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Estimation of the Bivariate Stable Spectral Representation by the Projection Method

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Abstract. A method of estimating the spectral representation of a generalized bivariate stable distribution is presented, based on a series of maximum likelihood (ML) estimates of the stable parameters of univariate projections of the data. The corresponding stable spectral density is obtained by solving a quadratic program. The proposed method avoids the often arduous task of computing the multivariate stable density, relying instead on the standard univariate stable density. The paper applies this projection procedure, under the simplifying assumption of symmetry, to simulated data as well as to foreign exchange return data, with favorable results. Kanter projection coefficients governing conditional expectations are computed from the estimated spectral density. For the simulated data these compare well to their known true values.

Key words: estimation of bivariate stable spectral representation, projection method, foreign exchange rates, Kanter projection coefficient

1. Introduction

Economic and financial data often arise as the aggregation of a vast number of more or less independent, unobserved factors. According to the Generalized Central Limit Theorem (e.g., Zolotarev, 1986), if the sum of a large number of i.i.d. random variables has a limiting distribution after appropriate shifting and scaling, the limiting distribution must be a member of the *stable* class. The normal or Gaussian distribution is the most familiar member of this class and is frequently used in econometric and financial models. However, the errors in such models are often too leptokurtic to be consistent with normality. In such cases, the heavier-tailed non-Gaussian stable distributions are the natural extension of the popular normality assumption.

The theory of univariate estimation of stable parameters is well worked out – see McCulloch (1996a) for extended references. However, multivariate stable parameter estimation remains in its infancy. Mittnik and Rachev (1993, pp. 365–366) suggest a method of estimating the characteristic exponent and spectral representation of a generalized bivariate stable distribution, but employ only a small subset of the data, drawn from the extreme tails. This method has been implemented by Cheng and Rachev (1996). The present paper proposes an alternative method, based on likelihood maximization, that uses the entire data set in a series of univariate projections. This method avoids the often arduous task of computing the MV stable density (see Nolan and Rajput, 1993), relying instead on the standard univariate stable density.

The projection procedure is applied here, in the simplifying case of symmetry, to simulated data and to actual financial data on German mark and Japanese yen returns. Kanter projection coefficients governing conditional expectations are computed from the estimated stable density. In the case of the simulated data, these estimates compare well to their known true values.

2. Bivariate Stable Distributions

A scalar random variable X has a stable distribution with characteristic exponent $\alpha \in (0, 2]$, skewness parameter $\beta \in [-1, 1]$, scale parameter $c > 0$, and location parameter $\delta \in \Re$ if and only if its log characteristic function may be written

$$\log E \exp(iXt) = i\delta t + \psi_{\alpha\beta}(ct),$$

where

$$\psi_{\alpha\beta}(t) = \begin{cases} -|t|^{\alpha}\left[1 - i\beta \operatorname{sign}(t) \tan \dfrac{\pi\alpha}{2}\right], & \alpha \neq 1, \\ -|t|\left[1 + i\beta \dfrac{2}{\pi} \operatorname{sign}(t) \log|t|\right], & \alpha = 1, \end{cases} \tag{1}$$

is the standard ($c = 1$, $\delta = 0$) univariate stable log c.f. (see McCulloch, 1996b). The Gaussian distribution arises for $\alpha = 2$, in which case the variance is $2c^2$ and β loses its effect. In non-Gaussian stable cases, the tails are thicker, and the variance is infinite. The univariate stable cumulative distribution functions and probability density functions may be computed by means of proper integral representations given by Zolotarev (1986; see also Nolan, 1998).

A vector random variable $\mathbf{x} = (x_1, x_2)'$ has a bivariate stable distribution, with $\alpha \neq 1$, if its log characteristic function may be written as

$$\log E \exp(i\mathbf{x}'\mathbf{t}) = i\,\boldsymbol{\delta}'\mathbf{t} + \int_0^{2\pi} \psi_{\alpha 1}(\mathbf{s}_\theta'\mathbf{t}) \mathrm{d}\Gamma(\theta), \tag{2}$$

where $\mathbf{s}_\theta = (\cos\theta, \sin\theta)'$ is the point of the unit circle at angle θ, $\boldsymbol{\delta}$ is a 2-vector of location parameters, and $\Gamma(\theta)$ is a non-decreasing, left-continuous function which may be normalized to 0 at any convenient angle, commonly, but not necessarily, 0 (cf. Hardin, Samorodnitsky and Taqqu, 1991, p. 585; Mittnik and Rachev, 1993, pp. 355–356; Wu and Cambanis, 1991, p. 86; McCulloch, 1996a, pp. 397–401). Equation (2) is called the spectral representation of the bivariate stable distribution. $\Gamma(\theta)$ is known as the spectral measure, and $\mathrm{d}\Gamma(\theta)$ is the spectral density.[1] These are ‘spectral’ in the sense of being defined on the unit circle, but are unrelated to the ‘spectrum’ of frequency-domain time series analysis.

Such a stable random vector $\mathbf{x}$ may be constructed from a maximally positively skewed ($\beta = 1$) α-stable Lévy motion $z(\theta)$, whose iid increments $\mathrm{d}z(\theta)$ have zero drift and scale $(\mathrm{d}\theta)^{1/\alpha}$, by

$$\mathbf{x} = \int_0^{2\pi} \mathbf{s}_\theta \frac{(\mathrm{d}\Gamma(\theta))^{1/\alpha}\mathrm{d}z(\theta)}{(\mathrm{d}\theta)^{1/\alpha}} + \boldsymbol{\delta} \tag{3}$$

(cf. Modarres and Nolan, 1992). This awkward-looking integrand has the following interpretation: If $\Gamma'(\theta)$ exists, θ contributes $(\Gamma'(\theta))^{1/\alpha}\mathrm{d}z(\theta)$ to the integral. If Γ instead jumps by $\Delta\Gamma$ at θ, θ contributes an atom $\mathbf{s}_\theta(\Delta\Gamma)^{1/\alpha}Z_\theta$ to the integral, where $Z_\theta = (\mathrm{d}\theta)^{-1/\alpha}\mathrm{d}z(\theta)$ is univariate α-stable with $\beta = 1$, $c = 1$, and $\delta = 0$.

