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Precise tabulation of the maximally-skewed stable distributions and densities

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Abstract

The cdf and pdf of the maximally skewed ($\beta = 1$) stable distributions are tabulated to high precision, by means of Zolotarev's integral representation, for $\alpha = 0.50$ (0.02) 2.00, at fractiles corresponding to $p = 0.0001$, 0.001, 0.005, 0.01 (0.01) 0.99, 0.995, 0.999, 0.9999. This tabulation is intended to be suitable for developing and calibrating a numerical approximation to these distributions. The probability at the tabulated fractiles is estimated to be accurate to within 4.1×10^{-10}. The densities have an absolute precision of 2.0×10^{-13} and a relative precision of 1.6×10^{-12}. Zolotarev's correction of the discontinuity at $\alpha = 1$ is graphically illustrated. The full tabulation, documented here, is available by anonymous FTP.

Keywords: Maximally-skewed stable distributions; Zolotarev integrals; Numerical quadrature

1. Introduction

A probability distribution with cumulative distribution function (cdf) $S(x)$ is said to belong to the *stable* family if and only if all linear combinations in positive coefficients of X_1 and X_2 have the same distribution, to within a location and scale shift, as do X_1 and X_2 themselves, when X_1 and X_2 are independent drawings from

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the distribution $S(x)$. According to the Generalized Central Limit Theorem, if the distribution of the sum of n independent and identically distributed (iid) random variables has a weak limit, after shifting and scaling, as n becomes large, the limiting distribution must be a member of this stable class (Zolotarev, 1986). In many applications, observed errors arise as the sum of a large number of more or less independent unobservable contributions. The normal or Gaussian distribution is the most familiar and tractable member of the stable class, and therefore is commonly assumed to approximate the actual distribution of errors. However, observed errors are often more leptokurtic than is consistent with normality. In such situations, the stable distributions become the natural extension of the normal.

The standard stable distributions, $S_{\alpha\beta}(x)$, are most usefully parameterized in terms of the log characteristic function (cf)

$$\log Ee^{iXt} = \psi_{\alpha\beta}(t) = \begin{cases} -|t|^\alpha[1 - i\beta\,\mathrm{sign}(t)\tan(\pi\alpha/2)], & \alpha \neq 1, \\ -|t|[1 + i\beta(2/\pi)\,\mathrm{sign}(t)\log|t|], & \alpha = 1, \end{cases} \tag{1}$$

The *characteristic exponent* $\alpha \in (0,2]$ determines the degree of leptokurtosis. When α equals 2, a normal distribution with variance 2 results. When α is less than 2, the cdf has at least one Paretian tail that behaves like $|x|^{-\alpha}$ for large $|x|$, and the *skewness parameter* $\beta \in [-1, 1]$ indicates the limiting ratio of the difference of the Paretian tail probabilities to their sum. As $\alpha \uparrow 2$, the Paretian tails vanish and β loses its effect. When (1) is used, $\beta > 0$ indicates positive skewness and $\beta < 0$ indicates negative skewness (see Zolotarev 1986; Hall, 1981).

Following DuMouchel (1975), a location parameter $\delta \in (-\infty, \infty)$ that shifts the distribution to the left or right, and a scale parameter $c \in (0, \infty)$ that expands or contracts it about δ may be added in such a way that the general stable cdf may be written as

$$S(x; \alpha, \beta, c, \delta) = S_{\alpha\beta}((x - \delta)/c). \tag{2}$$

The general stable log cf implied by (2) is

$$\log Ee^{iXt} = i\delta t + \psi_{\alpha\beta}(ct). \tag{3}$$

For further properties of stable distributions, the reader is referred to Zolotarev (1986), Cambanis et al. (1991), Samorodnitsky and Taqqu (1994), Janicki and Weron (1994), and McCulloch (1996a, b).

In the symmetric case $\beta = 0$, McCulloch (1996c) has developed a fast and reasonably accurate numerical approximation to the stable cdf and density. However, no comparable approximation is presently available for any other value of β. A practical approximation to the full class of stable distributions would greatly simplify studies like that of Buckle (1995), which deal with data that is skewed as well as leptokurtic.

Stable distributions are *maximally positively skewed* when β equals its upper bound of 1. In this case there is an upper Paretian tail but the lower tail has no Paretian component and falls off even faster than the normal. When $\alpha < 1$, $S_{\alpha 1}(x) = 0$ for $x \leq 0$, and the distribution is said to be *positive*. The thin lower tail of the maximally skewed stable distributions presents problems not present in the other cases. These problems must be solved before the general case can be addressed.

