J. Huston McCulloch*
Boston College

Continuous Time Processes with Stable Increments

It is well known that the continuous time sample paths of variables with stable increments contain infinitely many discontinuities. This paper derives the distribution of the largest discontinuity in the intermediate stable case and of all the discontinuities in the "worst" Cauchy case. The discontinuities prove not unmanageable and even economically attractive. The greater "cohesion" of stable processes actually makes arbitrage easier than with a diffusion process. Furthermore, the infinite expected returns with log-stable price changes are theoretically compatible with finite asset prices. We conclude that the stable assumption should be given serious attention in financial model building.

Benoit Mandlebrot, writing in this *Journal* in 1963, set off a burst of interest among financial economists in the symmetric stable Paretian distribution as a model for price changes. These distributions have the convenient and appealing property that when two random variables drawn from stable distributions having the same characteristic exponent α are added together, the resulting sum will have the same-shaped distribution, though with a different location and a wider spread. The normal distribution is a limiting case, with α equal to its maximum permissible value, 2. When α is less than 2, these distributions exhibit the fat tails or leptokurtosis that often characterizes price movements. This property makes them attractive for financial economics, in spite of the fact that their variance is infinite, along with all absolute moments greater than the αth.[1] Fama and Roll (1968, 1971) have gone to great lengths to tabulate the intermediate stable symmetric distributions and to establish simple procedures for estimating their parameters, with an

* Work on this paper was supported by a Scherman Research Fellowship at NBER-West. The author wishes to thank William H. DuMouchel, Michael Harrison, E. Philip Jones, Jr., and Merton H. Miller for helpful guidance and comments, and to absolve all of these individuals from responsibility for the views and any errors contained herein.

1. When α falls as low as 1 the Cauchy distribution results, which does not even have a mean. Stable distributions exist with $0 < \alpha < 1$, but the mean remains undefined. In financial applications it is universally assumed that $1 < \alpha \leq 2$.

(*Journal of Business,* 1978, vol. 51, no. 4)

0021-9398/78/5104-0001$01.51

eye to financial applications. DuMouchel (1973, 1975) has developed more sophisticated maximum-likelihood techniques for estimating these parameters. Stuck (1976) has further investigated the properties of the likelihood ratio.

However, the initial enthusiasm for these intermediate stable symmetric distributions has waned, in part because of their frightening properties in a continuous time context. It is well known that if a continuous time stochastic process has serially independent stable increments, in order for its sample path to be almost surely everywhere continuous the increments must be normal, that is, α must equal 2. If α is less than 2, any finite time interval almost surely contains an infinite number of discontinuities. Such a process does not seem natural, because, as Leibnitz (and Marshall after him) argued, *"Natura non facit saltus."* For this and other reasons to be discussed below, economists interested in continuous time applications have shunned intermediate stable processes in favor of "diffusion processes" that have normal increments and therefore continuous sample paths.

It is our contention in this paper that this aversion to continuous time stochastic processes with stable increments is misplaced. The discontinuities that result are not unmanageable statistically. In fact, the sample paths of these processes, in spite of the discontinuities, are actually in a sense more "cohesive" than those of diffusion processes. Furthermore, the discontinuities are not unreasonable economically. Nature may have continuous sample paths, but society, and the exchange economy in particular, is not part of "nature."[2] On the contrary, it is the interaction of millions of man-made and woman-made "unnatural" phenomena. There is therefore no reason why prices should not take leaps.

The Cauchy Distribution

Let us start with the "worst" case, namely, that of the Cauchy distribution, for which $\alpha = 1$. For median zero and "standard scale" c, a random variable with this distribution has the cumulative distribution function

$$F(x; c) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{c}\right) \tag{1}$$

and probability density function

$$f(x; c) = \frac{1}{\pi c\left[1 + \left(\frac{x}{c}\right)^2\right]}. \tag{2}$$

2. This is a point long insisted on by F. A. Hayek (1952, passim).

For a Cauchy distribution, the standard scale is exactly the semi-interquartile range. The probability that the absolute value of such a variable will be less than x is therefore

$$G(x; c) = \frac{2}{\pi} \arctan\left(\frac{x}{c}\right), \tag{3}$$

which has probability density function

$$g(x; c) = \frac{2}{\pi c\left[1 + \left(\frac{x}{c}\right)^2\right]}. \tag{4}$$

Now consider an interval over which the increment in a continuous time Cauchy process has standard scale c, which is to say that the median absolute increment is c. The standard scale of the sum of two Cauchy variables is the sum of their standard scales, so if we divide this interval into n equal subintervals, the increment across each subinterval will have standard scale c/n. The probability that *all* of these increments will be less than x in absolute value is $G(x; c/n)^n$. Therefore $\Phi(x; c)$, the probability that the *largest* discontinuity (in absolute value) is less than x, is

$$\begin{aligned} \Phi(x; c) &= \lim_{n\to\infty} G(x; c/n)^n \\ &= \lim_{n\to\infty} \left(\frac{2}{\pi} \arctan \frac{nx}{c}\right)^n. \end{aligned} \tag{5}$$

Setting $\nu = 1/n$, and using L'Hôpital's rule, we have

$$\begin{aligned} \log \Phi(x; c) &= \lim_{n\to\infty} n \log\left(\frac{2}{\pi} \arctan \frac{nx}{c}\right) \\ &= \lim_{\nu\to 0} \frac{\log\left(\frac{2}{\pi} \arctan \frac{x}{\nu c}\right)}{\nu} \\ &= \lim_{\nu\to 0} \frac{d}{d\nu} \log\left(\frac{2}{\pi} \arctan \frac{x}{\nu c}\right) \\ &= \lim_{\nu\to 0} \frac{(2/\pi)\{1/[1 + (x/\nu c)^2]\}(-x/c\nu^2)}{2/\pi \arctan x/\nu c}. \end{aligned}$$

The denominator of this expression goes to unity, so

$$\begin{aligned} \log \Phi(x; c) &= \lim_{\nu\to 0} \frac{-2x}{\pi c[\nu^2 + (x/c)^2]} \\ &= \frac{-2c}{\pi x}, \end{aligned}$$

whence

$$\Phi(x; c) = e^{-(2c/\pi x)}. \tag{6}$$

We see from equation (6) that the fractiles of the largest discontinuity in an interval are proportional to the standard scale of the total Cauchy increment. In fact, the median largest discontinuity is $0.918c$, so that the median largest discontinuity is actually smaller than c, the median absolute total Cauchy increment over the interval.[3]

