows us to characterize the equilibria of the NWG and the WG as finite-state Markov processes, from which we can study the long run probabilities of boom and recession. Intuitively, large n allows us to use the law of large numbers, thus getting rid of the randomness of beliefs in Regime 1 and Regime M. Meanwhile, large n also leads to nice cutoffs in the WG as shown by Lemma (3.5). Large ρ makes belief transitions across periods (in either Regime 0 or Regime 2) smoothly and guarantees that Regime 1 and Regime M occur when beliefs first cross the boundaries.

The Large, Persistent NWG The economy can be represented by $r^{NW} + 2 + b^{NW}$ Markov states. Regime 0 has r^{NW} states (state 1 to state r^{NW}). Regime 1 has two states: state $r^{NW} + 1$ (corresponding to a low investment return) and state $r^{NW} + 2$ (corresponding to a high investment return). Regime 2 has b^{NW} states (state $r^{NW} + 3$ to state $r^{NW} + 2 + b^{NW}$).

Since $n \to \infty$, by the law of large numbers, in Regime 1 the investment return is always fully revealed. Therefore, if the economy is in Markov state $r^{NW} + 1$, in the following period the economy will be in state 1 (corresponding to the beginning of Regime 0) for sure, with initial belief $1 - \rho$. Similarly, state $r^{NW} + 2$ transitions to state $r^{NW} + 3$ (corresponding to the beginning of Regime 2) for sure, with inital belief ρ. From state 1, the economy transitions through r^{NW} states of Regime 0 deterministically, until state r^{NW} is reached. State r^{NW} transitions to Regime 1 with probability one. Specifically, it transitions to state $r^{NW} + 1$ with probability $1 - \underline{p}^{NW}$, which is the probability of the investment return being low after r^{NW} periods, given that it was low initially; and it transitions to state $r^{NW} + 2$ with probability $\underline{p}^{NW}$. Similarly, once the economy reaches state $r^{NW} + 3$, it transitions through b^{NW} states of Regime 2 deterministically, until state $r^{NW} + 2 + b^{NW}$ is reached. State $r^{NW} + 2 + b^{NW}$ transitions to either state $r^{NW} + 1$ or state $r^{NW} + 2$. The probability of transitioning to state $r^{NW} + 2$, denoted by $\overline{p}^{NW}$, is the probability of the investment return being high after b^{NW} periods, given that it was high initially.

Note that the stationarity of the processes is guaranteed by the law of large numbers, as the investment return is fully revealed in regime 1. We summarize the state transitions in the NWG in the following transition probability matrix P^{NW} (from top to bottom rows and from left to right columns are state 1 to state $r^{NW} + 2 + b^{NW}$).

$$
P^{NW} = \begin{bmatrix}
 & 1 & & & & & & & & \\
 & & 1... & & & & & & & \\
 & & & 1 & & & & & & \\
 & & & & (1-\underline{p}^{NW}) & \underline{p}^{NW} & & & & \\
1 & & & & & & & & & \\
 & & & & & & 1 & & & \\
 & & & & & & & 1 & & \\
 & & & & & & & & 1... & \\
 & & & & & & & & & 1 \\
 & & & & (1-\overline{p}^{NW}) & \overline{p}^{NW} & & & &
\end{bmatrix}
$$

Since all of the Markov states are ergodic, we can calculate the stationary distribution of states π^{NW} (a row vector), which is defined as $\pi^{NW} P^{NW} = \pi^{NW}$. Using standard techniques, the fraction of time the economy spends in booms, π_B^{NW}, can be calculated as

$$\pi_B^{NW} = \frac{(b^{NW}+1)\underline{p}^{NW}}{(r^{NW}+1)(1-\overline{p}^{NW}) + (b^{NW}+1)\underline{p}^{NW}}, \tag{1}$$

with the probability of recession equal to $1 - \pi_B^{NW}$.

We can determine r^{NW} (b^{NW}) by calculating the number of periods that must pass, in order for the probability of the high investment return to first exceed $\underline{\mu}^{NW}$ (fall below $\overline{\mu}^{NW}$), given that the investment return was low (high) initially. Thus, we have[2]

$$r^{NW} = \frac{\log(1-2\underline{\mu}^{NW})}{\log(2\rho-1)}; \quad b^{NW} = \frac{\log(2\overline{\mu}^{NW}-1)}{\log(2\rho-1)}. \tag{2}$$

Equations (2) indicate that the number of periods in Regime 0 and Regime 2 grow without bound as $\rho \to 1$, but the ratio converges to a well defined limit. We can compute the limiting boom probability by noting that $\lim_{\rho\to1}(\underline{p}^{NW}) = \underline{\mu}^{NW}$ and $\lim_{\rho\to1}(\overline{p}^{NW}) = \overline{\mu}^{NW}$, yielding[3]

$$\pi_B^{NW} = \frac{1}{1 + \frac{\log(1-2\underline{\mu}^{NW})}{\log(2\overline{\mu}^{NW}-1)}\left[\frac{1-\overline{\mu}^{NW}}{\underline{\mu}^{NW}}\right]}. \tag{3}$$

The Large, Persistent WG For the WG of the large, persistent economy, we want to show that the dynamics are approximated by a first-order Markov process. Specifically, there are $r^W + 4 + b^W$ Markov states. States 1 to r^W correspond to Regime 0. Regime M has two states: state $r^W + 1$ (corresponding to a low investment return) and state $r^W + 2$ (corresponding to a high investment return). Regime 1 has two states: state $r^W + 3$ (corresponding to a low investment return) and state $r^W + 4$ (corresponding to a high investment return). States $r^W + 5$ to $r^W + 4 + b^W$ correspond to Regime 2.

The state transitions are as follows. By the law of large numbers, state $r^W + 3$ transitions to state 1 for sure, and state $r^W + 4$ transitions to state $r^W + 5$ for sure. From state 1, the economy goes through the r^W states of Regime 0. State r^W transitions to Regime M: to state $r^W + 1$ with probability $(1 - \underline{p}^W)$ and to state $r^W + 2$ with probability $\underline{p}^W$. From state $r^W + 5$, the economy goes through the b^W states of Regime 2. State $r^W + 4 + b^W$ transitions to Regime 1: to state $r^W + 3$ with probability $(1 - \overline{p}^W)$ and to state $r^W + 4$ with probability $\overline{p}^W$.

Now we specify transitions from Regime M. From state $r^W + 1$, with probability λ_0 the investment return is revealed to be low and the economy transitions to state 1; with probability $(1-\lambda_0)\rho$ it remains in state $r^W + 1$ next period, and with probability $(1-\lambda_0)(1-\rho)$ it switches to state $r^W + 2$. Similarly, from state $r^W + 2$, with probability λ_1 the investment return is revealed to be high and the economy transitions to state $r^W + 5$; with probability $(1-\lambda_1)(1-\rho)$ it switches to state $r^W + 1$ next period, and with probability $(1-\lambda_1)\rho$ it remains in state $r^W + 2$.

[2] Of course, the expressions (2) are not generally integers, so r^{NW} and b^{NW} are actually the smallest integers greater than or equal to the corresponding expressions.

[3] Clearly, if $c = 1/2$, then $\underline{\mu}^{NW} = 1 - \overline{\mu}^{NW}$ and $b^{NW} = r^{NW}$ holds. By (3), we have $\pi_B^{NW} = \pi_R^{NW} = 1/2$. As we will see, this symmetry of investment cycles for the symmetric model with $c = 1/2$ does not carry over to the WG.