M. Kanter (as reported by Hardin et al., 1991) has shown that if $\mathrm{d}\Gamma(\theta)$ is symmetrical and $\alpha > 1$, the projection of x_2 on x_1,

$$E(x_2 \mid x_1) = \boldsymbol{\kappa}_{2,1} x_1 ,$$

for example, is determined by the projection coefficient

$$\boldsymbol{\kappa}_{2,1} = \frac{1}{c^\alpha(x_1)} \int_0^{2\pi} \sin\theta(\cos\theta)^{\langle\alpha-1\rangle}\mathrm{d}\Gamma(\theta), \tag{4}$$

where

$$c^\alpha(x_1) = \int_0^{2\pi} |\cos\theta|^\alpha \mathrm{d}\Gamma(\theta),$$

$$x^{\langle\alpha\rangle} = \mathrm{sign}(x)|x|^\alpha.$$

Gamrowski and Rachev (1994) shows that, for stock returns, this Kanter projection coefficient may be interpreted as the ‘beta’ of the Capital Asset Pricing Model (CAPM) that in turn governs the relative returns of the stocks (see also McCulloch, 1996a). Hardin et al. (1991) – see also Samorodnitsky and Taqqu (1994) – demonstrates that if $\mathrm{d}\Gamma(\theta)$ is not symmetrical, $E(x_2 \mid x_1)$ is non-linear in x_1 but is still a simple function involving this $\boldsymbol{\kappa}_{2,1}$ and other functionals of $\mathrm{d}\Gamma$. The stable spectral density therefore has important practical implications.

3. Estimation by the Projection Method

For each $\omega \in [0, \pi)$, consider the projection

$$y(\omega) = \mathbf{s}'_\omega \mathbf{x} = \mathbf{s}'_\omega \boldsymbol{\delta} + \int_0^{2\pi} \cos(\theta - \omega) \frac{(\mathrm{d}\Gamma(\theta))^{1/\alpha}\mathrm{d}z(\theta)}{(\mathrm{d}\theta)^{1/\alpha}} . \tag{5}$$

This has scale $c(\omega)$, where

$$c^\alpha(\omega) = \int_0^{2\pi} |\cos(\theta - \omega)|^\alpha \mathrm{d}\Gamma(\theta). \tag{6}$$

By breaking the integral in (5) in half at $\omega + \pi/2$ and $\omega - \pi/2$ ($d\Gamma$ is cyclic by definition), $y(\omega)$ may be decomposed into the sum of its location parameter plus two maximally skewed zero-location stable variables, one with $\beta = 1$ and scale $C(\omega)$, and the other with $\beta = -1$ and scale $C(\omega + \pi)$, where

$$C^\alpha(\omega) = \int_{\omega-\pi/2}^{\omega+\pi/2} \cos(\theta - \omega)^\alpha d\Gamma(\theta), \quad \omega \in [0, 2\pi), \tag{7}$$

so that

$$c^\alpha(\omega) = C^\alpha(\omega) + C^\alpha(\omega + \pi). \tag{8}$$

The skewness of $y(\omega)$ is then given by

$$\beta(\omega) = \frac{C^\alpha(\omega) - C^\alpha(\omega + \pi)}{c^\alpha(\omega)}, \tag{9}$$

whence

$$C(\omega) = c(\omega)\left(\frac{1 + \beta(\omega)}{2}\right)^{1/\alpha} \tag{10}$$

and

$$C(\omega + \pi) = c(\omega)\left(\frac{1 - \beta(\omega)}{2}\right)^{1/\alpha}. \tag{11}$$

Now let $\mathbf{x}_i = (x_{1i}, x_{2i})'$, $i = 1, \ldots, n$, be a set of iid observations on the vector $\mathbf{x}$. The components x_1 and x_2 of $\mathbf{x}$ are each univariate stable with a common α. The parameters α, β_1, c_1, and δ_1 of x_1 may therefore be estimated consistently by univariate maximum likelihood (ML) from the n observations on x_1 using the methods of McCulloch (1998a) or Nolan (1998). Similarly, α, β_2, c_2, and δ_2 may be consistently estimated from the n observations on x_2. These two estimates of the common α do not ordinarily agree precisely. However, if the two log likelihoods are *pooled* by averaging, and the resulting average maximized subject to the restriction that the two have a common exponent, an even more efficient common estimate of α may be obtained. This is not a true full information ML estimate unless x_1 and x_2 are independent, but it shares the consistency of univariate ML and is far more efficient than the Mittnik and Rachev tail estimator of α when the true distribution is stable.

Next, center the $\mathbf{x}_i$ by subtracting the pooled ML estimate of $\boldsymbol{\delta}$. Set $\theta_j = \omega_j = 2\pi j/m$, $j = 0, \ldots, m - 1$, for some large integer m divisible by 4. For $j = 0, \ldots, m/2 - 1$, calculate $y_i(\omega_j)$ from the centered $\mathbf{x}_i$ as in (5) above, and then use these to estimate $\beta(\omega_j)$ and $c(\omega_j)$ by univariate ML, constraining α to be the pooled univariate estimate, and $\boldsymbol{\delta}$ to be $\mathbf{0}$.[2] Next estimate $C(\omega_j)$, $j = 0, \ldots, m-1$ using (10) and (11).