We can also show that the fractiles of the kth largest discontinuity fall toward zero as k becomes large. We again divide the entire interval into n subintervals and consider the distribution of the jth largest increment. The probability that some $j-1$ of the n increments will be *larger* than ξ is $[1-G(\xi; c/n)]^{j-1}$. The probability that all the others will be *smaller* than ξ is $G(\xi; c/n)^{n-j+1}$. The probability that the *largest* of these others is between ξ and $\xi + d\xi$ is $dG^{n-j+1} = (n-j+1)G^{n-j}gd\xi$. There are $\binom{n}{j-1}$ ways in which we can select the $j-1$ intervals in which the $j-1$ largest increments occur. Therefore the probability that the jth largest increment is less than x is $\binom{n}{j-1}\int_0^x (1-G)^{j-1}dG^{n-j+1}$, and $\Phi^{(j)}(x; c)$, the probability that the jth largest discontinuity is less than x, is

$$\Phi^{(j)}(x; c) = \lim_{n\to\infty} \binom{n}{j-1} \int_0^x (1-G)^{j-1}dG^{n-j+1}. \qquad (7)$$

Now consider the difference

$$\Delta_{j+1} = \Phi^{(j+1)} - \Phi^{(j)} \qquad (8)$$

$$= \lim_{n\to\infty} \left[\binom{n}{j}\int_0^x (1-G)^{j}dG^{n-j} - \binom{n}{j-1}\int_0^x (1-G)^{j-1}dG^{n-j+1}\right]. \qquad (9)$$

Performing integration by parts on the first integral in (9) with $u = (1-G)^j$ and $dv = dG^{n-j}$, we have

$$\Delta_{j+1} = \lim_{n\to\infty} \left\{\frac{n^{(j)}}{j!}\left[(1-G)^j G^{n-j} - \int_0^x (1-G)^{j-1}(-j)G^{n-j}gd\xi\right]\right.$$

$$\left. - \frac{n^{(j-1)}}{(j-1)!}\int_0^x (1-G)^{j-1}(n-j+1)G^{n-j}gd\xi\right\}$$

$$= \frac{1}{j!}\lim_{n\to\infty} \left\{n^{(j)}(1-G)^j G^{n-j}\right. \qquad (10)$$

$$\left. + [jn^{(j)} - jn^{(j-1)}(n-j+1)]\int_0^x (1-G)^{j-1}G^{n-j}gd\xi\right\}$$

$$= \frac{1}{j!}\lim_{n\to\infty} [n^{(j)}(1-G)^j G^{n-j} + 0],$$

since $(n-j+1)n^{(j-1)} = n^{(j)}$. Now, from (5) and (6),

$$\lim_{n\to\infty} G^{n-j} = \lim_{n\to\infty} G^n = e^{-(2c/\pi x)}. \qquad (11)$$

3. It can be shown that the limit as x approaches infinity of the ratio $(1-G)/(1-\Phi)$ is unity. Thus, an unusually large total Cauchy increment consists of a comparably large largest single discontinuity.

Furthermore, for $\nu = 1/n$,

$$\begin{aligned}
\lim_{n\to\infty} (n-i)(1-G) &= \lim_{n\to\infty} n[1-G(x;\, c/n)] \\
&= \lim_{\nu\to 0} \frac{1-G(x;\, \nu c)}{\nu} \\
&= \lim_{\nu\to 0} \frac{d}{d\nu}[1-G(x;\, \nu c)] \\
&= \lim_{\nu\to 0} \left(-\frac{2}{\pi}\right) \frac{1}{1+(x/\nu c)^2} \left(\frac{-x}{\nu^2 c}\right) \\
&= \frac{2}{\pi} \lim_{\nu\to 0} \frac{x}{\nu^2 + x^2/c} \\
&= \frac{2c}{\pi x}.
\end{aligned} \tag{12}$$

Therefore,

$$\Delta_{j+1} = \frac{1}{j!}(2c/\pi x)^j\, e^{-(2c/\pi x)}. \tag{13}$$

Given the definition of Δ_{j+1} in (8), we have

$$\begin{aligned}
\Phi^{(k)}(x;\, c) &= \Phi(x;\, c) + \sum_{j=1}^{k-1} \Delta_{j+1} \\
&= \sum_{j=0}^{k-1} \left[\frac{1}{j!}(2c/\pi x)^j\right] e^{-(2c/\pi x)}.
\end{aligned} \tag{14}$$

Thus, the distributions of the second and third largest discontinuities are

$$\Phi^{(2)}(x;\, c) = \left(1 + \frac{2c}{\pi x}\right) e^{-(2c/\pi x)} \tag{15}$$

and

$$\Phi^{(3)}(x;\, c) = \left[1 + \frac{2c}{\pi x} + \frac{1}{2!}\left(\frac{2c}{\pi x}\right)^2\right] e^{-(2c/\pi x)}, \tag{16}$$

which have medians $0.379c$ and $0.238c$, respectively. The distributions of G, Φ, $\Phi^{(2)}$, and $\Phi^{(3)}$ are shown in figure 1.

Equation (14) has the important implication that

$$\begin{aligned}
\lim_{k\to\infty} \Phi^{(k)}(x;\, c) &= e^{2c/\pi x}\, e^{-(2c/\pi x)} \\
&= 1,
\end{aligned} \tag{17}$$

for all $x > 0$. This equation demonstrates that even though there are, with probability 1, an infinite number of discontinuities on any interval,