The Markov transition matrix P^W is summarized below (from top to bottom rows and from left to right columns are state 1 to state $r^W + 4 + b^W$):

$$P^W = \left[\begin{array}{cccccccccccc}
 & 1 & & & & & & & & & & \\
 & & 1\ldots & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
 & & & & (1-\underline{p}^W) & \underline{p}^W & & & & & & \\
\lambda_0 & & & & (1-\lambda_0)\rho & (1-\lambda_0)(1-\rho) & & & & & & \\
 & & & & (1-\lambda_1)(1-\rho) & (1-\lambda_1)\rho & & & \lambda_1 & & & \\
1 & & & & & & & & & & & \\
 & & & & & & & & 1 & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & 1\ldots & \\
 & & & & & & & & & & & 1 \\
 & & & & & & (1-\overline{p}^W) & \overline{p}^W & & & &
\end{array}\right]$$

The law of large numbers guarantees that the investment return is revealed in Regime 1, or if the economy is in Regime M and k_1^t induces all remaining type-1 agents and no type-0 agents to invest in round 2. For ρ sufficiently close to one, it is clear that: when the investment return is revealed to be high (low), then the following period is in Regime 2 (0); during the first period that the economy leaves Regime 2, it is in Regime 1;[4] and during the first period that the economy leaves Regime 0, it is in Regime M. To finish our justification that P^W is an accurate approximation to the transition dynamics, we will show that for ρ sufficiently close to one, (i) in Regime M, if one agent invests in round 1, $k_1^t = 1$, then the remaining type-1 agents and none of the type-0 agents will invest in round 2, thereby revealing the investment return, (ii) in Regime M, the probability of more than one agent investing in round 1, $k_1^t > 1$, is negligible, even as compared to the already small probability that exactly one agent invests, and (iii) in Regime M, the beginning of period beliefs are approximately constant, independent of the history.

It will be useful to define the following notation for the probabilities of aggregate round-1 investment in Regime M, given beginning of period beliefs, μ, and the investment return:

$$prk0(\mu) \equiv pr(k_1^t = k | S^t = 0) \quad \text{and} \quad prk1(\mu) \equiv pr(k_1^t = k | S^t = 1).$$

Also, based on the investment probability q (which implicitly depends on μ), we define $Q(\mu)$ to be the ratio of the probability that no one invests when the investment return is low, to the probability that no one invests when the investment return is high:

$$Q(\mu) \equiv \left(\frac{1-(1-\alpha)q}{1-\alpha q}\right)^n. \tag{4}$$

Lemma 4.1: *For the large, persistent WG in Regime M, let the beginning of period belief be given by $\mu(h^{t-1}) = \mu < 1$. Then in the limit, as $n \to \infty$, round-1 investment probabilities are characterized as follows.*

[4]Regime 2 and Regime 1 always exists (not bypassed) for fixed $\delta < 1$ and $\rho \to 1$. If $\delta \to 1$ as well, then to ensure Regime 2 and Regime 1 exist the limiting ratio $(1-\delta)/(1-\rho)$ must be sufficiently large.

$$
\begin{aligned}
pr00(\mu) &= Q(\mu)^{-(1-\alpha)/(2\alpha-1)} \\
pr10(\mu) &= \left(\frac{1-\alpha}{2\alpha-1}\right) Q(\mu)^{-(1-\alpha)/(2\alpha-1)} \log(Q(\mu)) \\
pr01(\mu) &= Q(\mu)^{-\alpha/(2\alpha-1)} \\
pr11(\mu) &= \left(\frac{\alpha}{2\alpha-1}\right) Q(\mu)^{-\alpha/(2\alpha-1)} \log(Q(\mu))
\end{aligned}
$$

Also, if $n \to \infty$ and $\mu \to \underline{\mu}^W$, then $Q(\mu)$, $pr00(\mu)$, and $pr01(\mu)$ converge to one. Although the probability of $k_1^t = 1$ is converging to zero, the probability of $k_1^t > 1$, relative to the probability of $k_1^t = 1$, also converges to zero.

If beliefs in Regime M are close to $\underline{\mu}^W$, Lemma (4.1) allows us to ignore the possibility of more than one agent investing in round 1. Also, if $k_1^t = 1$ holds, then from (**??**) and Lemma (4.1), a type-1 agent finds it profitable to invest and a type-0 agent does not; thus, only the remaining type-1 agents invest in round 2, thereby revealing the investment return. Lemma (4.2) below completes our justification that P^W is an accurate approximation to the transition dynamics.

Lemma 4.2: *Consider the limiting large, persistent WG as $n \to \infty$, for fixed ρ close to one. Also assume that $\frac{1}{2} < \delta < 1$ holds. Then for all histories such that $k_1^t = 0$ in Regime M, in the next period $\mu(h^t)$ will be in Regime M, in the interval, $[\underline{\mu}^W, \rho\underline{\mu}^W + (1-\rho)(1-\underline{\mu}^W)]$. Furthermore, the beginning of period beliefs converge to a constant, μ^{fix}, following a sequence of periods of $k_1^t = 0$ in Regime M.*

Standard techniques allow us to compute the steady state distribution of Markov states, π^W, which is defined as $\pi^W P^W = \pi^W$. To simplify notation, we denote the probability of one of the Regime 0 (2) states as π_0^W (π_2^W), the probability of Regime M, low (high) investment return as π_{M0}^W (π_{M1}^W), and the probability of Regime 1, low (high) investment return as π_{10}^W (π_{11}^W). After much manipulation, we can solve the following equations for π_{M1}^W and π_{M0}^W:

$$1 = \pi_{M1}^W\left(\frac{b^W+1}{1-\overline{p}^W}\lambda_1 + 1 + \lambda_1 r^W\right) + \pi_{M0}^W(1+\lambda_0 r^W), \tag{5}$$

$$\pi_{M1}^W = \frac{1-\rho+\lambda_0[\rho-(1-\underline{p}^W)]}{1-\rho+\lambda_1(\rho-\underline{p}^W)}\pi_{M0}^W. \tag{6}$$

Next, the steady-state probability of being in one of the Markov states corresponding to a boom can be written as

$$\pi_B^W = \frac{b^W+1}{1-\overline{p}^W}\lambda_1\pi_{M1}^W. \tag{7}$$

Finally, we take limit as $\rho \to 1$, yielding

$$\underline{p}^W = \underline{\mu}^W = \frac{1}{1+(\frac{\alpha}{1-\alpha})(\frac{1-c}{c})}, \quad \overline{p}^W = \overline{\mu}^W = \frac{1}{1+(\frac{1-\alpha}{\alpha})(\frac{1-c}{c})(1-\delta)}, \quad \frac{b^W}{r^W} = \frac{\log(2\,\overline{\mu}^W-1)}{\log(1-2\underline{\mu}^W)},$$

and $\frac{\lambda_0}{\lambda_1}$, $\frac{1-\rho}{\lambda_1}$, and $\lambda_0 r^W$ are computed in the proof of Proposition (4.3) below. This allows us to compute the limiting boom probability, and the recession probability $1-\pi_B^W$.[5]

In Chamley and Gale (1994), when the number of agents is large and agents are extremely patient, underinvestment is the only source of inefficiency; there is no overinvestment, in the sense that when the investment return is low, the probability that an agent ever invests is zero. This finding seems to contradict our finding for the large, persistent economy, that the economy spends a positive fraction of periods in Regime 2 with all agents investing. In fact, our analysis shows that the Chamley-Gale no-overinvestment result is robust to our setting. In the large, persistent economy as $\delta \to 1$, the length of a single Regime 0 cascade approaches infinity, but the length of a single Regime 2 cascade is bounded.[6] Thus, booms are characterized by a sequence of many Regime 2 cascades, which repeat themselves as long as the investment return remains high. Overinvestment during the entire cycle only occurs during Regime M periods with a single round 1 investor, or (almost certainly) during at most one Regime 2 cascade, which together comprise a negligible fraction of the length of that cycle.

Comparing the Long Run Dynamics Across Games In this subsection, based on the Markov matrices P^{NW} and P^W, we compare the long-run dynamics of the NWG and the WG, for the large, persistent economy. We demonstrate that the expected length of a boom is shorter and the expected length of a recession is longer in the WG. The probability of being in a recession is greater in the WG than in the NWG. We also show that overinvestment (investing when the return is low) is more prevalent in the NWG, and underinvestment (not investing when the return is high) is more prevalent in the WG.

Let L_B^{NW} be the expected length of a boom for the NWG. The actual length of a boom is a random variable that can take one of the values, $b^{NW}+1$, $2(b^{NW}+1)$, $3(b^{NW}+1)$, and so on. The probability that a boom lasts for $k(b^{NW}+1)$ periods is $(1-\overline{p}^{NW})(\overline{p}^{NW})^{k-1}$. Thus, we have

$$L_B^{NW} = (b^{NW}+1)\sum_{k=1}^{\infty} k(1-\overline{p}^{NW})(\overline{p}^{NW})^{k-1} = (b^{NW}+1)\frac{1}{(1-\overline{p}^{NW})} = (b^{NW}+1)\frac{2}{1-(2\rho-1)^{b^{NW}+1}} \tag{8}$$

Similarly, the expected length of recessions L_R^{NW} is

$$L_R^{NW} = (r^{NW}+1)\frac{1}{\underline{p}^{NW}} = (r^{NW}+1)\frac{2}{1-(2\rho-1)^{r^{NW}+1}}. \tag{9}$$

Although the expected lengths of booms and recessions grow without bound as $\rho \to 1$, the ratios have well defined limits.