The maximally skewed stable distributions are important in their own right. They arise in the fields of nuclear engineering (Zolotarev, 1986), quantum statistics (Weron and Weron, 1985), dielectric relaxation (Weron and Jurlewicz, 1993), hydrology (Gupta and Waymire, 1990), and astronomy (Marcus, 1965). They govern the asymptotic properties of certain statistics that depend on a sum of squared errors, such as the Durbin–Watson ratio or the common tests for covariance stationarity, when errors have infinite absolute fourth moments. (Phillips and Loretan, 1991; Loretan and Phillips, 1994). Because of their role in subordinated stable distributions (Samorodnitsky and Taqqu, 1994, 20–21), they govern the density of the elliptical multivariate stable distributions (McCulloch, 1996b), as well as the scale of the GARCH-like subordinated stable conditional heteroskedastic processes (de Vries, 1990). In financial economics, apparently stable asset returns are ordinarily intermediately skewed (e.g. Buckle, 1995), if not symmetrical. Even so, the maximally skewed stable distributions are required for evaluating options on assets with intermediately skewed and even symmetrical log-stable returns (McCulloch, 1996b).

The present study tabulates the maximally positively skewed stable distributions $S_{\alpha 1}(x)$ for $\alpha = 0.50$ (0.02) 2.00 to high precision, in a form suitable for developing and calibrating a numerical approximation to these distributions. In order to space the calibration points well across the distribution, the cdf itself is given in the form of fractiles $x = S_{\alpha 1}^{-1}(p)$ at the cumulative probability values $p = 0.0001, 0.001, 0.005, 0.01$ (0.01) 0.99, 0.995, 0.999, 0.9999. The probability density function (pdf) $s_{\alpha 1}(x)$ is then evaluated at each of these fractile values.

The full table is too massive (133 pages; 439 KB) to publish here, but instead is available by anonymous FTP at *ecolan.sbs.ohio-state.edu/pub/skewstable/*, file *fracden*, and also at *ftp.uta.edu/pub/projects/skewstable/*. Table 1 gives an illustrative subset of the full table, for $\alpha = 0.50$, 1.00, and 1.50, and selected p values. The fractiles are tabulated to 11 digits of precision and the densities to 15 digits of precision, in exponential format. The probability at the reported fractile is estimated to be accurate to within 4.1×10^{-10}. The density at the reported fractile is estimated to be accurate to within an absolute error of 2.0×10^{-13}, and a relative error of 1.6×10^{-12}. The fractiles themselves are accurate to the probability precision divided by the local density. Previous tabulations of the maximally skewed stable distribution (Bol'shev et al., 1970, 4 place; DuMouchel, 1971, 5 place) and density (Holt and Crow, 1973, 4-place) have far less precision. Other stable tabulations (Fama and Roll, 1971; Panton, 1992, 1993) deal only with the symmetric case.

2. Discussion

Holding β, c and δ constant, the stable characteristic function (3), and therefore the distribution itself, undergoes a discontinuity as α passes 1 unless $\beta = 0$. Fig. 1 shows the standard cumulative probability distributions $S_{\alpha 1}(x)$ as plotted from our

Table 1
Maximally skewed stable fractiles and fractile densities: illustrative subset of full tabulation in file *fracden*