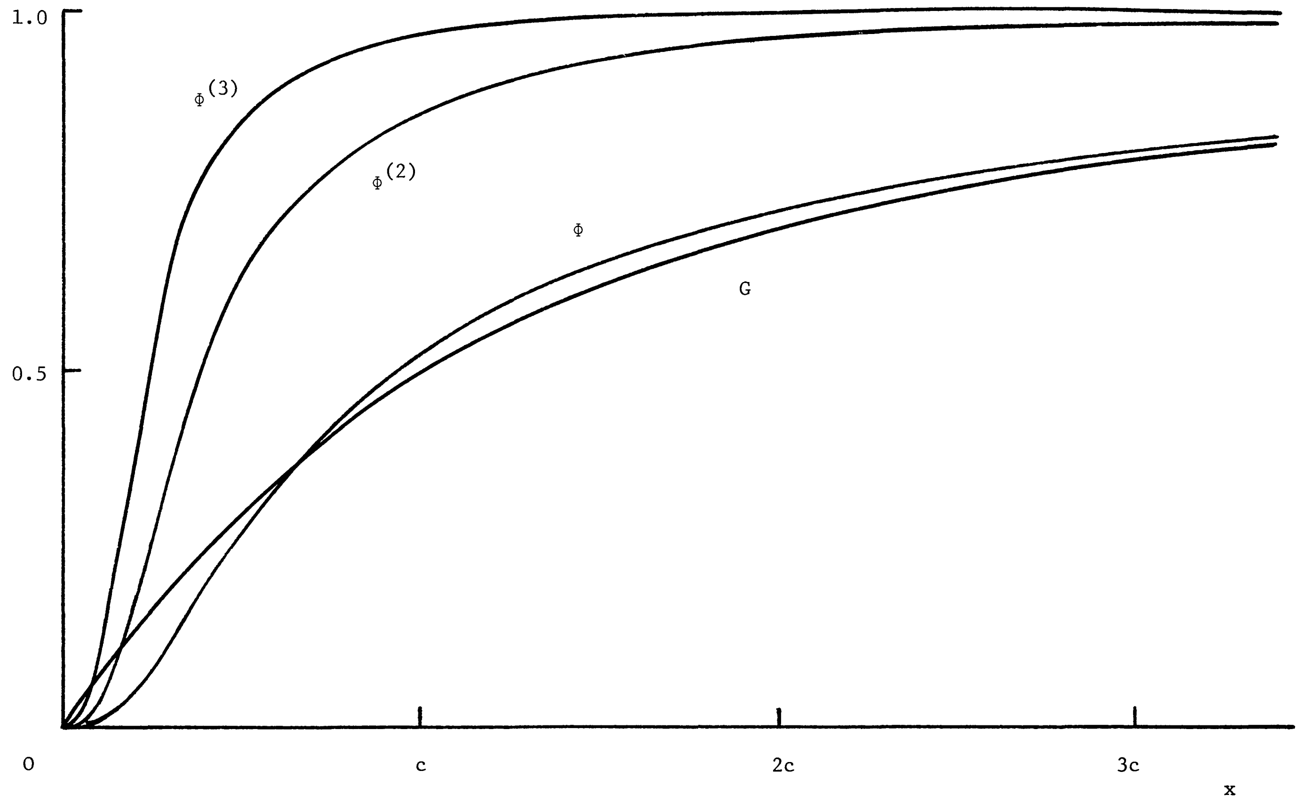

FIG. 1.—Cauchy process: G, Φ, $\Phi^{(2)}$, and $\Phi^{(3)}$ represent, respectively, the probability, over an interval with cumulative standard scale c, that the absolute cumulative increment and three largest discontinuities will be less than x.

they vanish in size. Furthermore, since c is proportional to the length of the interval, the size of the discontinuities we expect to find is also proportional to the length of the interval, and therefore they go to zero as the length of the interval goes to zero. Thus the continuous time Cauchy process is everywhere almost surely continuous, even though it is not almost surely everywhere continuous. To illustrate this distinction, suppose we throw a countable infinity of darts at a board. Although the darts will not (with probability 1) hit any particular point, they will (with probability 1) hit some point in any region with positive area; in fact, an infinite number of points. Although any point chosen randomly in time is almost surely not a discontinuity in the process, it is almost surely a limit point of discontinuity points, but whose jumps almost surely get smaller and smaller, approaching zero, as the point in question is approached. There are an infinite number of points for which this is not true, but they form a set of measure zero.

Figure 2 shows a computer-generated zero-median Cauchy random walk with small time increments, approximating continuous time. The unit time interval indicated was divided into 1,000 increments. (Equations [5] and [7] for $j = 2$ and 3 were found to converge well by $n = 1{,}000$.) For each time increment a pseudorandom number was generated and transformed in accordance with (1) to produce a Cauchy random variable. The standard scale c for each drawing was set equal to 1/1,000 so that the median cumulative Cauchy increment at the end of the unit time interval is unity in terms of the units indicated on the vertical axis. Only every tenth point was actually plotted in figure 2.

The three largest "discontinuities" in absolute value were -1.308, 0.516, and -0.147, at $t = 0.941$, 0.992, and 0.115, respectively. (Recall the median values to be expected were 0.918, 0.379, and 0.238.) Their size fell off quickly, with only seven of the 1,000 individual drawings greater than 0.100. The total cumulative Cauchy increment over the entire unit interval was -1.020, not far in absolute value from the median to be anticipated of 1.000. Most of this increment is due to a few of the largest discontinuities. The three largest sum algebraically to -0.940, and the seven largest to -1.1534.

The Intermediate Symmetric Stable Case

Figure 2 does not much resemble a price series, and indeed few proponents of nonnormal stable distributions believe that α is as low as 1 in economic applications. Roll (1970, pp. 70–71) does find a few cases where his best estimate of α is 1.00 in the context of interest rate movements, but most of his estimates are in the range 1.2–1.5.

There is no simple formula like (1) for the cumulative distribution function for an intermediate symmetric stable random variable. However, Fama and Roll have calculated these distributions by indirect