Similarly, the expected length of booms in the WG, denoted by L_B^W, is given by

$$L_B^W = (b^W+1)\sum_{k=1}^{\infty} k(1-\overline{p}^W)(\overline{p}^W)^{k-1} = \frac{b^W+1}{1-\overline{p}^W}. \tag{10}$$

[5] For the large, patient, persistent economy ($\delta \to 1$) the limiting probability of boom is $\pi_B^W = \frac{2c(\alpha-c)}{6c\alpha-2c-4c^2\alpha-(\alpha-c)\log(\frac{\alpha-c}{c+\alpha-2c\alpha})}$.

[6] The limiting ratio $(1-\delta)/(1-\rho)$ must be sufficiently large to ensure that there is a Regime 2.

Denote the expected length of recessions in the WG by L_R^W. It will be convenient to keep track of the expected length of recessions starting from Regime M when the investment return is high, which we denote by ℓ_1, and starting from Regime M when the investment return is low, which we denote by ℓ_0. From P^W, we have the following equations:

$$\begin{aligned} L_R^W &= (r^W + 1) + \underline{p}^W \ell_1 + (1 - \underline{p}^W)\ell_0 && (11) \\ \ell_1 &= (1 - \lambda_1)[1 + \rho\ell_1 + (1 - \rho)\ell_0] && (12) \\ \ell_0 &= (1 - \lambda_0)[1 + \rho\ell_0 + (1 - \rho)\ell_1] + \lambda_0 L_R^W && (13) \end{aligned}$$

Solving the above equations simultaneously, we can compute L_R^W.

Proposition 4.3: *In the large, persistent economy, the expected length of a boom is shorter, and the expected length of a recession is longer, in the WG than in the NWG. That is,* $L_B^W < L_B^{NW}$ *and* $L_R^W > L_R^{NW}$.

The average length of a boom is shorter in the WG than in the NWG, because Regime 2 cascades are shorter with the possibility of waiting. The shorter Regime 2 cascades reduce the chance that the investment return switches from high to low without being detected. The average length of a recession is longer in the WG, because of the presence of Regime M. Suppose the investment return switches to high during a Regime 0 cascade. In the NWG, the economy moves to Regime 1, and the high investment return is revealed, ending the recession. However, in the WG, the economy moves to Regime M, and is likely to stay there for many periods. This directly prolongs the recession, and also allows for the possibility that the investment return switches back to low before the high return is detected.

Next we consider long-run probabilities of boom and recession.

Proposition 4.4: *In the large, persistent economy, the long-run probability of being in a recession in the WG is greater than in the NWG,* $\pi_R^W > \pi_R^{NW}$ *and* $\pi_B^W < \pi_B^{NW}$.

The intuition for Proposition (4.4) is the following. The economy is oscillating between boom and recession. Given that the average length of a boom is shorter and the average length of a recession is longer in the WG than in the NWG (Proposition (4.3)), the economy must spend relatively more time in a recession in the WG.

Finally, we show that the possibility of waiting reduces expected overinvestment and increases expected underinvestment. Let O (U) be the overinvestment (underinvestment) index, measuring the average investment (lack of investment) when the return is low (high). More specifically,

$$O = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}[\frac{I^t}{n}|S^t = 0]; \;\; U = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}[\frac{n - I^t}{n}|S^t = 1].$$

For the large, persistent NWG, O^{NW} and U^{NW} can be expressed as

$$\begin{aligned} O^{NW} &= \{\frac{b^{NW}}{2} - \frac{2\rho - 1}{4(1-\rho)}[1 - (2\rho - 1)^{b^{NW}}]\}\pi_2 + (1-\alpha)\pi_0, && (14) \\ U^{NW} &= \{\frac{r^{NW}}{2} - \frac{2\rho - 1}{4(1-\rho)}[1 - (2\rho - 1)^{r^{NW}}]\}\pi_0 + (1-\alpha)\pi_{11}. && (15) \end{aligned}$$

To understand (14), note that the term in braces is the expected number of periods that the investment return is low during b^{NW} consecutive periods of Regime 2. The second term is the probability that the investment return is low during Regime 1, multiplied by the fraction of agents that invest.

For the large and persistent WG, O^W and U^W can be computed as:

$$O^W = (1-\alpha)(\lambda_1\pi^W_{M1} + \lambda_0\pi^W_{M0}) + \frac{\lambda_1\pi^W_{M1}}{1-\overline{p}^W}\{\frac{b^W}{2} - \frac{2\rho-1}{4(1-\rho)}[1-(2\rho-1)^{b^W}]\} \quad (16)$$

$$U^W = \pi^W_{M1}[(1-\lambda_1)+\lambda_1(1-\alpha)] + \pi^W_0\{\frac{r^W}{2} - \frac{2\rho-1}{4(1-\rho)}[1-(2\rho-1)^{r^W}]\} \quad (17)$$

Proposition 4.5: *In the large, persistent economy, (i) the overinvestment index is higher for the NWG than for the WG, $O^{NW} > O^W$, and (ii) the underinvestment index is higher for the WG than for the NWG, $U^{NW} < U^W$.*

The reason for part (i) of Proposition (4.5) is that Regime 2 cascades are longer in the NWG than in the WG. Longer Regime 2 cascades on average lead to higher probabilities of overinvestment, since it becomes more likely that the investment return has switched from high to low during a Regime 2 cascade. The result of part (ii) is due to the presence of Regime M in the WG. The presence of Regime M increases the probability that no agent invests, even though the investment return has switched from low to high.

Proposition 4.6: *Consider the large, persistent WG with $\delta' > \delta''$. Then we have (i) $L^W_B(\delta') < L^W_B(\delta'')$ and $L^W_R(\delta') = L^W_R(\delta'')$; (ii) $\pi^W_R(\delta') > \pi^W_R(\delta'')$; (iii) $O^W(\delta') < O^W(\delta'')$ and $U^W(\delta') > U^W(\delta'')$.*

Propsoition (4.6) shows that as δ decreases, the long-run dynamics of the WG become closer to those of the NWG. The underlying intuition is that, as δ decreases, Regime 2 cascades become longer, because $\overline{\mu}^W$ decreases, which tends to increase the average length of booms. On the other hand, since ρ is very close to one, the transition probability from Regime M to Regime 2 does not depend on δ. This means that the average length of recessions remains the same as δ changes. Combining these two effects, as δ decreases the economy spends more and more time in booms rather than in recessions. As a result, the overinvestment probability increases and underinvestment probability decreases. If ρ is strictly less than 1, then a decrease in δ would lead to a higher investing probability for type-1 agents in Regime M, which in general shortens the average length of recessions.

Proposition (4.6) has some potentially testable implications. We can interpret a larger δ as a smaller cost of waiting. Then our model would predict that, as the cost of delaying investment decreases, booms will tend to be shorter and recessions will tend to be longer; that the economy will tend to spend less time in booms; and that the averge investment (or output) will decrease.

Our results are consistent with the empirical evidence provided by Van Nieuwerburgh and Veldkamp (2006), that analysts' forecasts of real GDP are both less accurate and more dispersed near business cycle troughs. Suppose that forecasters are the agents (or outsiders who observe a signal) of our large, persistent economy, and forecasts are the conditional beliefs of the investment return. Then measuring the inaccuracy of forecasts as the squared deviation between the forecast and the true investment return, the average inaccuracy is a function of α and the beginning of period belief μ, given by

$$\mu[\alpha(1-\mu_1)^2 + (1-\alpha)(1-\mu_0)^2] + (1-\mu)[\alpha{\mu_0}^2 + (1-\alpha){\mu_1}^2],$$

which is symmetric in μ with a peak at $\mu = 0.5$. Assume that Regime 0 cascades are longer than Regime 2 cascades, which occurs if the model parameters are symmetric (i.e., $c = 0.5$) or if agents are reasonably patient. Then forecasts are most accurate during Regime 2 (boom), somewhat less accurate during Regime 0 (recession), and far less accurate during Regime M (also recession). In terms of dispersion, type-0 and type-1 forecasters will have almost the same beliefs during Regime 2, somewhat more dispersed beliefs during Regime 0, and far more dispersed beliefs during Regime M.