α	p	$x = S_{\alpha 1}^{-1}(p)$	$s_{\alpha 1}(x)$
0.50	0.0001	6.6064575687E − 02	1.21356583204336E − 02
0.50	0.0010	9.2356859706E − 02	6.33184437322565E − 02
0.50	0.0100	1.5071824938E − 01	2.47122229847038E − 01
0.50	0.0200	1.8477817997E − 01	3.35549115138592E − 01
0.50	0.0500	2.6031777170E − 01	4.40039998765221E − 01
0.50	0.1000	3.6961151008E − 01	4.58976595025777E − 01
0.50	0.1500	4.8256705888E − 01	4.22269779307600E − 01
0.50	0.2000	6.0887456127E − 01	3.69386695471860E − 01
0.50	0.2500	7.5568443071E − 01	3.13362921467179E − 01
0.50	0.3000	9.3093039189E − 01	2.59582823630614E − 01
0.50	0.3500	1.1448758626E + 00	2.10428063372091E − 01
0.50	0.4000	1.4117787229E + 00	1.66897186487796E − 01
0.50	0.4500	1.7523819106E + 00	1.29285984338474E − 01
0.50	0.5000	2.1981093390E + 00	9.75097267942318E − 02
0.50	0.5500	2.7986340681E + 00	7.12689653221390E − 02
0.50	0.6000	3.6364178825E + 00	5.01400529409011E − 02
0.50	0.6500	4.8567236131E + 00	3.36267068458687E − 02
0.50	0.7000	6.7352829538E + 00	2.11902481460247E − 02
0.50	0.7500	9.8492043223E + 00	1.22676438492458E − 02
0.50	0.8000	1.5580023718E + 01	6.28232430507648E − 03
0.50	0.8500	2.7959687266E + 01	2.65060590648705E − 03
0.50	0.9000	6.3328117678E + 01	7.85391638751667E − 04
0.50	0.9500	2.5431444455E + 02	9.81747198329580E − 05
0.50	0.9800	1.5912160766E + 03	6.28318522468520E − 06
0.50	0.9900	6.3658643851E + 03	7.85398162752572E − 07
0.50	0.9990	6.3661943867E + 05	7.85398164071347E − 10
0.50	0.9999	6.3661976900E + 07	7.85398163460882E − 13
1.00	0.0001	− 2.1849120477E + 00	1.20913237657303E − 03
1.00	0.0010	− 1.9612653085E + 00	8.69618603024061E − 03
1.00	0.0100	− 1.6275061066E + 00	5.38427032673220E − 02
1.00	0.0200	− 1.4855280174E + 00	8.82959908692096E − 02
1.00	0.0500	− 1.2413046955E + 00	1.58266246832091E − 01
1.00	0.1000	− 9.8283730757E − 01	2.25589278887450E − 01
1.00	0.1500	− 7.7888323692E − 01	2.61775165571778E − 01
1.00	0.2000	− 5.9483193480E − 01	2.79206987456308E − 01
1.00	0.2500	− 4.1776476362E − 01	2.83752942618803E − 01
1.00	0.3000	− 2.4045468918E − 01	2.78899325157886E − 01
1.00	0.3500	− 5.7580616609E − 02	2.66995257732917E − 01
1.00	0.4000	1.3571822207E − 01	2.49764217228393E − 01
1.00	0.4500	3.4467428392E − 01	2.28555138795696E − 01
1.00	0.5000	5.7563014450E − 01	2.04481748174078E − 01
1.00	0.5500	8.3697041069E − 01	1.78507735792456E − 01
1.00	0.6000	1.1405756677E + 00	1.51503576055258E − 01
1.00	0.6500	1.5043951879E + 00	1.24287934789192E − 01
1.00	0.7000	1.9574701760E + 00	9.76608543330775E − 02
1.00	0.7500	2.5508156833E + 00	7.24331302773577E − 02

Table 1 Continued

α	p	$x = S_{\alpha 1}^{-1}(p)$	$s_{\alpha 1}(x)$
1.00	0.8000	3.3842882767E + 00	4.94548057250199E − 02
1.00	0.8500	4.6862628047E + 00	2.96445410639818E − 02
1.00	0.9000	7.1286784854E + 00	1.40195490964922E − 02
1.00	0.9500	1.4004804442E + 01	3.71974588044426E − 03
1.00	0.9800	3.3732198367E + 01	6.15399149070384E − 04
1.00	0.9900	6.6020512869E + 01	1.55484962873149E − 04
1.00	0.9990	6.4045906557E + 02	1.56922296700933E − 06
1.00	0.9999	6.3715044008E + 03	1.57063922092402E − 08
1.50	0.0001	− 4.5665389704E + 00	4.92901468494204E − 04
1.50	0.0010	− 4.0473367935E + 00	3.96110251555282E − 03
1.50	0.0100	− 3.3711334460E + 00	2.88364271498797E − 02
1.50	0.0200	− 3.1147755100E + 00	5.05779088384112E − 02
1.50	0.0500	− 2.7117446662E + 00	1.01499738963684E − 01
1.50	0.1000	− 2.3312357815E + 00	1.62356069148668E − 01
1.50	0.1500	− 2.0599692455E + 00	2.05635718920809E − 01
1.50	0.2000	− 1.8344267374E + 00	2.36695367319111E − 01
1.50	0.2500	− 1.6328124097E + 00	2.58167516187233E − 01
1.50	0.3000	− 1.4444625303E + 00	2.71641625296920E − 01
1.50	0.3500	− 1.2629375020E + 00	2.78190471434978E − 01
1.50	0.4000	− 1.0836467501E + 00	2.78592974837143E − 01
1.50	0.4500	− 9.0278392285E − 01	2.73446655242114E − 01
1.50	0.5000	− 7.1671068575E − 01	2.63232189368862E − 01
1.50	0.5500	− 5.2147017792E − 01	2.48355183903037E − 01
1.50	0.6000	− 3.1224904768E − 01	2.29177339185606E − 01
1.50	0.6500	− 8.2596962340E − 02	2.06044582055238E − 01
1.50	0.7000	1.7693386413E − 01	1.79319175558072E − 01
1.50	0.7500	4.8151210882E − 01	1.49426020153363E − 01
1.50	0.8000	8.5829482041E − 01	1.16933807050329E − 01
1.50	0.8500	1.3636800656E + 00	8.27224685036277E − 02
1.50	0.9000	2.1457331050E + 00	4.83899144035980E − 02
1.50	0.9500	3.8242359660E + 00	1.74533201706293E − 02
1.50	0.9800	7.3019969444E + 00	4.02208425595022E − 03
1.50	0.9900	1.1654134354E + 01	1.28010348607252E − 03
1.50	0.9990	5.4191613924E + 01	2.76780378815650E − 05
1.50	0.9999	2.5153975348E + 02	5.96326884561649E − 07

full tabulation. It may be seen that as α increases from 0.5 toward 1.0, the fractiles all increase without limit. As α decreases from 2.0 toward 1.0, the fractiles below approximately $p = 0.87$ all decrease without limit. The $\alpha = 1$ distribution stands by itself in the middle. Fig. 2 shows the corresponding densities $s_{\alpha 1}(x)$, with the $\alpha = 2.0$ normal density (with variance 2) in the foreground and the $\alpha = 0.5$ Lévy or inverse $\chi^2(1)$ density in the background.