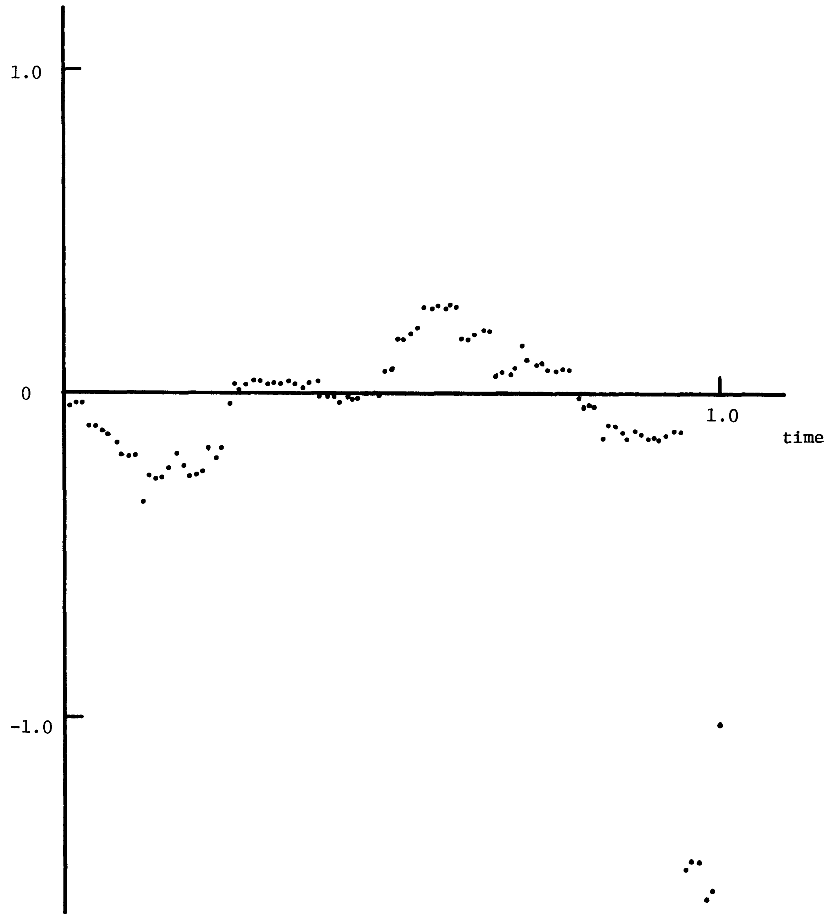

FIG. 2.—A Cauchy process ($\alpha = 1$) with small increments approximating continuous time. After 1.0 time unit, the standard scale of the cumulative increment is indicated by one unit on the vertical scale.

means, and have tabulated the results for several values of α (1968, pp. 820–21).

Figure 3 shows a "continuous time" computer-generated symmetric stable random walk with characteristic exponent 1.5. In order to make figure 3 comparable in scale with figure 2, the standard scale of each of 1,000 increments was set equal to $1{,}000^{-1/1.5}$. Because of the way the standard scale accumulates, this makes the standard scale of the total cumulated stable increment just equal to 1.000 in terms of the units shown on the vertical axis. (For $\alpha = 1.5$, the median absolute increment is only 0.969 times the standard scale rather than exactly 1.000 as with the Cauchy, but this discrepancy is negligible for our present

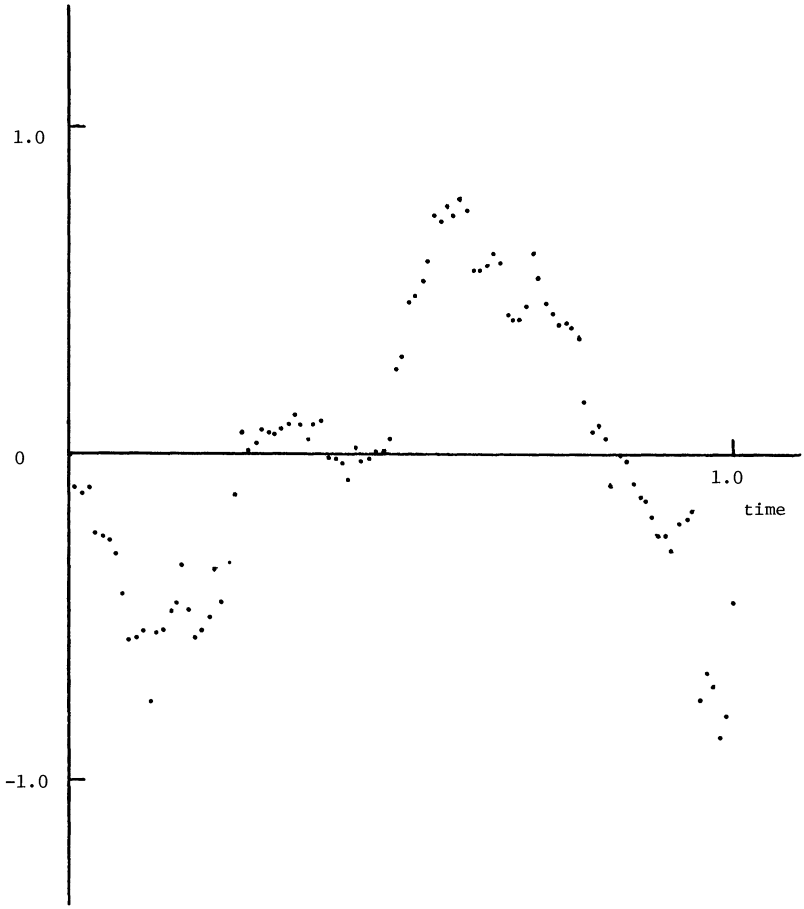

FIG. 3.—A stable process ($\alpha = 1.5$) with small increments approximating continuous time. After 1.0 time unit, the standard scale of the cumulative increment is indicated by one unit on the vertical scale.

purposes.) Again, only every tenth point of the 1,000 underlying points is actually plotted.

In this case the three largest "discontinuities" were −0.478, 0.354, and −0.210. Since the same seed was used for the random number generator, these occurred at the same points in time, namely, 0.941, 0.992, and 0.115. Fourteen of the 1,000 random increments were greater than 0.100, as contrasted with only seven in the Cauchy case.[4] The total increment is −0.457, whose absolute value is only the 28.9th

4. It was necessary to extrapolate somewhat from the Fama-Roll table to obtain our value for the largest "discontinuity" in fig. 3. The figure is nevertheless adequate for illustrative purposes.

percentile of the distribution of absolute increments with unit standard scale. In spite of using underlying "events" with exactly the same probabilities, we end up with a different fractile when we add up the 1,000 increments, because the stable 1.5 distribution produces bigger increments for the medium-likelihood events and relatively smaller increments for the highly unusual events.

Figure 3 looks far more like an economic time series than figure 2. Even though it is "full" of discontinuities, most of them are quite small, and even the large ones do not offend the eye. It does a lot of wandering that is not due to the large discontinuities. The three largest discontinuities here sum (algebraically) to -0.324, and the 14 largest to -0.530. Yet the process drifts as high as 0.885 and as low as -0.913, a total spread of 1.798.

Even though the cumulative distribution function $F_\alpha(x; c)$ for intermediate symmetric stable distributions is not known, we can derive the distribution $\Phi_\alpha(x; c)$ of the largest discontinuity that occurs in an interval over which standard scale c accumulates, using only what we know about how the standard scale accumulates. It can be shown from the (known) characteristic function for stable symmetric distributions that when two such independent variables having the same characteristic exponent α are added together, the standard scale of the sum, raised to the α power, is the sum of the α powers of the standard scales of the two variables added. By extension to continuous time, if c_0 is the standard scale that accumulates in 1.0 time units, the standard scale after t time units will be

$$c(t) = c_0 t^{1/\alpha}. \tag{18}$$

This rule is illustrated in figure 4 for $\alpha = 1$ (Cauchy), 2 (normal), and 1.5, where the standard scales are chosen so that in each case unit standard scale accumulates after one time unit.