Equation (7) states that $C^\alpha(\omega)$ is a moving average of $\mathrm{d}\Gamma(\theta)$. This moving average may be numerically approximated by

$$\gamma_j = C^\alpha(\omega_j) \approx \sum_{h=j-m/4}^{j+m/4} \cos(\omega_j - \theta_h)^\alpha \Delta_h \,, \tag{12}$$

where

$$\Delta_h = \Gamma(\theta_h + \pi/m) - \Gamma(\theta_h - \pi/m). \tag{13}$$

Now (12) is a system of m equations in m unknowns of the form $\boldsymbol{\gamma} \approx \mathbf{A}\boldsymbol{\Delta}$ that may in principle be solved for $\boldsymbol{\Delta} \approx \mathbf{A}^{-1}\boldsymbol{\gamma}$ so long as $\alpha < 2$ and $\mathbf{A}$ is non-singular.[3] However, this calculation may be quite ill-conditioned, so that sampling error may cause some of the Δ_h estimates to be negative. This can be prevented by solving the following quadratic program instead:[4] Find

$$\boldsymbol{\Delta} \geq \mathbf{0}$$

such that

$$(\boldsymbol{\gamma} - \mathbf{A}\boldsymbol{\Delta})'(\boldsymbol{\gamma} - \mathbf{A}\boldsymbol{\Delta}) = \min .$$

Having thus estimated $\boldsymbol{\Delta}$ from $\boldsymbol{\gamma}$, $\Gamma(\theta_j + \pi/m)$ may be estimated by summing the Δ_h from 0 to j. The offset of π/m in (13) is desirable, because the axes are often prime candidates for atoms. With this offset, such atoms will tend to show up uniquely in Δ_0, $\Delta_{m/4}$, $\Delta_{m/2}$, and $\Delta_{3m/4}$, rather than being split in two.

Because the matrix $\mathbf{A}$ does not depend on the data, m is not limited by the sample size n, and ordinarily may be set as high as desired without $\mathbf{A}$ becoming singular. However, in the Gaussian case $\alpha = 2$, two atoms are sufficient to generate the joint distribution, and these may be selected in an infinite number of ways. In this case $\mathbf{A}$ is singular and Γ is not identified. This is not in itself a problem, since then we have only a bivariate normal distribution to estimate. However, it does suggest that even the quadratic programming estimates of the Δ_h will behave increasingly erratically as $\alpha \uparrow 2$ with any fixed m and sample size n. In such a case, it may be desirable to impose some prior smoothness or discreteness restrictions on the spectral measure, such as the elliptical restriction considered by Press (1982: 158; case $m = 1$), the ‘diagonal model’ considered by Fama (1965), or the state-space model of Oh (1994) or Bidarkota and McCulloch (1998). Marine Carrasco has suggested that smoothness could be introduced into the above quadratic program by adding $\boldsymbol{\lambda}\boldsymbol{\Delta}'\boldsymbol{\Delta}$ to the minimand for some selected positive value of a smoothing parameter $\boldsymbol{\lambda}$. A similar effect, with perhaps even more smoothness, would be achieved by adding

$$\lambda \sum_{h=1}^{m} (\Delta_h - \Delta_{h-1})^2.$$

However, the present paper employs the pure quadratic programming method, having neither of these smoothing factors, with satisfactory results.

4. The Projection Method in the Symmetric Case

The general projection method described above requires univariate ML estimation of α and of $c(\theta)$. Although the general stable density can be calculated by means of the proper integral representations of Zolotarev (1986), computation is greatly accelerated if a numerical approximation is used in its place. McCulloch (1998b) provides a fast numerical approximation to the symmetric stable density that is the basis of the univariate symmetric stable ML program of McCulloch (1998a).[5] I therefore restrict the implementation of the projection method in this paper to the symmetric case.

When $\mathrm{d}\Gamma(\theta)$ is symmetrical, i.e., $\mathrm{d}\Gamma(\theta) = \mathrm{d}\Gamma(\theta + \pi)$, (2) becomes

$$\log E \exp(i\mathbf{x}'\mathbf{t}) = i\,\boldsymbol{\delta}'\mathbf{t} + \int_0^\pi \psi_{\alpha 0}(\mathbf{s}_\theta'\mathbf{t})\mathrm{d}\Gamma^*(\theta),$$

where

$$\psi_{\alpha 0}(t) = -|t|^\alpha$$

is the standard univariate symmetric stable log characteristic function and $\mathrm{d}\Gamma^*(\theta) = 2\mathrm{d}\Gamma(\theta)$. Such a vector $\mathbf{x}$ may be generated by

$$\mathbf{x} = \int_0^\pi \mathbf{s}_\theta \frac{(\mathrm{d}\Gamma^*(\theta))^{1/\alpha}\mathrm{d}z(\theta)}{(\mathrm{d}\theta)^{1/\alpha}} + \boldsymbol{\delta},$$

where now $\mathrm{d}z(\theta)$ is a standard symmetric ($\beta = 0$) α-stable Lévy motion. The projection equation (5) becomes

$$y(\omega) = \mathbf{s}_\omega'\boldsymbol{\delta} + \int_0^\pi \cos(\theta - \omega)\frac{(\mathrm{d}\Gamma^*(\theta))^{1/\alpha}\mathrm{d}z(\theta)}{(\mathrm{d}\theta)^{1/\alpha}},$$

while the scale $c(\omega)$ of $y(\omega)$ is determined by

$$c^\alpha(\omega) = \int_0^\pi |\cos(\theta - \omega)|^\alpha \mathrm{d}\Gamma^*(\theta), \omega \in [0, \pi).$$

The scale $c(\omega_j)$ may be estimated from the observed values of $y_i(\omega_j)$ by symmetric stable univariate ML, at $\omega_j = \theta_j = \pi j/m$, for $j = 0, \ldots, m-1$, for some large m preferably divisible by 2. The symmetric spectral density may then be estimated either by inverting

$$\gamma_j^* = c(\omega_j)^\alpha \approx \sum_{h=0}^{m-1} |\cos(\theta_h - \omega_j)|^\alpha \Delta_h^*$$

or by solving the corresponding quadratic program.