The discontinuity that is apparent in Figs. 1 and 2 may in fact be thought of as a discontinuity in the *focus of stability*, which we have somewhat artificially fixed

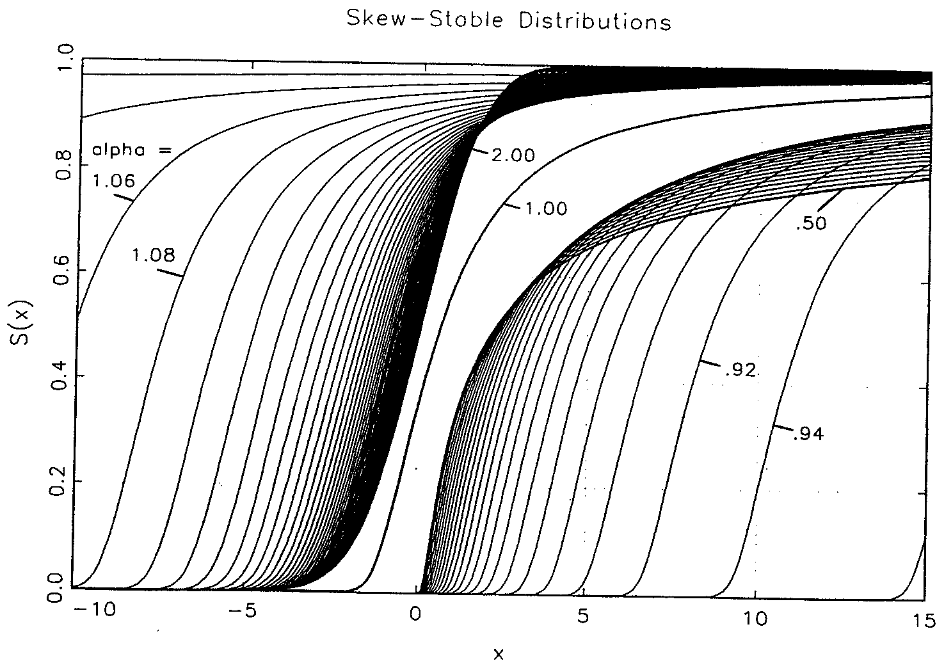

Fig. 1. Maximally skew-stable cumulative probability distribution functions for $0.50 \leq \alpha \leq 2.00$ in increments of 0.02. Bold lines represent $\alpha = 0.50$, 1.00, 1.50 and 2.00.

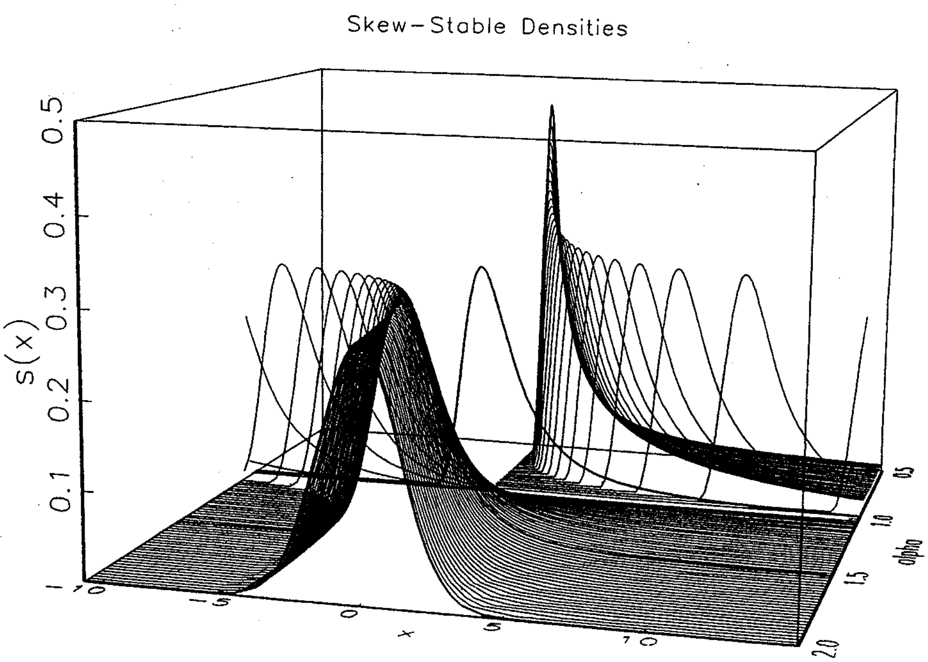

Fig. 2. Maximally skew-stable probability density functions. The normal density $s_{2,\bullet}(x)$ is in the foreground, while the Lévy density $s_{0.5,1}(x)$ is in the background.