Consider a process with standard scale c after one time unit. After n time units, it will have standard scale $n^{1/\alpha}c$, so the probability that the largest discontinuity in n time units will be less than x is $\Phi_\alpha(x; n^{1/\alpha}c)$. However, this probability is also the nth power of the probability $\Phi_\alpha(x; c)$ that the largest discontinuity in each one-unit time increment is less than x. Furthermore, $F_\alpha(x; c)$ and therefore $G_\alpha(x; c)$ and $\Phi_\alpha(x; c)$ are homogeneous of degree zero in x and c. Therefore $\Phi_\alpha(xn^{-1/\alpha}; c) = \Phi_\alpha(x; n^{1/\alpha}c) = \Phi_\alpha(x; c)^n$. Similarly, by dividing the n time units into m equal intervals, we have

$$\Phi_\alpha(xr^{-1/\alpha}; c) = \Phi_\alpha(x; c)^r, \tag{19}$$

for any positive rational $r = n/m$. By continuity, (19) is also valid for any positive real r, whether rational or not. Setting $\lambda = r^{-1/\alpha}$, we have

$$\log \Phi_\alpha(\lambda x; c) = \lambda^{-\alpha} \log \Phi_\alpha(x; c), \ \lambda > 0. \tag{20}$$

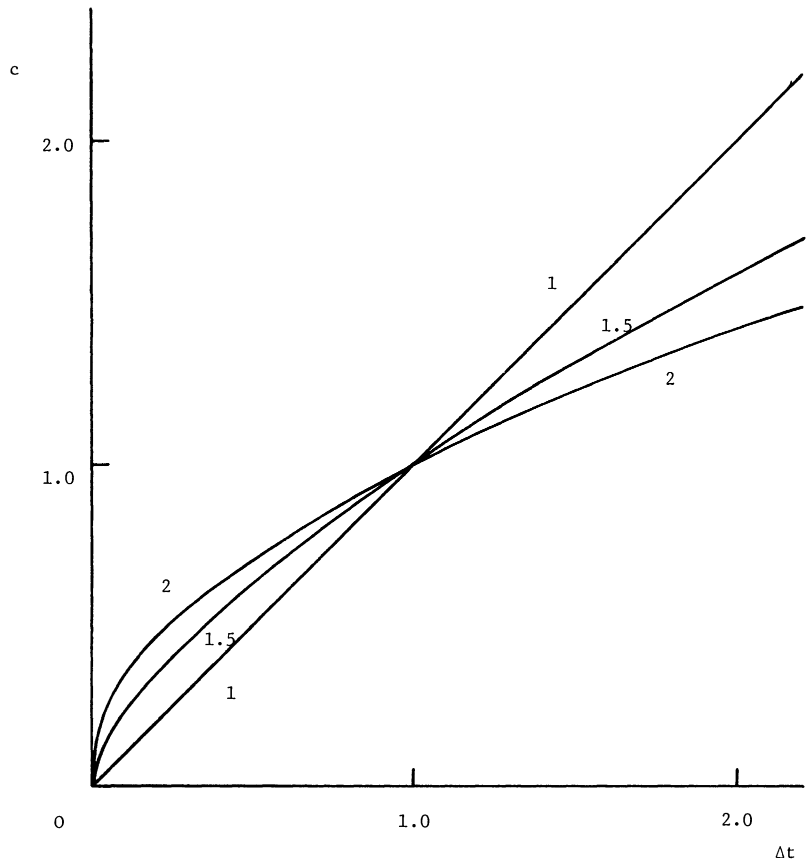

FIG. 4.—The standard scale c of the increment that accumulates over an interval Δt in length, given unit standard scale after 1.0 time unit, for Cauchy, stable 1.5, and normal processes.

Equation (20) states that the logarithm of Φ_α is homogeneous of degree $-\alpha$ in x, which implies that it must have the functional form

$$\log \Phi_\alpha(x;\, c) \;=\; \log \Phi_\alpha\!\left(\frac{x}{c};\, 1\right) \;=\; -k_\alpha (x/c)^{-\alpha} \tag{21}$$

for some (necessarily positive) constant k_α dependent on α. Therefore the probability that the largest discontinuity in an interval with standard scale c is less than x must have the functional form

$$\Phi_\alpha(x;\, c) \;=\; e^{-k_\alpha (c/x)^\alpha}.[5] \tag{22}$$

5. The distribution in (22) is a log gompit. Gray (1970) uses it as a functional form to describe military enlistment rates as a function of military pay relative to civilian pay. The gompit distribution $\exp\,[-a \cdot \exp\,(-bx)]$ for positive a and b is named after a Mr. Gompertz. Cf. also the Gumbel distribution.

We know from (6) that $k_1 = 2/\pi$. Furthermore, since a normal continuous time process is almost surely everywhere continuous, we must have $\Phi_2(x; c) = 1, x > 0$, so that $k_2 = 0$. Intermediate values of k_α may be calculated using the formula

$$\begin{aligned}\Phi_\alpha(x;\ 1) &= \lim_{n\to\infty} G_\alpha(x;\ n^{-1/\alpha})^n \\ &= \lim_{n\to\infty} G_\alpha(xn^{1/\alpha};\ 1)^n,\end{aligned} \tag{23}$$

where

$$\begin{aligned}k_\alpha &= -\log \Phi\ (1;\ 1) \\ &= -\lim_{n\to\infty} n \log G_\alpha(n^{1/\alpha};\ 1),\end{aligned} \tag{24}$$

in conjunction with the lead term of Bergstrom's asymptotic expansion for the stable cumulative distribution function (Fama and Roll 1968, p. 819; DuMouchel 1973, p. 471):

$$F_\alpha(x;\ 1) = 1 - \frac{1}{\pi}\Gamma\ (\alpha) \sin\left(\frac{\pi\alpha}{2}\right)x^{-\alpha} + O(x^{-2\alpha}),\ x\to\infty. \tag{25}$$