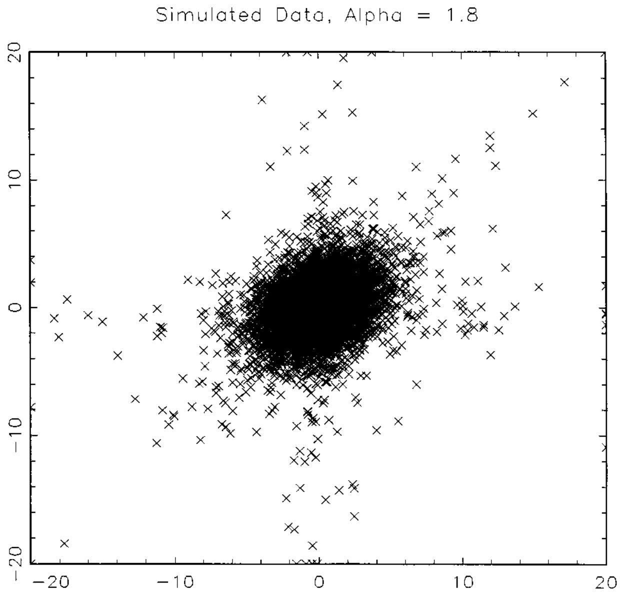

Figure 1. Simulated bivariate symmetric stable data set, with $\alpha = 1.8$, and three symmetric atoms on the unit circle at 0°, 45°, and 90°. Sample size 10 000.

Figure 1 shows a simulated bivariate distribution generated by 10 000 independent draws of a 2×1 vector $\mathbf{x}_i$,

$$\mathbf{x}_i = \mathbf{M}\mathbf{z}_i ,$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 1 \end{pmatrix} ,$$

and $\mathbf{z}_i$ is a 3×1 vector of iid standard symmetric stable random variables with $\alpha = 1.8$, generated by the method Chambers, Mallows and Stuck (1976).

This example contains three symmetric atoms, in the directions 0, $\pi/4$, and $\pi/2$ with unit mass each, or, equivalently, six maximally skewed atoms in the directions 0, $\pi/4$, $\pi/2$, π, $5\pi/4$, $3\pi/2$, with mass 1/2 each. This distribution in turn can be represented either by a general spectral measure on the whole circle with steps of 1/2 at each of these six angles or by a symmetric spectral measure on the half circle with three steps of one each.

It can be seen in Figure 1 that an approximately elliptical mass of points lies near the center of the distribution, with strings of outliers in each of the six indicated directions. For plotting purposes only, the observations are cropped at ±20. These points appear at the edges of the plot.

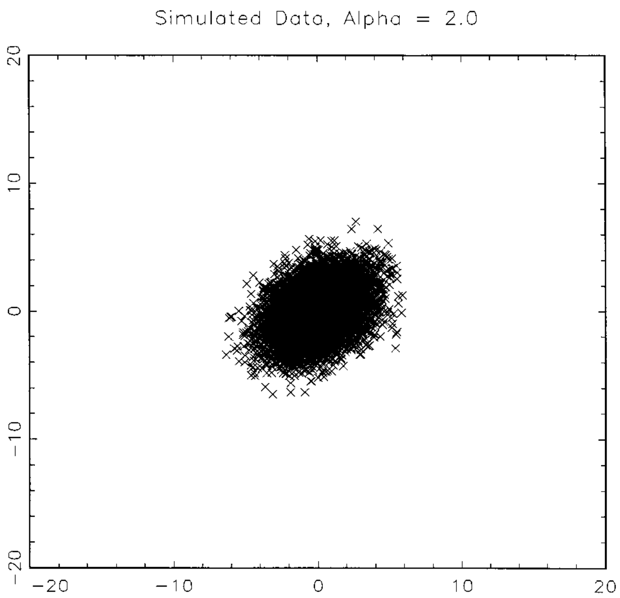

Figure 2. Simulated Gaussian data set constructed as in Figure 1, but with $\alpha = 2$.

For comparison, Figure 2 shows a joint distribution generated in exactly the same manner, but for the Gaussian case $\alpha = 2$. This distribution is entirely determined by its covariance matrix

$$\Sigma = 2\mathbf{M}\mathbf{M}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} .$$

Exactly the same bivariate normal distribution could have been generated by two independent normal random variables using, for example,

$$\mathbf{M}^* = \begin{pmatrix} \sqrt{3/2} & 0 \\ 1/\sqrt{6} & 2/\sqrt{3} \end{pmatrix} .$$

In the non-Gaussian stable cases, however, this matrix would generate a distinctly different distribution.