The WG can also shed some light on the timing of goverment policy to pull the economy out of recession. Suppose the government only observes the history of aggregate investments. Our model predicts that an investment subsidy will be most effective when the economy transitions to Regime M, i.e., after the recession has lasted for some time. At that point, beliefs are reasonably optimistic, and the subsidy needed to induce type-1 agents to invest (with a high probability) is small. Moreover, it is reasonably likely that market activity will reveal the investment return to be high (in a large economy) and that the recession can be ended. On the other hand, if there is a subsidy at the beginning of Regime 0, agents are very pessimistic, which means that a large subsidy rate is required to induce type-1 agents to invest, and the investment return is very likely to be low. As a result, even if type-1 agents are induced to invest, the revealed information is very likely to be bad news, and thus the recession will continue.

Proof of Lemma 4.1. For a type-1 agent that does not invest in round 1 and observes $k_1^t = 0$, the probability of the high investment return, $\mu_1^{0,q}$, is

$$\mu_1^{0,q} = \frac{1}{1 + \frac{1-\mu}{\mu}(\frac{1-\alpha}{\alpha})(\frac{1-(1-\alpha)q}{1-\alpha q})^{n-1}} = \frac{1}{1 + \frac{1-\mu}{\mu}(\frac{1-\alpha}{\alpha})Q}. \tag{18}$$

The last equality holds because q must be near zero as n approaches infinity.[7] From (4), the investment probability q can be written as $q = \frac{Q^{\frac{1}{n}}-1}{\alpha Q^{\frac{1}{n}}-(1-\alpha)}$. The probability of $k_1^t = 0$ in the high investment return is given by

$$pr01 = (1-\alpha q)^n = [\frac{\alpha Q^{\frac{1}{n}} - (1-\alpha)}{2\alpha - 1}]^{-n}.$$

Taking the limit of $\log(pr01)$, as n approaches infinity, yields

$$\lim_{n\to\infty} \log(pr01) = -\lim_{n\to\infty} \frac{\log[\frac{\alpha Q^{\frac{1}{n}}-(1-\alpha)}{2\alpha-1}]}{1/n} = -\frac{\alpha}{2\alpha-1}\log Q.$$

Thus, we have

$$\lim_{n\to\infty} pr01 = Q^{-\alpha/(2\alpha-1)}.$$

[7] To economize on clutter, we suppress the dependence of Q, $pr00$, $pr01$, etc. on beginning of period beliefs, μ.

By a similar computation, one can show that the probability of $k_1^t = 0$ in the low investment return is given by

$$\lim_{n\to\infty} pr00 = Q^{-(1-\alpha)/(2\alpha-1)}.$$

The probability of $k_1^t = 1$ in the high investment return is given by $n\alpha q(1-\alpha q)^{n-1}$. Therefore, we have

$$\begin{aligned}\lim_{n\to\infty} pr11 &= \lim_{n\to\infty}(n\alpha q(pr01)) = \lim_{n\to\infty}(n\alpha\left[\frac{Q^{\frac{1}{n}}-1}{\alpha Q^{\frac{1}{n}}-(1-\alpha)}\right]pr01) \\ &= \left(\frac{\alpha}{2\alpha-1}\right)Q^{-\alpha/(2\alpha-1)}\log(Q).\end{aligned}$$

By a similar computation, one can show that the probability of $k_1^t = 1$ in the low investment return is given by

$$\lim_{n\to\infty} pr10 = \left(\frac{1-\alpha}{2\alpha-1}\right)Q^{-(1-\alpha)/(2\alpha-1)}\log(Q).$$

If μ is close to $\underline{\mu}^W$, it follows that round 1 investment is only slightly profitable for a type-1 agent. For a type-1 agent to be indifferent between investing in round 1 and waiting, the option value of not having to invest if $k_1^t = 0$ must be small. It follows that profits from investing in round 2 are only slightly negative if $k_1^t = 0$, and are positive if $k_1^t > 0$. The indifference equation can therefore be written as

$$(1-\delta)(\mu_1 - c)/\delta = Pr(k_1^t = 0|s = 1, q, \mu_1)(\mu_1^{0,q} - c),$$

which can be written as

$$\frac{(1-\delta)}{\delta}\left[1 - c - c(\frac{1-\alpha}{\alpha})(\frac{1-\mu}{\mu})\right]Q^{\alpha/(2\alpha-1)} = \left[c - 1 + cQ(\frac{1-\alpha}{\alpha})(\frac{1-\mu}{\mu})\right]. \tag{19}$$

As $n\to\infty$ and $\mu\to\underline{\mu}^W$, the left side of (19) converges to zero, which implies $Q\to 1$. Therefore, we have $\lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty} pr01 = 1$ and $\lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty} pr00 = 1$ hold. Although we have $\lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty} pr11 = 0$ and $\lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty} pr10 = 0$, having one agent invest is infinitely more likely than having more than one agent invest. To see this, note that

$$\lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty}\left(\sum_{k=2}^{n} prk1\right) = 1 - \lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty} pr11 - \lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty} pr01 = 1 - \lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty} pr11 - Q^{-\frac{\alpha}{2\alpha-1}}.$$

Thus, we have

$$\begin{aligned}\lim_{\mu\to\underline{\mu}^W}\lim_{n\to\infty}\left(\frac{\sum_{k=2}^{n} prk1}{pr11}\right) &= \lim_{Q\to 1}[\frac{1-Q^{-\frac{\alpha}{2\alpha-1}}}{\frac{\alpha}{2\alpha-1}Q^{-\frac{\alpha}{2\alpha-1}}\log Q}] - 1 = \lim_{Q\to 1}[\frac{Q^{\frac{\alpha}{2\alpha-1}}-1}{\frac{\alpha}{2\alpha-1}\log Q}] - 1 \\ &= \lim_{Q\to 1}\frac{\frac{\alpha}{2\alpha-1}Q^{\frac{\alpha}{2\alpha-1}-1}}{\frac{\alpha}{2\alpha-1}\frac{1}{Q}} - 1 = 0.\end{aligned}$$

A similar calculation yields

$$\lim_{n\to\infty}\left(\frac{\sum_{k=2}^{n} prk0}{pr10}\right) = 0,$$

which completes the proof. □

Proof of Lemma 4.2. During the first period that the economy emerges from Regime 0 into Regime M, clearly beliefs must satisfy $\mu \in [\underline{\mu}^W, \rho\underline{\mu}^W + (1-\rho)(1-\underline{\mu}^W)]$. For beginning of period t beliefs in this interval, consider the mapping to beginning of period $t+1$ beliefs, $\Psi(\mu)$, based on the equilibrium mixing probability and outcome $k_1^t = 0$. For ρ sufficiently close to one, from the proof of Lemma (4.1) the equilibrium mixing condition is (19). Denoting the left hand side of (19) as LHS, we have

$$Q = \frac{LHS + 1 - c}{c(\frac{1-\alpha}{\alpha})(\frac{1-\mu}{\mu})}. \tag{20}$$

Using the expression of $\mu_i^{k,q}$ and (20), $\mu^{0,q}$ can be simplified to

$$\mu^{0,q} = \frac{1}{1 + (1+D)(\frac{1-c}{c})(\frac{\alpha}{1-\alpha}) - D(\frac{1-\mu}{\mu})}, \tag{21}$$

where $D = \frac{(1-\delta)}{\delta} Q^{\alpha/(2\alpha-1)}$ is governed by the discount factor, since $Q^{\alpha/(2\alpha-1)}$ is nearly one for all $\mu \in [\underline{\mu}^W, \rho\underline{\mu}^W + (1-\rho)(1-\underline{\mu}^W)]$. Since no one invests in round 2 when $k_1^t = 0$, beliefs at the beginning of period $t+1$ are given by

$$\Psi(\mu) = \frac{2\rho - 1}{1 + (1+D)(\frac{1-c}{c})(\frac{\alpha}{1-\alpha}) - D(\frac{1-\mu}{\mu})} + 1 - \rho. \tag{22}$$