upon as our location parameter for $\alpha \neq 1$, rather than in the distribution itself. McCulloch (1986) defines a *focus of stability* to be any fractile of a stable distribution that is invariant under averaging of iid contributions. If n iid stable variates with scale c are averaged, the average has scale $n^{1/\alpha - 1} c$. Unless $\alpha = 1$, the scale of the average is different than that of the contributions, and therefore the focus of stability is unique. In the *convergent* cases $\alpha > 1$, the scale of the average is smaller than that of the contributions, so that the distribution of the average converges in toward the unique focus of stability at $\delta = Ex$. In the *divergent* cases $\alpha < 1$, the scale of the average is greater than that of the contributions, so that the Law of Large Numbers works in reverse: The distribution of the average diverges out *away* from the unique focus of stability at δ, while the mean is undefined. In the *Cauchy* case $\alpha = 1$, $\beta = 0$, the distribution of the average coincides with that of the contributions, so that *every* fractile is a focus of stability. In the *afocal* cases $\alpha = 1$ and $\beta \neq 0$, however, no focus of stability exists and δ is just an arbitrary fractile that happens to simplify the cf.

If we define the modified location parameter

$$\zeta = \begin{cases} \delta + \beta c \tan(\pi\alpha/2), & \alpha \neq 1, \\ \delta, & \alpha = 1, \end{cases} \tag{4}$$

then Zolotarev (1986, 11) has shown that the characteristic function and therefore the distribution of the new variable $z = x - \zeta$ undergoes no discontinuity as α passes unity. Fig. 3 shows the maximally skewed stable cdfs as a function of z rather than x, while Fig. 4 shows the corresponding densities as a function of z. In order to obtain a good perspective, Fig. 3 was drawn with the α axis reversed from Figs. 2 and 4, so that the $\alpha = 0.5$ Lévy distribution is in the foreground, and the normal distribution is in the background. Unfortunately, Fig. 1 does not lend itself to a perspective view from any angle. In Figs. 2–4, the cdf and density are not plotted outside their support.

Janicki and Weron (1994, 23) take the position that for $\alpha = 1$, the only acceptable value of β is 0. In fact, Figs. 3 and 4 demonstrate that the afocal stable distributions fit in smoothly between the convergent and divergent cases, if we just know where to look for them. With finite samples, they will be statistically indistinguishable from their immediate neighbors.

Clearly any numerical approximation to $S_{\alpha\beta}(x)$ should be based internally on z rather than x. Externally, however, it is ordinarily expedient to retain δ rather than ζ as the "official" location parameter, since δ has an important interpretation for $\alpha \neq 1$, whereas ζ has no known significance other than that it removes the discontinuity while retaining a simple relation to the cf when $\alpha = 1$. Nevertheless, if α is insignificantly different from 1 and the distribution is clearly skewed, little can be said about δ other than that it lies in a confidence set of the form $(-\infty, a] \cup [b, \infty)$. In such a case, ζ will still be well-identified, and will provide good information as to the location of the distribution.

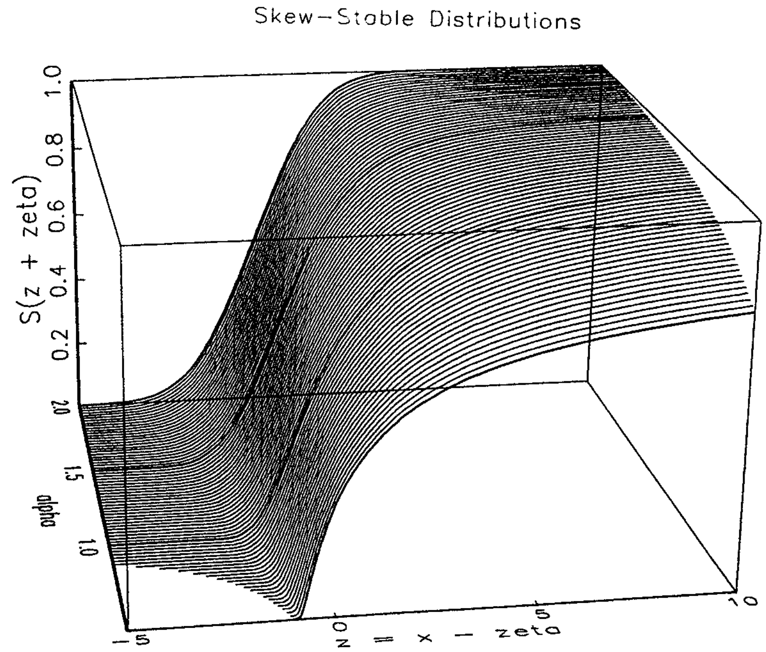

Fig. 3. Maximally skewed stable cumulative probability distribution functions, as a function of $z = x - \zeta$. The Lévy distribution $S_{0.5,1}(z+\zeta)$ is in the foreground, while the normal distribution $S_{2,\bullet}(z+\zeta)$ is in the background.