Pooled maximum likelihood gives an estimate of α of 1.792, which compares favorably to the true value of 1.800. The two estimated scales are

$$c_1 = 1.256$$
$$c_2 = 1.250\,,$$

which compares favorably to their common true value,

$$c_1 = c_2 = (1 + (\sqrt{2}/2)^{1.8})^{1/1.8} = 1.269.$$

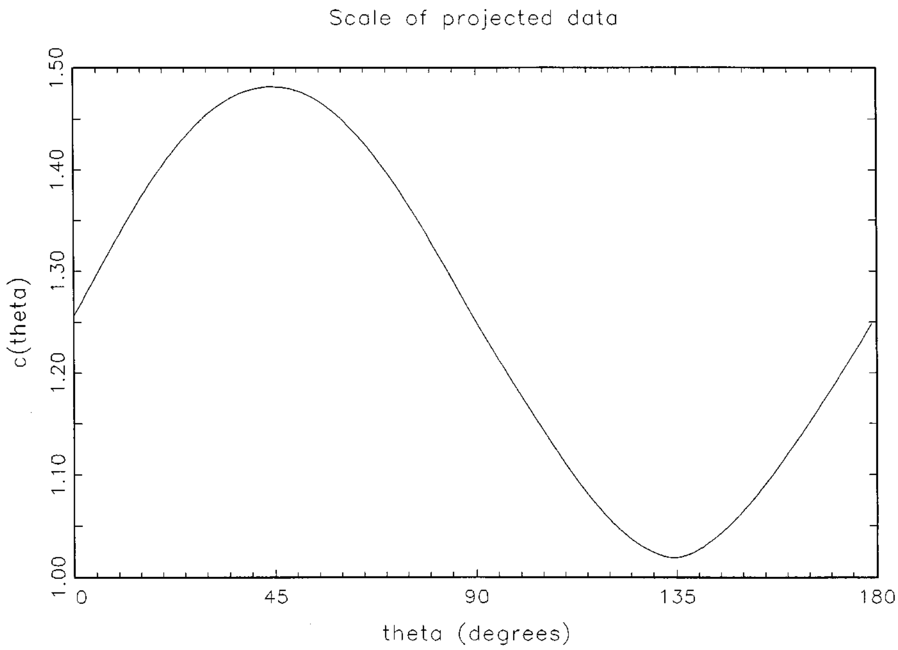

Figure 3. Estimated $c(\theta)$ for simulated data of Figure 1.

The pooled likelihood ratio statistic for the hypothesis $\alpha = 2$ is 3965.46. This overwhelmingly rejects normality, using the Monte Carlo critical values given in McCulloch (1997).

Figure 3 shows the estimated scale $c(\theta_j)$, using $m = 180$ points on the half-circle, with the angles plotted in degrees for convenience of exposition. As we would expect from Figure 1, the scales peak near the center of the first quadrant at 44° and have a minimum near the center of the second quadrant at 134°.

Figure 4 shows the estimated discrete approximation to the symmetric spectral density $d\Gamma^*(\theta)$ obtained by solving the quadratic program with no smoothness priors using a GAUSS quadratic programming routine kindly provided by Robert D. Dittmar. Each density contribution has been represented as a rectangle with height Δ_j^* and angular width π/m, centered on θ_j. Since we expect a mass point near 0 with little, if any, mass in the second quadrant, the density has been plotted from –45° to 134°. The nearly equal mass contributions near 0°, 45°, and 90° show quite clearly, with only a little noise around –18° and +30°.

Figure 5 shows the estimate of the symmetric spectral measure $\Gamma^*(\theta)$, computed as the accumulation of the density of Figure 4. There are, as expected, approximately unit steps at the three expected angles. In order to see these steps more clearly, the measure is accumulated from –45°, rather than 0°.

The precise shape of the bivariate spectral density is of little practical consequence. However, the projection coefficients determined by the spectral density,

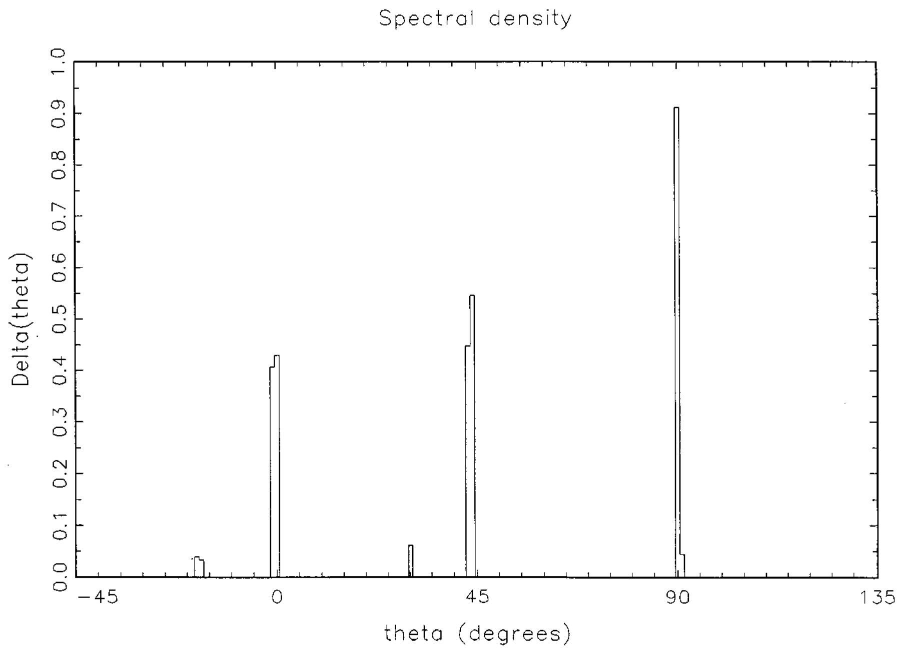

Figure 4. Estimated symmetric spectral density $\Delta^*(\theta)$ for simulated data of Figure 1.