It is straightforward to check that $\Psi(\underline{\mu}^W) > \underline{\mu}^W$ and $\Psi(\rho\underline{\mu}^W + (1-\rho)(1-\underline{\mu}^W)) < \rho\underline{\mu}^W + (1-\rho)(1-\underline{\mu}^W)$ hold, so Ψ must have a fixed point within the interval, which we denote by μ^{fix}. From (22), the slope of the mapping is $-D(2\rho-1)$. Therefore, for $\delta > \frac{1}{2}$ and ρ close to one, it can be shown that beliefs converge to μ^{fix} over time (in an oscillatory fashion) and remain within the interval, $[\underline{\mu}^W, \rho\underline{\mu}^W + (1-\rho)(1-\underline{\mu}^W)]$, as long as no one invests in round 1.[8] □

Proof of Proposition 4.3. Since $r^W = r^{NW}$ and $\underline{p}^W = \underline{p}^{NW}$, we will drop the superscripts without causing confusion. It will also be convenient to adopt the shorthand notation,

$$z = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-c}{c}\right), \tag{23}$$

$z > 1$ by $\alpha > c$. From (2), we have

$$\lim_{\rho\to1}(r+1)(1-\rho) = \lim_{\rho\to1} r(1-\rho) = \lim_{\rho\to1} \frac{(1-\rho)\log[\frac{z-1}{z+1}]}{\log(2\rho-1)} = \frac{1}{2}\log\frac{1+z}{z-1}. \tag{24}$$

From Lemma (4.1), we have[9]

$$\lim_{\rho\to1}\frac{\lambda_0}{\lambda_1} = \lim_{\rho\to1}\frac{pr10(\mu^{fix})}{pr11(\mu^{fix})} = \frac{1-\alpha}{\alpha}, \tag{25}$$

$$\lim_{\rho\to1}\frac{(1-\rho)}{\lambda_1} = \lim_{\rho\to1}\frac{1-\rho}{1-Q(\mu^{fix})^{\frac{-\alpha}{2\alpha-1}}} = \lim_{\rho\to1}\frac{-1}{\frac{\alpha}{2\alpha-1}Q'(\mu^{fix})} = \frac{2\alpha-1}{\alpha}\frac{z}{z^2-1}. \tag{26}$$

[8] To demonstrate that we remain within the interval (and therefore do not drop out of Regime M), solve the quadratic equation based on (22) for μ^{fix}, then show that $(\mu^{fix} - \underline{\mu}^W)/(\rho\underline{\mu}^W + (1-\rho)(1-\underline{\mu}^W) - \mu^{fix})$ is greater in absolute value than the slope of Ψ. Computations were performed using Maple 10.

[9] In (26), we use l'Hopital's rule and analytically evaluate $\lim_{\rho\to1}[Q'(\mu^{fix})]$ using Maple 10 software.

By using the fact that λ_1 goes to 0 as ρ goes to 1 and (24)-(26) , we have

$$\lim_{\rho\to 1} r\lambda_1 = \lim_{\rho\to 1}(r+1)\lambda_1 = \lim_{\rho\to 1}(r+1)(1-\rho)\frac{\lambda_1}{1-\rho} = \frac{1}{2}\frac{\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1}, \quad (27)$$

$$\lim_{\rho\to 1} r\lambda_0 = \lim_{\rho\to 1} r\lambda_1\frac{\lambda_0}{\lambda_1} = \frac{1}{2}\frac{1-\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1}, \quad (28)$$

$$\lim_{\rho\to 1}\frac{\pi^W_{M0}}{\pi^W_{M1}} = \lim_{\rho\to 1}\frac{\frac{(1-\rho)}{\lambda_1}+(1-\frac{1}{1+z})}{\frac{(1-\rho)}{\lambda_1}+\frac{\lambda_0}{\lambda_1}\frac{1}{1+z}} = \frac{1+\frac{\alpha}{2\alpha-1}(z-1)}{1+\frac{1-\alpha}{2\alpha-1}\frac{z-1}{z}}. \quad (29)$$

By (8) and (10), we have

$$L^{NW}_B - L^W_B = \frac{b^{NW}+1}{1-\overline{p}^{NW}} - \frac{b^W+1}{1-\overline{p}^W}. \quad (30)$$

To show that (30) is positive, we define the function, $p^{10}(b) \equiv \frac{1}{2} - \frac{1}{2}(2\rho-1)^{b+1}$, which is the probability of the investment return switching from high to low after b periods. Note that $p^{10}(b^{NW}) = 1-\overline{p}^{NW}$ and $p^{10}(b^W) = 1-\overline{p}^W$ hold. We will now show that $\frac{b+1}{p^{10}(b)}$ is increasing in b. We have

$$\begin{aligned}\frac{b+1}{p^{10}(b)} - \frac{b}{p^{10}(b-1)} &\propto [1-(b+1)(2\rho-1)^b + b(2\rho-1)^{b+1}]\\ &= [1-(2\rho-1)]\{[1+(2\rho-1)+...+(2\rho-1)^b] - (b+1)(2\rho-1)^b]\}\\ &> [1-(2\rho-1)]\{(b+1)(2\rho-1)^b - (b+1)(2\rho-1)^b\} = 0. \quad (31)\end{aligned}$$

Thus, $\frac{b+1}{p^{10}(b)}$ is increasing in b, so we have $L^{NW}_B > L^W_B$.

To show $L^W_R > L^{NW}_R$, first recall that $L^{NW}_R = \frac{r+1}{\underline{p}}$ holds. By (11), (12), and (13), $L^W_R - L^{NW}_R$ can be simplified to

$$\begin{aligned}L^W_R - L^{NW}_R \propto\ & \underline{p}^2(1-\lambda_1)[\lambda_0+(1-\lambda_0)\frac{(1-\rho)\lambda_1}{1-\rho+\rho\lambda_1}] + (1-\lambda_0)\underline{p}[1-\rho+\rho\lambda_1-\underline{p}\lambda_1](2-\frac{\lambda_1}{1-\rho+\rho\lambda_1})\\ & -(r+1)(1-\rho)[\lambda_1(1-\lambda_0)(1-\underline{p}) - \lambda_0(1-\lambda_1)\underline{p}].\end{aligned}$$

Since $\frac{\lambda_1}{1-\rho+\rho\lambda_1} < 1$ holds, to show $L^{NW}_R < L^W_R$, it is sufficient to show that the third term is smaller than the first term in the above expression, which is implied by the following condition:

$$(r+1)(1-\rho)[1-\frac{\underline{p}}{1-\underline{p}}\frac{1-\lambda_1}{1-\lambda_0}\frac{\lambda_0}{\lambda_1}] \le \underline{p} + \frac{\underline{p}}{1-\underline{p}}\frac{1-\lambda_1}{\lambda_1}(1-\rho) + \frac{\underline{p}^2}{1-\underline{p}}\frac{1-\lambda_1}{1-\lambda_0}\frac{\lambda_0}{\lambda_1}. \quad (32)$$

Using the limits (25)-(29), when ρ converges to 1, (32) becomes

$$\begin{aligned}\frac{1}{2}(1-\frac{1-\alpha}{\alpha}\frac{1}{z})\log\frac{1+z}{z-1} &\le \frac{1}{1+z} + \frac{1}{z^2-1}\frac{2\alpha-1}{\alpha} + \frac{1}{z(z+1)}\frac{1-\alpha}{\alpha}\\ \Leftrightarrow\ \frac{1}{2}\log\frac{1+z}{z-1} &\le \frac{1}{z^2-1}\frac{z^2-\frac{1-\alpha}{\alpha}}{z-\frac{1-\alpha}{\alpha}}.\end{aligned}$$

since $z > 1$ holds, the following inequality is sufficient to show $L^{NW}_R < L^W_R$:

$$\frac{2z}{z^2-1} - \log\frac{1+z}{z-1} \ge 0. \quad (33)$$

Given z is bounded, (33) holds. To verify the condition, the derivative of the expression with respect to z is $\frac{-4}{(z^2-1)^2}$, which is negative, so the left side of (33) is decreasing in z. Moreover, we have $\lim_{z\to\infty}[\frac{2z}{z^2-1} - \log\frac{1+z}{z-1}] = 0$. Therefore, inequality (33) holds. □

Proof of Proposition 4.4. To establish part (i), it is sufficient to show

$$\lim_{\rho\to 1}\frac{\pi_R^{NW}}{\pi_B^{NW}} < \lim_{\rho\to 1}\frac{\pi_R^W}{\pi_B^W}, \tag{34}$$

because $\pi_R^{NW} + \pi_B^{NW} = 1$ and $\pi_R^W + \pi_B^W = 1$ hold.