The order operator $O(x^{-2\alpha})$ indicates an unspecified function $f(x)$ such that $\lim_{x\to\infty} [f(x)/x^{-2\alpha}]$ is bounded. Equation (25) implies that $G_\alpha(n^{1/\alpha}; 1) = 1 - \frac{2}{\pi}\Gamma\ (\alpha) \sin\left(\frac{\pi\alpha}{2}\right)n^{-1} + O(n^{-2})$, so that

$$\begin{aligned}k_\alpha &= \lim_{n\to\infty}\left[\frac{2}{\pi}\Gamma\ (\alpha) \sin\left(\frac{\pi\alpha}{2}\right) - O(n^{-1})\right] \\ &= \frac{2}{\pi}\Gamma\ (\alpha) \sin\left(\frac{\pi\alpha}{2}\right).\end{aligned} \tag{26}$$

Table 1 tabulates selected values of k_α, employing Dwight's tabulation (1941, pp. 208–9) of the gamma function. Since $\Gamma(1) = \Gamma(2) = 1$, and since the gamma function does not drop below 0.8856 on the interval (1, 2) our constant behaves very much like the sine function. Also indicated are the median $x_{0.5}$ and the ninety-ninth percentile $x_{0.99}$ of $\Phi_\alpha(x;\ 1)$, computed by inverting (22). We see from table 1 that the median largest discontinuity declines monotonically as the characteristic exponent increases. Furthermore, the α exponent in (22) implies that the distribution will be more concentrated about its median the higher α is. Therefore the ninety-ninth percentiles fall off even more rapidly than the medians.

Equation (22) has two ready implications that help one understand the nature of the discontinuities. Let c_0 be the standard scale of the

TABLE 1 **Selected Values of k_α and Fractiles of Φ_α**

α	k_α	$x_{.5}$	$x_{.99}$
1.0	.6366	.918	63.34
1.1	.5982	.875	41.05
1.2	.5559	.832	28.34
1.3	.5091	.789	20.48
1.4	.4570	.743	15.28
1.5	.3989	.692	11.64
1.6	.3344	.634	8.94
1.7	.2626	.565	6.82
1.8	.1832	.478	5.02
1.9	.0958	.353	3.28
1.95	.0489	.257	2.25
1.99	.0100	.119	1.00
1.999	.0010	.038	.32
2.0	0	0	0

total stable increment that accumulates by the end of 1.0 time units. Substituting (18) into (22), we find that the probability that no discontinuity greater than x_0 will have occurred in t time units is given by the exponential distribution, with average rate of occurrence of discontinuities larger than x_0 equal to $k_\alpha(c_0/x_0)^\alpha$:

$$\Phi_\alpha[x_0; c(t)] = e^{-k_\alpha(c_0/x_0)^\alpha t}. \tag{27}$$

Therefore the appearance of discontinuities larger than the threshold x_0 is governed by a Poisson-driven process.

We can also find the distribution of the size of these Poisson-driven "important" discontinuities. From (22) we can easily show that the probability that the largest discontinuity in an interval with cumulative standard scale c is smaller (in absolute value) than x, given that it is larger than x_0, is

$$\frac{e^{-k_\alpha(c/x)^\alpha} - e^{-k_\alpha(c/x_0)^\alpha}}{1 - e^{-k_\alpha(c/x_0)^\alpha}}. \tag{28}$$

This formula is complicated by the possibility that there may be more than one discontinuity greater than x_0 in the interval. Note, however, that every discontinuity greater than x_0 is the *only* discontinuity larger than x_0, for some sufficiently short time interval. Therefore, in order to find the distribution of a single discontinuity larger than x_0, we merely take an interval so short that c, in accordance with (18), goes to zero. Taking this limit in (28) with the help of L'Hôpital's rule, we find that the distribution of single discontinuities larger than x_0 is given by the simple formula

$$1 - \left(\frac{x_0}{x}\right)^\alpha \tag{29}$$

for x greater than x_0, and by zero for x less than x_0. This is a Pareto (not to be confused with Paretian stable) distribution which, as is well known, has expected value $\alpha x_0/(\alpha-1)$.

The Normal Case

Figure 5 shows a quasi-continuous time stochastic process with normal increments. Each of the 1,000 increments was given a standard scale of $1/1{,}000^{1/2}$, so that the total standard scale at the end of the unit time period would be unity. Note that for a normal distribution the standard

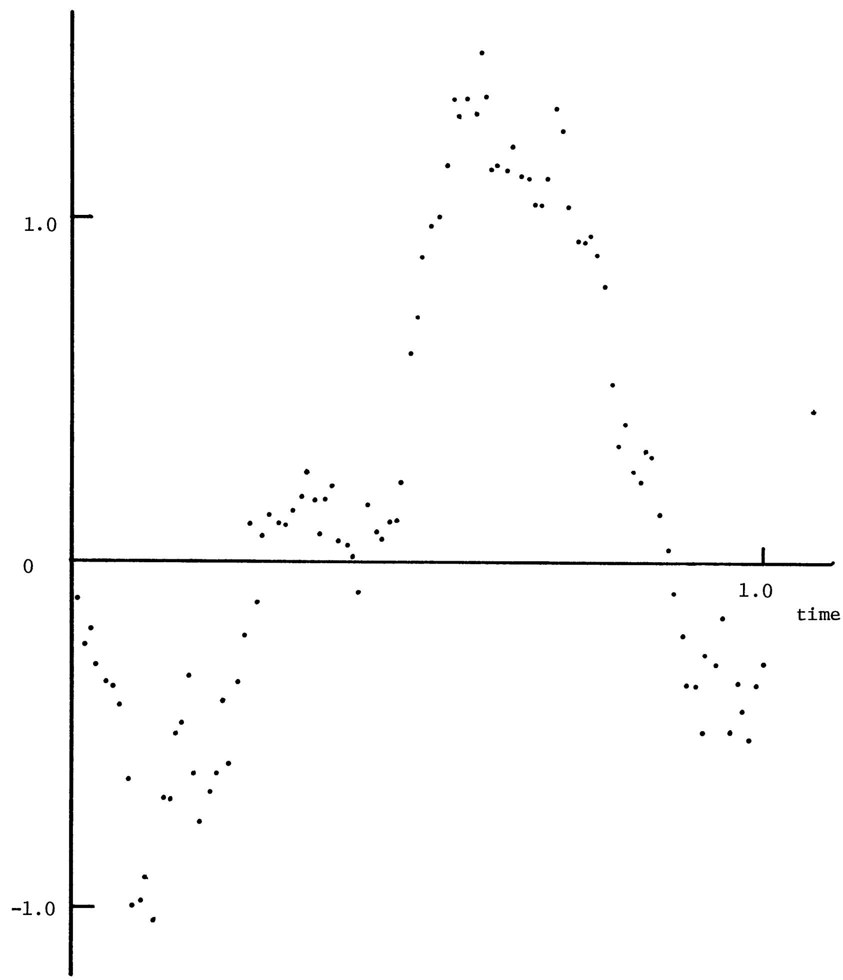

FIG. 5.—A normal process (α = 2) with small increments approximating continuous time. After 1.0 time unit, the standard scale of the cumulative increment is indicated by one unit on the vertical scale.