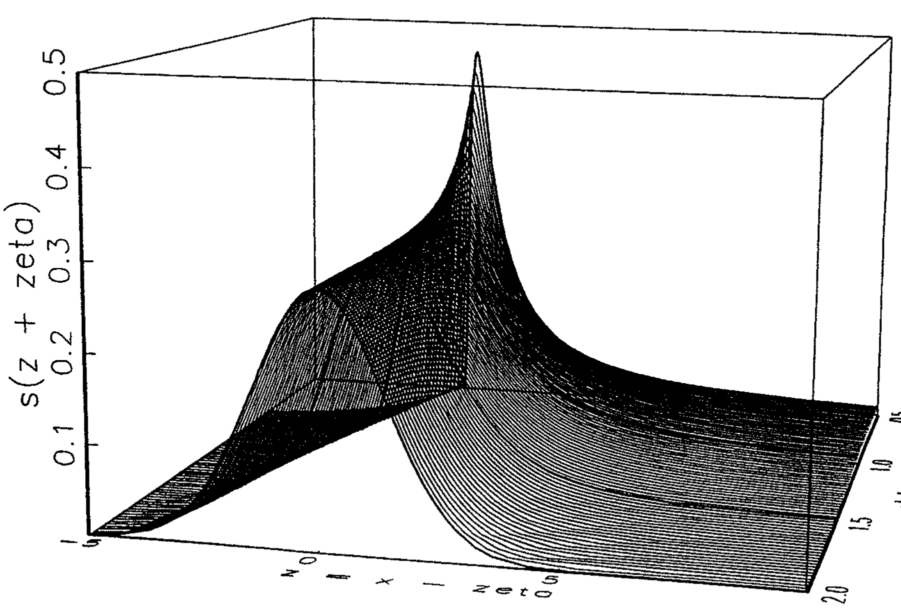

Fig. 4. Maximally skewed stable probability density functions, as a function of $z = x - \zeta$. The normal density $s_{2,\bullet}(z+\zeta)$ is in the foreground, while the Lévy density $s_{0.5,1}(z+\zeta)$ is in the background.

3. Computation

The standard stable cumulative probabilities and densities were computed by means of the proper integral representations due to Vladimir M. Zolotarev (1986, 74, 78). For $\alpha \neq 1$ and $x > 0$, the cumulative probability determined by characteristic function (1) is, in our notation (see section on alternative notations below), given by

$$S_{\alpha\beta}(x) = C(\alpha,\theta) + \frac{sign(1-\alpha)}{2}\int_{-\theta}^{1} \exp\left[-x^{*\frac{\alpha}{\alpha-1}}\, U_\alpha(\varphi,\theta)\right] d\varphi, \qquad (5)$$

where

$$x^* = c^* x, \qquad (6)$$

$$c^* = \left[1 + \left(\beta \tan\frac{\pi\alpha}{2}\right)^2\right]^{-1/2\alpha}, \qquad (7)$$

$$\theta = \frac{2}{\pi\alpha}\tan^{-1}\left[\beta\tan\frac{\pi\alpha}{2}\right], \qquad (8)$$

$$C(\alpha,\theta) = \begin{cases} 1, & \alpha > 1 \\ (1-\theta)/2, & \alpha < 1, \end{cases}$$

$$U_\alpha(\varphi,\theta) = \left[\frac{\sin\frac{1}{2}\pi\alpha(\varphi+\theta)}{\cos\frac{1}{2}\pi\varphi}\right]^{\alpha/(1-\alpha)} \frac{\cos\frac{1}{2}\pi((\alpha-1)\varphi + \alpha\theta)}{\cos\frac{1}{2}\pi\varphi}.$$

For $\alpha = 1$ and $\beta > 0$, the cdf becomes

$$S_{\alpha 1}(x) = \frac{1}{2}\int_{-1}^{1} \exp(-e^{-x^*/\beta}\, U_1(\varphi,\beta))\, d\varphi, \qquad (9)$$

where

$$x^* = \tfrac{1}{2}\pi x + \beta \log\tfrac{1}{2}\pi \qquad (10)$$

$$U_1(\varphi,\beta) = \frac{\pi(1+\beta\varphi)}{2\cos\frac{1}{2}\pi\varphi}\exp\left[\frac{\pi}{2}\left(\varphi + \frac{1}{\beta}\right)\tan\frac{\pi}{2}\varphi\right].$$