By (1) and (7) we have

$$\lim_{\rho\to 1}\frac{\pi_R^{NW}}{\pi_B^{NW}} = \lim_{\rho\to 1}\frac{\frac{r+1}{\underline{p}}}{\frac{b^{NW}+1}{1-\overline{p}^{NW}}} = \lim_{\rho\to 1}\frac{\frac{r+1}{\underline{p}}\lambda_1}{\frac{b^{NW}+1}{1-\overline{p}^{NW}}\lambda_1} \quad \text{and}$$

$$\lim_{\rho\to 1}\frac{\pi_R^W}{\pi_B^W} = \lim_{\rho\to 1}\frac{(r\lambda_1+1)+(1+r\lambda_0)\frac{\pi_{M0}^W}{\pi_{M1}^W}}{\frac{b^W+1}{1-\overline{p}^W}\lambda_1} > \lim_{\rho\to 1}\frac{(r\lambda_1+1)+(1+r\lambda_0)\frac{\pi_{M0}^W}{\pi_{M1}^W}}{\frac{b^{NW}+1}{1-\overline{p}^{NW}}\lambda_1}. \tag{35}$$

Inequality (35) comes from the fact that $\frac{b^W+1}{1-\overline{p}^W} < \frac{b^{NW}+1}{1-\overline{p}^{NW}}$ holds. To show (34), it is sufficient to show

$$\lim_{\rho\to 1}(r\lambda_1+1)+(1+r\lambda_0)\frac{\pi_{M0}^W}{\pi_{M1}^W} \geq \lim_{\rho\to 1}\frac{r+1}{\underline{p}}\lambda_1. \tag{36}$$

Using the previous limiting results (25)-(29), inequality (36) is equivalent to

$$4+2\frac{1-\alpha}{2\alpha-1}\frac{z-1}{z}+2\frac{\alpha}{2\alpha-1}(z-1)+\frac{1-\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1}-\frac{\alpha}{2\alpha-1}(z^2-1)\log\frac{z+1}{z-1} \geq 0. \tag{37}$$

It is easy to verify that inequality (37) holds if the following condition holds

$$\frac{1}{2}\log\frac{z+1}{z-1} - \frac{1}{z-1} < 0. \tag{38}$$

To see (38) holds, note that the derivative with respect to z is positive, which implies that the left side of (38) is increasing in z. Moreover, we have $\lim_{z\to\infty}[\frac{1}{2}\log\frac{z+1}{z-1} - \frac{1}{z-1}] = 0$. Therefore, (38) holds. □

Proof of Proposition 4.5. First, define the function, $A(b)$, by

$$A(b) \equiv \frac{1}{b}\{\frac{b}{2} - \frac{2\rho-1}{4(1-\rho)}[1-(2\rho-1)^b]\} = \frac{1}{2} - \frac{2\rho-1}{4(1-\rho)}\frac{[1-(2\rho-1)^b]}{b}.$$

That is, $A(b)$ is the probability that the investment return is low during one of the b periods of Regime 2, chosen at random, and $A(r)$ is the probability that the investment return is high during one of the r periods of Regime 0, chosen at random. We show that $A(\cdot)$ is an increasing function. To see this, it is sufficient to show that $\frac{[1-(2\rho-1)^b]}{b}$ is decreasing in b. This condition is satisfied, since we have

$$\frac{1-(2\rho-1)^b}{b} - \frac{1-(2\rho-1)^{b+1}}{b+1} \propto [1-(2\rho-1)^{b+1}-(b+1)(2\rho-1)^b(1-(2\rho-1))] > 0,$$

where the last inequality follows from (31).

By (16) and the fact that both λ_1 and λ_0 go to 0 as ρ goes to 1, we have

$$\lim_{\rho\to 1} O^W = \lim_{\rho\to 1} \frac{b^w+1}{1-\overline{p}^W}\lambda_1 \pi_{M1}^W A(b^W) = \lim_{\rho\to 1} \pi_B^W A(b^W). \tag{39}$$

By (14) and the fact that r and b^{NW} go to infinity as ρ goes to 1, we have

$$\lim_{\rho\to 1} O^{NW} = \lim_{\rho\to 1} A(b^{NW}) \frac{(b^{NW}+1)\underline{p}}{(r+1)\overline{p}^{NW} + (b^{NW}+1)\underline{p}} = \lim_{\rho\to 1} \pi_B^{NW} A(b^{NW}). \tag{40}$$

Now we compare (39) and (40). From Proposition (4.4), we have $\lim_{\rho\to 1} \pi_B^{NW} > \lim_{\rho\to 1} \pi_B^W$. And by the fact $b^{NW} > b^W$, we have $A(b^{NW}) > A(b^W)$. Therefore, $\lim_{\rho\to 1} O^W < \lim_{\rho\to 1} O^{NW}$. This proves part (i).

Now we show part (ii). By (15), we have

$$\lim_{\rho\to 1} U^{NW} = \lim_{\rho\to 1} \frac{r\overline{p}^{NW}}{(r+1)\overline{p}^{NW} + (b^{NW}+1)\underline{p}} A(r) = \lim_{\rho\to 1} \pi_R^{NW} A(r). \tag{41}$$

On the other hand, by (17), we have

$$\begin{aligned} \lim_{\rho\to 1} U^W &= \lim_{\rho\to 1} \pi_{M1}^W + (r\lambda_1 \pi_{M1}^W + r\lambda_0 \pi_{M0}^W) A(r) \\ &= \lim_{\rho\to 1} [(r\lambda_1+1)\pi_{M1}^W + (r\lambda_0+1)\pi_{M0}^W] A(r) + [1-A(r)]\pi_{M1}^W - A(r)\pi_{M0}^W \\ &= \lim_{\rho\to 1} \pi_R^W A(r) + \lim_{\rho\to 1} [1-A(r)]\pi_{M1}^W - A(r)\pi_{M0}^W, \end{aligned} \tag{42}$$

where the last equality follows from (7). Now we compare (41) and (42). Since by Proposition (4.3), $\lim_{\rho\to 1} \pi_R^W > \lim_{\rho\to 1} \pi_R^{NW}$ holds, the following condition is sufficient to show $\lim_{\rho\to 1}(U^W - U^{NW}) > 0$:

$$\lim_{\rho\to 1} [1-A(r)]\pi_{M1} - A(r)\pi_{M0} \geq 0, \tag{43}$$

After using (29) and simplifying, we can rewrite (43) as

$$\frac{1}{2}\left(\frac{1-\alpha}{2\alpha-1}\right)\frac{z-1}{z} + \frac{1}{(z+1)\log\frac{z+1}{z-1}}[2 + \frac{1-\alpha}{2\alpha-1}\frac{z-1}{z} + \frac{\alpha}{2\alpha-1}(z-1)] - \frac{1}{2}\frac{\alpha}{2\alpha-1}(z-1) \geq 0.$$

The above inequality holds since $\frac{1}{2}\log\frac{z+1}{z-1} < \frac{1}{z-1}$, by (38). Therefore, inequality (43) holds. □

Proof of Proposition 4.6. By Proposition (3.5), we have $\overline{\mu}^W(\delta') > \overline{\mu}^W(\delta'')$. This implies that $b^W(\delta') < b^W(\delta'')$. On the other hand, r^W does not depend on δ. The limit $\lim_{\rho\to 1} \pi_{M1}^W/\pi_{M0}^W$ does not depend on δ either (see (29)). Following the proof of Proposition 4.3, We have

$$L_B^W(\delta') - L_B^W(\delta'') = \frac{b^W(\delta')+1}{1-\overline{p}^W(\delta')} - \frac{b^W(\delta'')+1}{1-\overline{p}^W(\delta'')} < 0.$$

Inspecting the proof of Proposition (4.3), one can see that L_R^W does not depend on δ, thus $L_R^W(\delta') = L_R^W(\delta'')$. This proves part (i).