scale c equals the standard deviation σ divided by the square root of 2 (i.e., c^2 is half the variance). A little over half, 52.04% to be more precise, of the density lies within c units of the mean, so the standard scale almost exactly equals the old-fashioned "probable error," or semi-interquartile range. We standardized figure 5 in terms of this standard scale rather than in terms of the more familiar standard deviation in order to make it readily comparable with figures 2 and 3. Again, only every tenth point is actually plotted here.

Of the 1,000 increments making up this random walk, the three largest were −0.156, 0.145, and −0.128, again at t = 0.941, 0.992, and 0.115. Thirty of the 1,000 increments were greater than 0.100, compared with 14 for the intermediate stable process and seven for the Cauchy process. The increments decline much more slowly in size; 87 are greater than 0.080, compared with 25 for the intermediate stable and only nine for the Cauchy.

The Economic Appeal of Stable Processes

Comparing figures 2, 3, and 5, we see that the intermediate stable and Cauchy processes, in spite of their discontinuities, seem in some sense to be more cohesive than the normal. Except at the big discontinuities, the points stick together better than they do in the normal process, which appears fuzzy. This is because of the way the increments accumulate, as indicated already in figure 4. We see from this figure that, for multiples of the unit time interval, the smaller α is, the faster the standard scale of the accumulated increments goes to infinity, so in that sense the Cauchy and intermediate stables are more poorly behaved than the normal. But the same "$1/\alpha$" rule implies that when we consider shorter and shorter intervals they are actually better behaved; the smaller α is, the faster the standard scale of the subincrements goes to zero. This implies that as the length of a given time interval goes to zero, the probability of a given size stable increment eventually becomes smaller than the probability of the same-sized normal increment. This is why the nonnormal processes seem—and actually are in this objective sense—more cohesive. Furthermore, this cohesion is actually greater the further from the normal case we go. We are still left with the discontinuities. But they are, for any given point, events with probability zero. In spite of the large size of a few of them, they do not interfere with cohesion, because of the rapidity with which the distribution of the kth largest one converges on zero as k increases.

The discontinuities are actually appealing in their own right, for a price series. We must consider first that transactions prices are not really defined in continuous time, since only a finite number of transactions take place in any time interval. Buying and selling offers, on the other hand, do exist continuously in time, so by a "continuous time price series'' we must mean a bid or an asked or a bid-asked-mean price offer series, rather than an actual transaction price series.[6] However, these offers are, in fact, discontinuous. When an extremal offer is taken up, withdrawn, or supplanted by a better offer, the ''price'' of the security or commodity in question changes discontinuously the instant the alteration occurs. Most of these discontinuities are very small, but then so are almost all of the discontinuities in a stable process. Occasionally some important news arrives, causing a larger discontinuity.

One of the foremost advocates of normal increment diffusion processes, Robert C. Merton, has pointed out in a recent paper (1976) that the continuity of a pure diffusion process is actually one of its drawbacks. He suggests that financial models introduce discontinuities by adding to a background diffusion process a Poisson-driven process that provides occasional discontinuities at irregular intervals. We have two reservations about this procedure. First, it does not provide the inherent cohesion of an intermediate stable process. And second, it requires a much more complicated specification than does a pure Paretian process. Merton's specification requires five parameters at the minimum: a mean drift per unit time and a variance per unit time for the diffusion process, an average frequency for the Poisson process that times the discontinuities, and finally a mean and variance for the discontinuities themselves. A continuous time stable process, on the other hand, requires only three parameters: an average drift per unit time, a standard scale at the end of a unit time interval, and a characteristic exponent. At the same time, it preserves the attractive property of Merton's compound hypothesis that there will, with probability 1, be a little bit of change during any positive time interval, no matter how short.

If we were to set the threshhold x_0 in equations (27) and (29) low enough, we could approximate the stable process arbitrarily closely as just the sum of the ''important'' jumps greater than x_0 which, as we have demonstrated above, can be regarded as constituting a special case of the class of Poisson-driven processes that Merton describes. Note, however, that the frequency of the stable-induced jumps and the shape of the distribution of these jumps are simultaneously determined by the underlying stable process, whence the greater economy of parameters.

The Feasibility of Arbitrage

It is sometimes argued that the discontinuities and infinite variances of a stable process rule out the practicality of arbitrage, since it is actually

6. For the same reason, transaction prices are unsatisfactory when we need to establish simultaneous prices of different securities, as, for example, to estimate the term structure of the interest rates (see McCulloch 1971, pp. 20–21).

impossible to buy and sell simultaneously in two markets. In practice an arbitrageur ascertains the price in one market, makes a transaction in a second market, and then quickly returns to the first market to close out his position. He is speculating a little but nevertheless differs from a true speculator in that he closes his position so quickly that he can rely on his actual closing price in the first market being "virtually the same" as the price he originally ascertained. This argument maintains that a continuous sample path (and therefore a normal process) is necessary to make this operation practical. In fact, however, no arbitrageur can act with infinite speed to take advantage of the actual continuity of a normal process. The sort of problem that really confronts him is whether he can act fast enough so that the probability is very high that the price in the first market will not have moved by more than just a little bit. Because of the greater cohesion of stable processes, the speed necessary is actually less for a stable process than for a normal process. Even though a normal process is everywhere continuous, there is no limit to how far the price can move in 30 seconds, say. In fact, for most of the sizes of change that are ever likely to be observed, a given-sized normal increment will have a higher probability. It is only for exceedingly large (and rare) rates of change that the stable assumption predicts higher probabilities. But then there actually are, in the real world, instants when news of war or assassination reaches the trading floor, and prices move by 5% or 10%, if trading is not halted altogether (an infinite change in price!).[7]

Consider the stable and normal processes of figures 3 and 5. There are roughly 1,000 business hours in a ½-year period, so we can consider the time unit as a "½-year" and each of the increments as an "hour" of trading. Suppose that in this market the median absolute price change to be anticipated over 6 months is 10% one way or the other. In the stable process, there was 1 hour in which the price moved by 4.78% but only 14 hours in which it moved by more than 1.00%. In the normal process the largest single hourly change was only 1.56%, but there were 30 hours in which the price changed by more than 1.00%. There were only 13 hours in the entire 6-month period in which the normal price did not change by more than the stable price. The "continuities" of the normal process are almost always larger, for all practical purposes, than the discontinuities of the stable process. In which market is it easier to perform arbitrage? Which seems more realistic?