The cdf for $\alpha \neq 1$ with $x < 0$ or for $\alpha = 1$ with $\beta < 0$ may be computed using the identity

$$S_{\alpha\beta}(-x) = 1 - S_{\alpha,-\beta}(x).$$

The densities may be computed by differentiating (5) and (9). For $\alpha \neq 1$ and any $x \neq 0$, this may be written

$$s_{\alpha\beta}(x) = \frac{\alpha |x^*|^{1/(\alpha-1)}}{2|1-\alpha| c^*}\int_{-\theta^*}^{1} U_\alpha(\varphi,\theta^*)\exp\left[-|x^*|^{\alpha/(\alpha-1)}\, U_\alpha(\varphi,\theta^*)\right] d\varphi, \qquad (11)$$

where $\theta^* = \theta \operatorname{sign}(x)$. For $\alpha = 1$, $\beta \neq 0$, and any x,

$$s_{\alpha 1}(x) = \frac{1}{\pi|\beta|} e^{-x^*/\beta} \int_{-1}^{1} U_1(\varphi,\beta) \exp(-e^{-x^*/\beta} U_1(\varphi,\beta))\, d\varphi. \tag{12}$$

For our special case $\beta = 1$, the following simplifications are useful:

$$c^* = \left|\cos\frac{\pi\alpha}{2}\right|^{1/\alpha}, \quad \theta = \begin{cases} -(2-\alpha)/\alpha, & \alpha > 1 \\ 1, & \alpha < 1, \end{cases} \quad C(\alpha,\theta) = 0,\ \alpha < 1. \tag{13}$$

At the limits of integration, the function $U_\alpha(\varphi,\theta)$ is incalculable for $\beta = \pm 1$ as given above. The limiting values given in Table 2 must be employed in these cases. At the infinite limits, both the cdf integrand and the density integrand are 0. At the 0 limit the cdf integrand is 1, but the density integrand is 0.

Numerical integration was performed by $n + 1$ point Simpson's rule quadrature and checked variously with Gaussian, Romberg, and Richardson extrapolation quadrature methods. The precision was estimated under the assumption that the remaining error using Simpson's rule with $n + 1$ points is bounded above by the absolute difference between $n + 1$ point quadrature and $(n/2) + 1$ point quadrature. Fractiles were found by a numerical search procedure. Precision rather than efficiency was the primary concern.

For $\beta = \pm 1$, $U_\alpha(\varphi,\theta)$ and $U_1(\varphi,\beta)$, and therefore the cumulative probability integrands, are monotonic functions of φ. Furthermore, the cdf integrands are bounded below by 0 and above by 1. The only problematic cases are those in which the integrand falls (or rises) abruptly from 1 to 0 (or from 0 to 1), and elsewhere is virtually 1 or 0. This tends to occur when α is near 1 or 2 and/or x is near 0 or very large. In these cases, it is useful to first isolate the non-trivial region with a preliminary grid search before attempting numerical integration. For example, with $\alpha = 0.98$ and $p = 0.9999$ ($x = 7738.83\ldots$), the integrand is within 10^{-10} of 1 for $\varphi < 0.99968$, and within 10^{-10} of 0 for $\varphi > 0.99982$. Even within this reduced range, Simpson's rule with $n = 10^6$ was required in order to achieve the desired relative precision, though in most cases $n = 10\,000$ or $100\,000$ was adequate.

The most difficult density cases occur for the same α and x values, when the integrand becomes a narrow spike. These cases were handled with a cascading adaptive Simpson's rule procedure: The function was first integrated over the entire

Table 2
Limiting values of $U_\alpha(\varphi,\theta)$ and $U_1(\varphi,\beta)$ at lower and upper limits of integration

Parameter values	Lower limit	Upper Limit
$\alpha > 1, \beta = 1\ (\theta = (\alpha - 2)/\alpha)$	∞	0
$\alpha > 1, \beta = -1\ (\theta = (2 - \alpha)/\alpha)$	∞	$(\alpha - 1)\,\alpha^{\alpha/(1-\alpha)}$
$\alpha = 1, \beta = 1$	$1/e$	∞
$\alpha = 1, \beta = -1$	∞	$1/e$
$\alpha < 1, \beta = 1\ (\theta = 1)$	$(1 - \alpha)\,\alpha^{\alpha/(1-\alpha)}$	∞

interval with $n + 1$ points. If this was above tolerance, the interval was split in half, and an additional $n/2$ points evaluated in each half. If these subintervals were individually above half the total tolerance allowed, they were again split in half, iteratively, up to r times, giving a local grid spacing as fine as $1/(n2^r)$ times the full interval. In most cases $r = 15$ was adequate with $n = 10\,000$. However, with $\alpha = 1$ and $p = 0.9999$, $r = 30$ was required. With $\alpha = 1.02$ and $p = 0.9999$, the spike near $\varphi = 0.9998$ fell completely between the initial grid points with $n = 10\,000$, so it was necessary to adjust the lower limit by hand to compensate for this.