To show part (ii), it is sufficient to show that

$$\lim_{\rho\to 1}\frac{\pi_R^W(\delta')}{\pi_B^W(\delta')} > \lim_{\rho\to 1}\frac{\pi_R^W(\delta'')}{\pi_B^W(\delta'')},$$

because $\pi_R^W + \pi_B^W = 1$ hold. By the proof in Proposition (4.4),

$$\lim_{\rho\to 1}\frac{\pi_R^W(\delta')}{\pi_B^W(\delta')} - \lim_{\rho\to 1}\frac{\pi_R^W(\delta'')}{\pi_B^W(\delta'')} = \lim_{\rho\to 1}\frac{(r\lambda_1+1)+(1+r\lambda_0)\frac{\pi_{M0}^W}{\pi_{M1}^W}}{\lambda_1}\left[\frac{1}{\frac{b^W(\delta')+1}{1-\overline{p}^W(\delta')}} - \frac{1}{\frac{b^W(\delta'')+1}{1-\overline{p}^W(\delta'')}}\right] > 0.$$

The last inequality follows part (i) and $\lim_{\rho\to 1}\pi_{M1}^W/\pi_{M0}^W$ does not depend on δ.

By the proof in Proposition (4.5), $\lim_{\rho\to 1} O^W(\delta) = \lim_{\rho\to 1}\pi_B^W(\delta)A(b^W(\delta))$. Moreover, $A(b^W)$ is increasing in b^W. Given that $b^W(\delta') < b^W(\delta'')$, we have $A(b^W(\delta')) < A(b^W(\delta''))$. In addition, from part (ii) we have $\pi_B^W(\delta') < \pi_B^W(\delta'')$. Thereforefore, $\lim_{\rho\to 1} O^W(\delta') < \lim_{\rho\to 1} O^W(\delta'')$.

Similarly,

$$\lim_{\rho\to 1} U^W(\delta) = \lim_{\rho\to 1}\pi_{M1}^W(\delta) + [r\lambda_1\pi_{M1}^W(\delta) + r\lambda_0 k\pi_{M1}^W(\delta)]A(r),$$

where $k \equiv \frac{\pi_{M0}^W}{\pi_{M1}^W}$. Note that when ρ goes to 1, by (26)-(29) the limits of $r\lambda_1$, $r\lambda_0$ and k do not depend on δ. Therefore, to show $U^W(\delta') > U^W(\delta'')$ it is sufficient to show that $\lim_{\rho\to 1}\pi_{M1}^W(\delta') > \lim_{\rho\to 1}\pi_{M1}^W(\delta'')$. By (5) and (6),

$$\lim_{\rho\to 1}\pi_{M1}^W(\delta) = \frac{1}{\lim_{\rho\to 1}[\frac{b^W(\delta)+1}{1-\overline{p}^W(\delta)}\lambda_1 + 1 + k + r\lambda_1 + rk\lambda_0]}.$$

Since $\frac{b^W(\delta)+1}{1-\overline{p}^W(\delta)}$ is decreasing in δ, we have $\lim_{\rho\to 1}\pi_{M1}^W(\delta') > \lim_{\rho\to 1}\pi_{M1}^W(\delta'')$. This proves part (iii).□

3. More Simulation Results

Propositions (2.1) and (3.2) suggest a procedure to compute equilibrium trajectories numerically, for both the NWG and the WG. Basically, the computation is feasible because expectations of what future agents might do have no impact on current choices. Equilibrium can be computed, history by history, by updating beliefs, determining the regime, computing the investment probability if necessary, and so on. Although the number of histories to compute grows exponentially as the number of periods increases, it is relatively easy to compute an equilibrium trajectory, by drawing a realization of the investment state in each period, then drawing signals, drawing the outcome of each type-1 agent's mixing (if necessary), and so on. Figures 1-6 demonstrate equilibrium trajectories for the two games for various discount factors, with other parameters fixed at $\alpha = 0.65$, $c = 0.5$, $n = 100$, $\rho = 0.95$, and a time horizon of 1000 periods.[10]

For the NWG, the symmetry of parameters leads to symmetric cycles. The economy starts in Regime 1, but the large number of agents yields almost perfect information about the investment

[10] Programming was done using SAS version 9.1. Source code is available upon request. The same "seed" was used in both games to draw random numbers, so the investment state and signal realizations are the same across the two games. We are grateful to Hammad Qureshi for doing an excellent job programming the algorithms.

state. Beliefs jump into either Regime 2 or Regime 0, with $\mu(h^t) \simeq 1$ if about 65 agents invest in period t, and $\mu(h^t) \simeq 0$ if about 35 agents invest in period t. Figure 2 shows the self-correcting nature of the cascades, as type-1 agents occasionally invest, or type-0 agents occasionally choose not to invest, as the probability of an unobserved change in the investment return builds up. For the WG, parameters are symmetric, but the pattern of cycles is not. The Regime 0 cascade lasts for the same number of periods in the WG as in the NWG, but Regime 2 cascades are much shorter, as seen in Figure 3.

Table 1 shows averages for investment, overinvestment, underinvestment, and welfare.[11] Several features emerge from this example. First, the average length of a recession is longer and the average length of a boom is shorter under the WG than under the NWG. Second, under the WG the economy spends more time in recession than in a boom. Third, there is less overinvestment and more underinvestment under the WG than under the NWG. These features generalize across all parameter values for which we have performed simulations. Table 1 indicates that either the NWG or the WG can provide higher welfare, and that the tradeoff is non-monotonic in δ.

Table 1: Simulation Statistics

	NWG	WG ($\delta = 0.9$)	WG ($\delta = 0.7$)	WG ($\delta = 0.5$)
Mean of Investment	54.941	33.255	38.255	43.167
Overinvestment Index	0.13446	0.01584	0.04652	0.06419
Underinvestment Index	0.09405	0.19229	0.17297	0.14152
Welfare	14.0245	14.75895	13.9886	14.5535

Booms are shorter in the Waiting Game, because the shorter Regime 2 cascade reduces the chance that the investment return changes undetected, from high to low and back again to high, all within the same boom. Also, when the investment state has changed from high to low, the economy will learn about the change more quickly, on average. Recessions are longer in the Waiting Game, because when the economy moves out of Regime 0, we move into Regime M. Thus, while the No-Waiting Game moves into Regime 1 and activity almost reveals the state, the Waiting Game moves into Regime M and activity reveals little, on average. Figure 3 shows that the economy must wait several periods until a single type-1 agent decides to invest, and having two or more agents invest in round 1 of any one period is very unlikely. This pattern is established analytically, for the large, patient, persistent economy, in the next section.

We conjecture that, in the Waiting Game, recessions will become shorter and booms longer as δ decreases. Intuitively, the Waiting Game with $\delta = 0$ has the same cyclical behavior as the No-Waiting Game, because round-1 behavior and the resulting information flow will be the same. As δ decreases, the Waiting Game should look more and more like the No-Waiting Game, with longer booms and shorter recessions. Another intuition, based on Proposition (3.5), is that as δ decreases, Regime 2 cascades are longer. Also, type-1 agents are more likely to invest in Regime M, which provides more information to the market and shortens the amount of time before a switch to the high investment state is detected, thereby shortening the average length of recessions. Our numerical results exhibit this pattern.