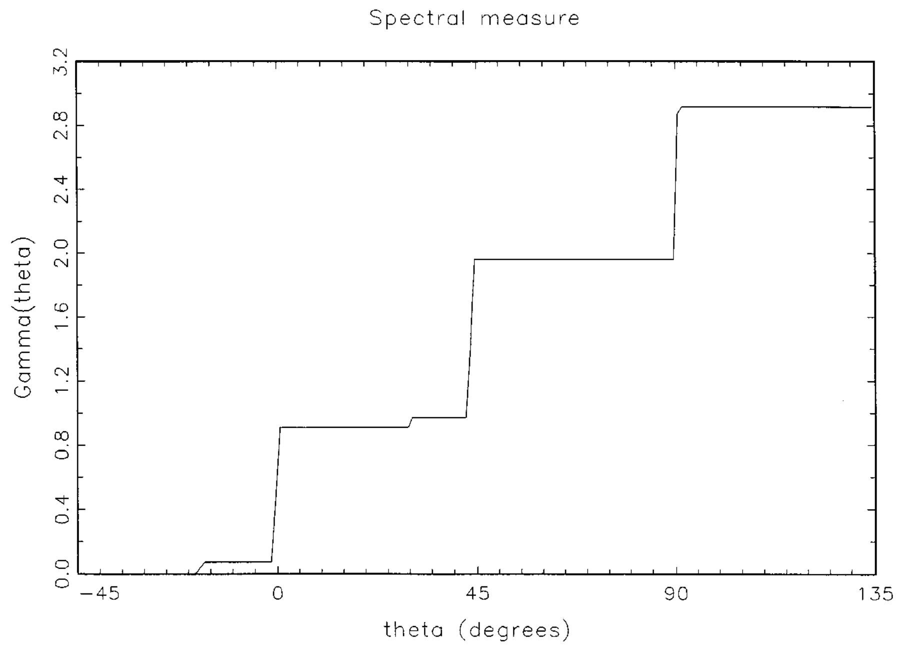

Figure 5. Estimated symmetric spectral measure $\Gamma^*(\theta)$ for simulated data of Figure 1.

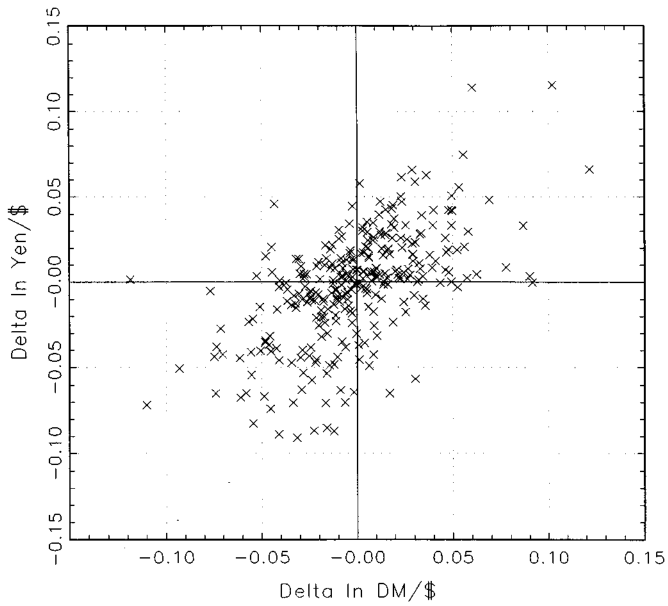

Figure 6. Monthly changes in the log DM/$ and Yen/$ exchange rates, April 1973–April 1998. Sample size $n = 301$.

as given by (4), are of greater operational significance. For the simulated data of Figure 1, the true values of the two Kanter projection coefficients are

$$\kappa_{21} = \kappa_{12} = \frac{(\sqrt{2}/2)^{1.8}}{1 + (\sqrt{2}/2)^{1.8}} = 0.3255,$$

while the estimated values computed from the estimated spectral density of Figure 4 are

$$\hat{\kappa}_{21} = 0.3499$$
$$\hat{\kappa}_{12} = 0.3501$$

The ability of multivariate stable distribution estimation methods to estimate such projection coefficients correctly is an important criterion by which to judge their relative merits.

5. Application to Foreign Exchange Returns

Figure 6 shows actual data on $n = 301$ pairs of monthly changes in the natural logarithms of the DM/$ and Yen/$ exchange rates, during the floating exchange rate period, April 1973–April 1998.[6] There is a central cluster of points with clearly positive correlation, plus a number of outliers reminiscent of Figure 1. The distribution shows some signs of positive skewness but is approximately symmetrical.