7. Mandlebrot (1963, p. 417) interprets the discontinuities as spontaneous changes in price that leave supply and demand behind and cites them as justification for closing markets when the price changes by more than a certain amount. I would, rather, interpret them as discontinuous shifts in supply and/or demand that take the price with it. Closing the market simply aggravates the unpredictability of the terms on which the security or commodity can be bought and sold, since it means one cannot sell for any price above zero or buy for any price below infinity.

Log-Stable Processes

One of the most inconvenient implications of a stable process as a model for movements in a speculative price is that if the logarithm of the price has stable increments (a natural assumption to keep the price from ever going negative), the expected value of the price itself at any future time will be infinite. Its median and other fractiles will all exist and be reasonable, but the expected value will not exist.

However, it has been known for centuries that a risky asset can have an infinite expected value and still have a finite market price. In 1738 Daniel Bernoulli solved the most elementary type of "St. Petersburg Paradox" by assuming logarithmic utility. Diminishing marginal utility makes most of these paradoxes disappear, and in 1871 Carl Menger provided economic grounds for believing that indeed marginal utility does, as a rule, diminish.[8] In 1934, C. Menger's son Karl demonstrated that bounded utility is all that is necessary to rule out all conceivable "Super-Petersburg Paradoxes." That is, if utility is bounded, any risky asset with finite fractiles will have a finite certainty-equivalent price.[9] Any particular infinite-expected-value asset may have a finite price even if utility is not bounded.

There is therefore no theoretical reason to reject log-stable price movements on the grounds that they give infinite expected future prices and expected rates of return. However, almost all of finance theory, with only a few exceptions (e.g., Samuelson 1976), is based on the assumption that variances and expected values of rates of return are finite. If prices really do move in a log-stable manner, most of this literature is only of heuristic value, with little rigorous applicability to the real world. In particular, the global shape of the utility function may become relevant, instead of just its local properties.

Conclusion

While the purpose of this paper is to defend the plausibility of the intermediate stable model in a continuous time framework, we are not wed to the assumption that price changes really are intermediate stable. As Press (1967) and Praetz (1972) have demonstrated in two earlier papers in this *Journal*, mixtures of variables drawn from distributions having different variances can mimic the leptokurtosis characteristic of intermediate stable distributions and still have tractable finite variances. Indeed, in my own earlier work (1975, pp. 99–104), I demonstrated that, while unanticipated changes in interest rates pooled over the interval 1951–66 are significantly nonnormal (with estimates of α in the range 1.34–1.47), when shorter time periods are considered

8. See McCulloch (1977) for a restatement and extension of C. Menger's theory.
9. See Samuelson (1977) for a survey of the St. Petersburg Paradox literature.

separately and allowed to have different variances, the hypothesis that changes are heteroskedastically normal cannot be rejected. The true distribution of price changes remains to be settled, as does the issue of whether stable distributions are adequate approximations to distributions that are strongly leptokurtic, yet not actually stable, such as the Student's distribution suggested recently in this *Journal* by Blattberg and Gonedes (1974). We are simply arguing that stable distributions should not be dismissed out of hand just because of their interesting behavior in continuous time and when exponentiated.

References

Blattberg, Robert C., and Gonedes, Nicholas J. 1974. A comparison of the stable and student distributions as statistical models for stock prices. *Journal of Business* 47 (April): 244–80.

DuMouchel, W. H. 1973. On the asymptotic normality of the maximum likelihood estimate when sampling from a stable distribution. *Annals of Statistics* 1:948–57.

DuMouchel, W. H. 1975. Stable distributions in statistical inference. II. Information from stably distributed samples. *Journal of the American Statistical Association* 70 (June): 386–93.

Dwight, H. B. 1941. *Mathematical Tables*. New York: McGraw-Hill.

Fama, Eugene F., and Roll, Richard. 1968. Some properties of symmetric stable distributions. *Journal of the American Statistical Association* 63 (September): 817–36.

Fama, Eugene F., and Roll, Richard. 1971. Parameter estimates for symmetric stable distributions. *Journal of the American Statistical Association* 66 (June): 331–38.

Gray, Burton C. 1970. Supply of first-term military enlistees: a cross-sectional analysis. In President's Commission on an All-Volunteer Armed Force, *Studies*. Vol. 1. Washington, D.C.: Government Printing Office.

Hayek, F. A. 1952. *The Counter-Revolution of Science: Studies on the Abuse of Reason*. Glencoe, Ill.: Free Press.

McCulloch, J. Huston. 1971. Measuring the term structure of interest rates. *Journal of Business* 44 (January): 19–31.

McCulloch, J. Huston. 1975. An estimate of the liquidity premium. *Journal of Political Economy* 83 (February): 95–119.

McCulloch, J. Huston. 1977. The Austrian theory of the marginal use and of ordinal marginal utility. *Zeitschrift für Nationalökonomie* 37:249–80.

Mandelbrot, Benoit. 1963. The variation of certain speculative prices. *Journal of Business* 36 (October): 394–419.

Merton, Robert C. 1976. Optimal pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3:125–44.

Praetz, Peter D. 1972. The distribution of share price changes. *Journal of Business* 45 (January): 49–55.

Press, S. James. 1967. A compound events model for security prices. *Journal of Business* 40 (July): 317–35.

Roll, Richard. 1970. *The Behavior of Interest Rates: The Application of the Efficient Market Model to U.S. Treasury Bills*. New York: Basic.

Samuelson, Paul A. 1976. Limited liability, short selling, bounded utility, and infinite-variance stable distributions. *Journal of Financial and Quantitative Analysis* 11 (September): 485–503.

Samuelson, Paul A. 1977. St. Petersburg paradoxes: defanged, dissected, and historically described. *Journal of Economic Literature* 15 (March): 24–55.

Stuck, Bart. 1976. Distinguishing stable probability measures. I. Discrete time. II. Continuous time. *Bell System Technical Journal* 55 (October): 1125–82, 1183–96.