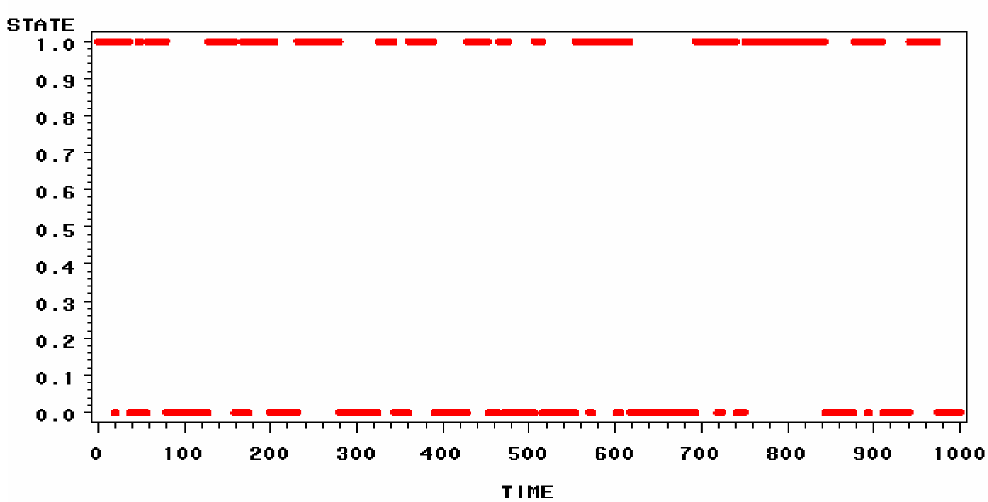

Figure 1: State vs Time

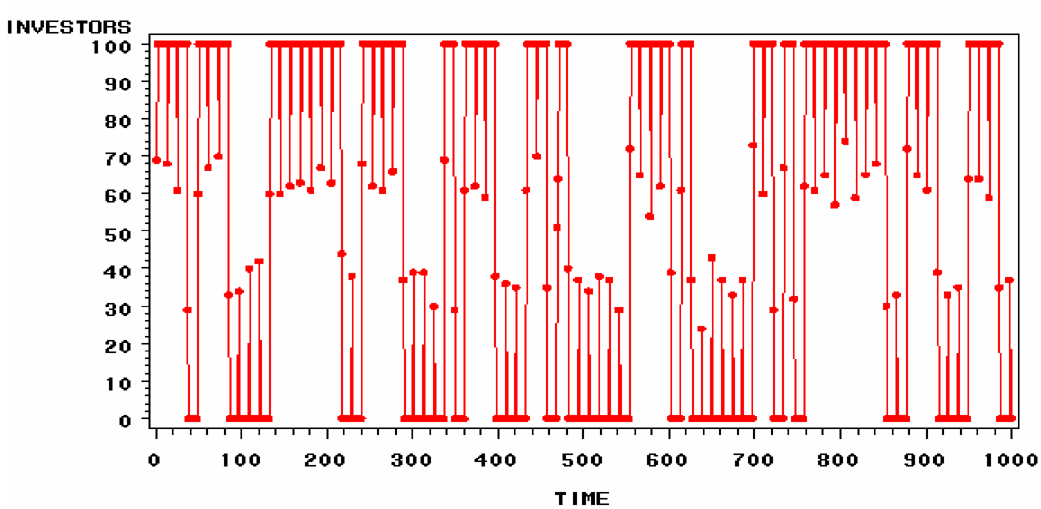

Figure 2: Total Investment in the NWG

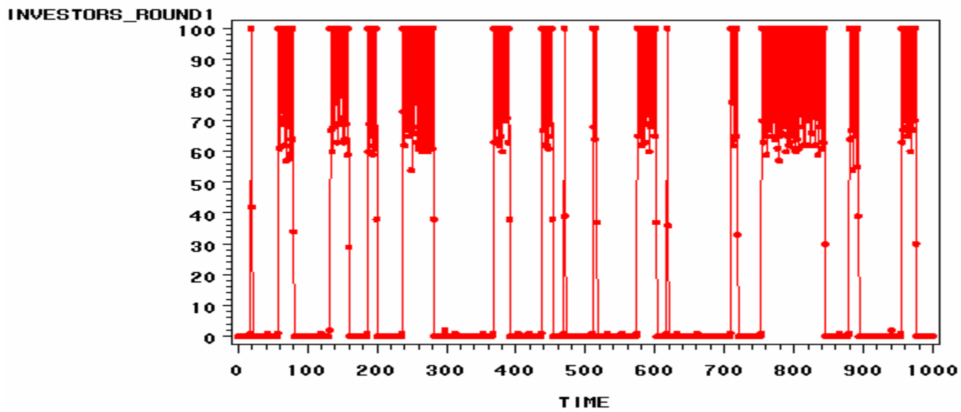

Figure 3: Round 1 Investment in the WG ($\delta = 0.9$)

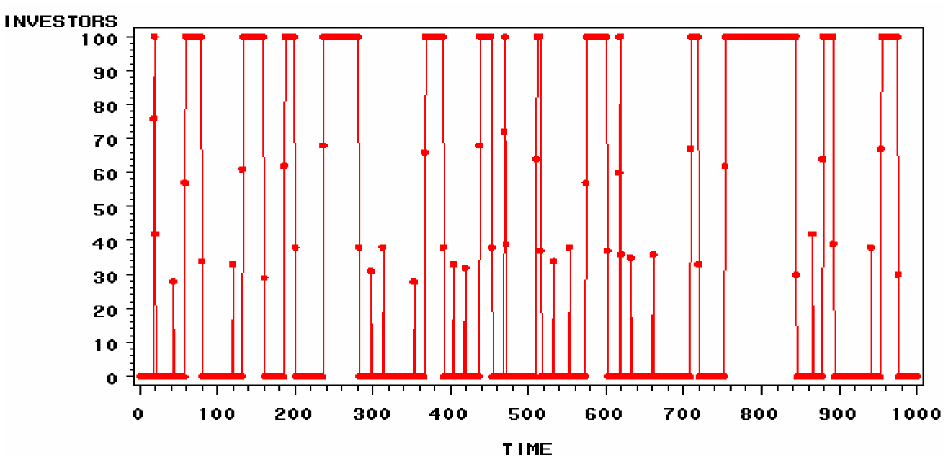

Figure 4: Total Investment in the WG ($\delta = 0.9$)

[11] The average overinvestment (underinvestment) is the probability that the state is low (high) and a random agent invests.

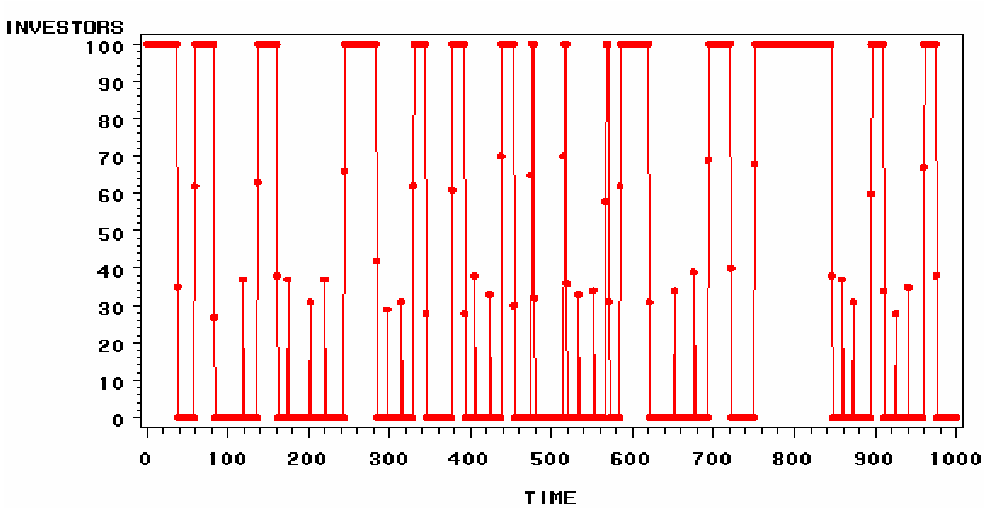

Figure 5: Total Investment in the WG ($\delta = 0.7$)

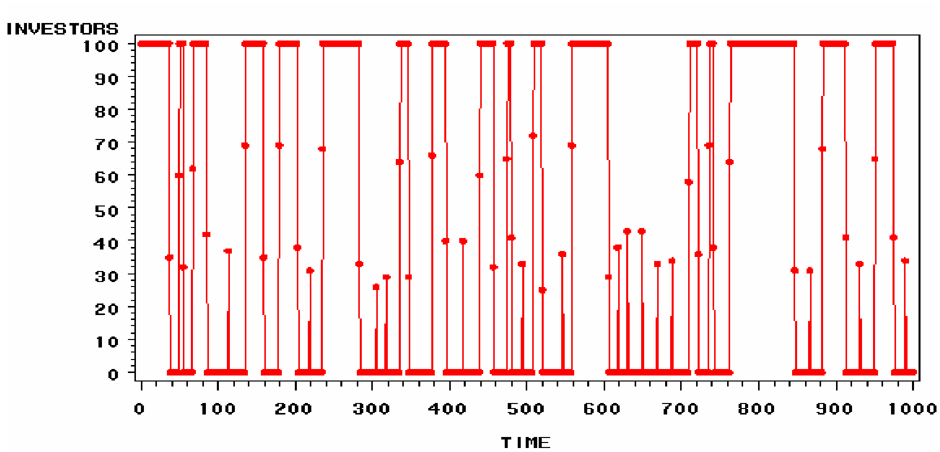

Figure 6: Total Investment in the WG ($\delta = 0.5$)