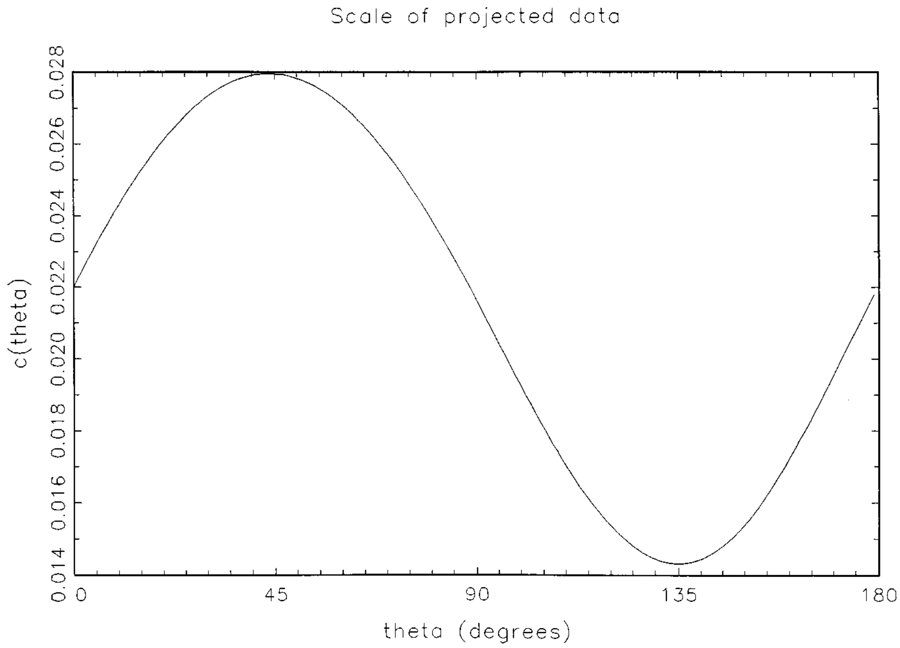

Figure 7. Estimated $c(\theta)$ for foreign exchange return data of Figure 6.

Univariate symmetric stable ML estimates of α are similar, 1.852 and 1.888, respectively, and normality may be rejected at the 0.01 and 0.05 levels, respectively, using the Monte Carlo critical values of the Likelihood Ratio statistic tabulated in McCulloch (1997), Table 4b. The LR statistics are 5.652 and 2.896, respectively.

A simple purchasing-power-parity model of foreign exchange rates would suggest three strong country-specific mass points: At 0° for Germany, at 90° for Japan, and at 45° for the U.S., since the dollar is an equal component of both these exchange rates. There might perhaps also be additional mass at other angles in the first quadrant, reflecting pairwise herd instincts of central bankers (see McCulloch, 1996a).

Pooled ML gives an estimated common α of 1.866. The pooled LR statistic for normality is 4.24, which exceeds the 0.02 Monte Carlo critical value. Figure 7 shows the estimated scale of the projected data, again using $m = 180$ points on the half-circle, with the DM returns at 0° and the Yen returns at 90°. Again, there is a clear maximum in the first quadrant at 44° and a minimum in the second quadrant at 135°.

Figure 8 shows the estimated symmetric spectral density. Here, there are two clear spikes near 13° and 68°, curiously offset from the axes. Figure 9 accumulates the spectral density of Figure 8 into an estimate of the spectral measure. The total mass lying in a group of adjacents bins is subject to much less sampling error than the mass in an individual bin.

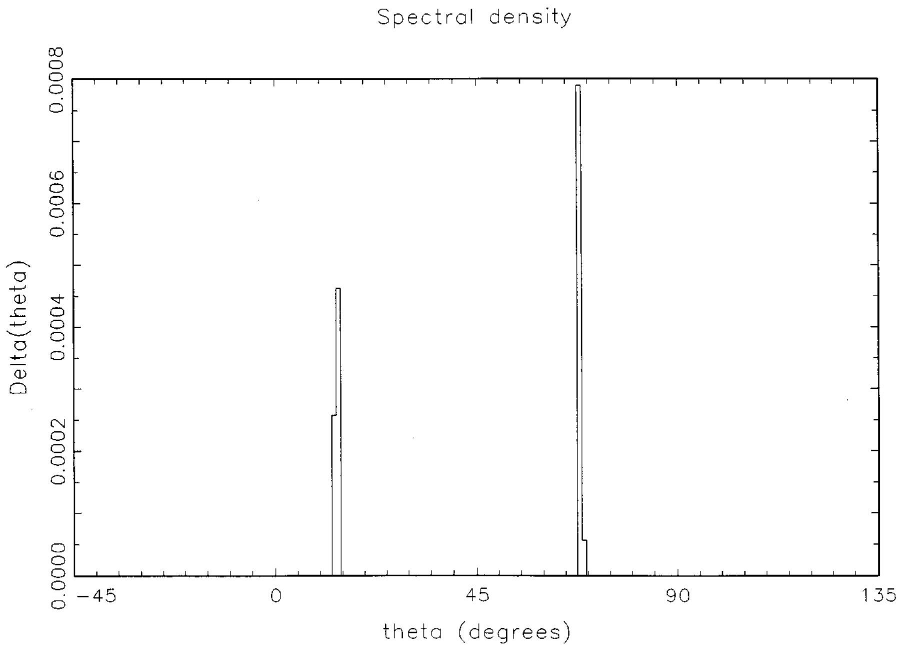

Figure 8. Estimated symmetrical spectral density $\Delta^*(\theta)$ for foreign exchange data of Figure 6.

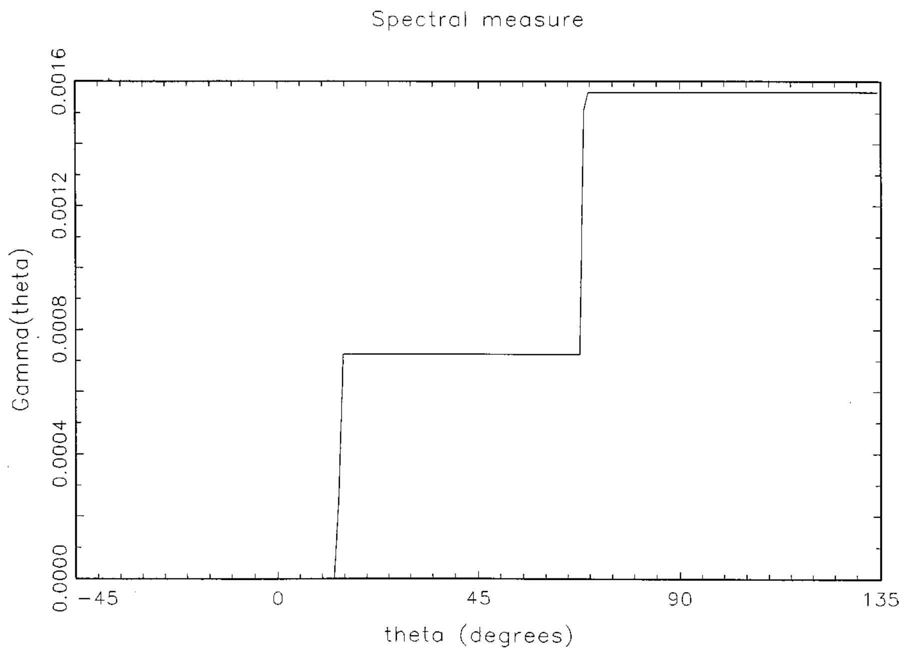

Figure 9. Estimated symmetrical spectral measure $\Gamma^*(\theta)$ for foreign exchange data of Figure 6.

The Kanter projection coefficients computed from the spectral density of Figure 8 are

$$\hat{\boldsymbol{\kappa}}_{\text{Yen, DM}} = 0.613$$
$$\hat{\boldsymbol{\kappa}}_{\text{DM, Yen}} = 0.633.$$

These are somewhat higher than the corresponding OLS regression coefficients obtained from the same data, namely 0.561 (s.e. = 0.046) and 0.599 (s.e. = 0.049), respectively. If the true distribution is stable, the Kanter coefficients are the more efficient measures of the conditional expectation coefficients.

The fact that we do not find atoms at the three expected primary angles is curious and deserves further research. Daily data with a greatly enlarged sample size would perhaps shed further light on this. However, daily foreign-exchange-rate data typically exhibit pronounced GARCH-like volatility-clustering effects that violate the i.i.d. assumption of our procedure. Correcting for such effects goes beyond the scope of the present paper.[7]

All estimation calculations for the simulated data ($n = 10\,000$) required 1110 seconds on a Pentium 100 processor using GAUSS 3.2.12 and the symmetric stable numerical density approximation of McCulloch (1997c). Pooled maximum likelihood estimation of the univariate scales, locations, and common exponent required 190 seconds using the Nelder-Mead polytope (downhill simplex) method. Estimating the 180 scales of the projected data required 875 seconds using a golden ratio hill-climbing procedure. The quadratic program required another 45 seconds. The first step should be roughly proportional to the sample size, and the second to m times the sample size. The third step is independent of the sample size but should increase with m. All calculations for the foreign-exchange-rate data ($n = 301$) required 49 seconds. These times could easily be improved using more efficient maximization routines. The computational burden is minor with the sample sizes and resolution employed in these examples.

6. Multivariate Stable Distributions

The bivariate method described in Section 3 above may be straightforwardly extended to the d-dimensional multivariate case by approximating the stable spectral measure defined on the unit sphere in d-space by a discrete measure in which a large but finite number of points on the unit sphere represent a small adjacent region (Modarres and Nolan, 1992). This may be implemented most efficiently by repeated geodesic triangulation, or, with little loss of efficiency and a great gain in simplicity, by rectangular patches in polar coordinates with approximately equal angular height and width. In either case, the number of unknowns is proportional to m^{d-1}, where $2\pi/m$ is the average angular distance between points. This quickly becomes unmanageable without special restrictions or programming considerations. The case $d > 2$ is not attempted here.

7. Conclusion

The projection method developed here, when coupled with the quadratic programming approach, provides a useful tool for the estimation of the spectral measure of bivariate non-Gaussian random variables and, in particular, of financial asset returns. The estimated spectral density is remarkably sharp and accurate with large simulated samples and produces good estimates of the Kanter projection coefficients that govern conditional expectations. It provides an informative characterization of the joint distribution of real-world data, as illustrated here with foreign exchange rate return data.

Notes

[1] In the ‘afocal’ cases with $\alpha = 1$ and $\beta \neq 0$, the location parameter is not additive, and the formulas (2) and (3) require modification. See McCulloch (1996b). These cases go beyond the scope of the present paper.

[2] Because $c(\omega)$ and $\beta(\omega)$ are continuous, the estimates for each j are good initial values for $j+1$.

[3] The matrix $\mathbf{A}$ is cyclic, as is $\mathbf{A}^{-1}$, i.e., each row is an image of the row above, offset by one, and wrapped. The inverse may therefore be stored as its first row, thus alleviating storage constraints with large m. The same inverse works for every problem involving the same α and m.

[4] This quadratic program projection method was first proposed by the author in an earlier draft of the present paper, and first implemented by John Nolan and Anna Panorska in an early draft of Nolan, Panorska and McCulloch (1996). The latter paper compares it to a characteristic function-based procedure, as well as to the Mittnik and Rachev procedure.

[5] Program STABLE.EXE (Nolan, 1998) now computes the general stable density and does univariate stable ML by a spline numerical approximation. However, this program is not yet available in a form compatible with GAUSS.

[6] Data source is the June 1998 International Financial Statistics CD-ROM, published by the International Monetary Fund. Month-end values for March 1973–April 1998 were obtained from series 134..AE.ZF... and 158..B..Z... (the April 1973 returns use the March 1973 exchange rates in their computation).

[7] McCulloch (1985) models bond returns with a univariate GARCH-stable process.

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