The argument of the exponential function in (5) and (9) often generates floating point overflows if calculated directly. These can be avoided by first computing the logarithm of the absolute value of the argument, and replacing it with a large value such as 700 if it is in excess of this value, before exponentiating, negating, and exponentiating again to obtain the integrand. This essentially sets the integrand to 0 in the overflow cases, and gives a better conditioned answer in many others. In the density integrals, the product of an overflow and an underflow sometimes arises if computed directly. In order to avoid this, it is expedient to move the terms in front of the integral inside the integral, compute the logarithm of the entire expression, and then exponentiate the result to obtain the integrand.

All calculations were performed on a (corrected) Intel P5 processor in GAUSS 3.2.12, and variously checked on an Intel 486 processor in PASCAL. The normal fractiles and densities in the tabulation were computed with the GAUSS CDNFI and EXP functions rather than with the integral representations.

4. Duality

An important duality relationship exists between each half of each convergent stable distribution with characteristic exponent $\alpha > 1$ and half of a divergent stable distribution with exponent $\alpha' = 1/\alpha \in [0.5, 1.0)$ (Zolotarev, 1986, 82). In our notation, this relationship implies that for $x > 0$ and $\alpha > 1$,

$$\alpha S_{\alpha 1}(-x/c_\alpha^*) = S_{1/\alpha,1}((x/c_{1/\alpha}^*)^{-\alpha}),$$

where c_α^* and $c_{1/\alpha}^*$ are computed as in (13) but with $\alpha' = 1/\alpha$ in place of α in the latter case. Thus, a numerical approximation to the convergent portion of our tabulation would make an approximation to the divergent portion redundant. The opposite is not true, since the upper halves of the convergent distributions in our tabulation are dual to the lower halves of intermediately skewed divergent distributions that we have not tabulated. However, a numerical approximation to all the divergent distributions with $\alpha \geq 0.5$ would make an approximation to the convergent cases redundant.

The simplest instance of this duality is that between the normal distribution $S_{2,\bullet}(x)$ and the Lévy distribution $S_{0.5,1}(x)$. The duality relationship implies that the latter is equivalent to the inverse χ^2 distribution with 1 degree of freedom. The Lévy values in Table 1 and in the full tabulation were computed numerically using

Zolotarev's integral representations, but the reader may check that they match the distribution of the inverse square of a standard (unit variance) normal variable to within the claimed precision.

5. Numerical approximation

As in McCulloch (1996c), any numerical approximation to the skew-stable density function should be constructed as the derivative of a proper approximation to the cdf, so as to guarantee that the density approximation integrates exactly to unity and therefore obeys the crucial information identity. Its parameters should then be chosen so as to minimize the average over α of the expectation (as roughly computed using the tabulated fractiles) of the squared deviation of the density approximation from the density calibration values given in our tabulation. This will give the approximation a very low maximal expected relative error, and hence place a low upper bound on the expected error in any log likelihood calculated from it. In order to control the relative error in the tails, the upper tail should asymptotically have the well-known Paretian form, while the lower tail should have the asymptotic form implied by Zolotarev's Eq. (2.5.21–22) (1986, 100). Any proper distribution approximation that has the correct tail behavior and whose derivative is a good fit to the density will also fit the cdf well, as may be confirmed from our fractile tabulation.

6. Alternative notations

Our parameterization (3) corresponds, in the standard case $c = 1$, $\delta = 0$ of (1), to Zolotarev's representation A (1986, 9), in which the complex portion of the standard log cf is given in Cartesian form. However, Zolotarev expresses his integral representation (1986, 74, 78) in terms of two alternative representations, B (1986, 12) and C (1986, 17), in which the complex portion is instead given in polar form. He indicates the skewness parameters of his A and B representations as β_A and β_B, respectively. However, throughout his Chapter 2, which presents the integral representations, "β" with no subscript is implicitly the polar β_B (see 1986, 59), whereas our "β" is the very different Cartesian β_A (see Samorodnitsky and Taqqu, 1994, 9). Zolotarev's newer C representation uses θ as its skewness parameter. This is less confusing and at the same time actually simplifies the formulas in which the polar notation is useful. We have therefore completely avoided the B representation, which may now be regarded as obsolete, in the preceding sections.

To see the equivalence of our formulas to Zolotarev's for $\alpha \neq 1$, note that θ in Zolotarev's (I.28) (1986, 17) is equivalent to his $\beta_B K(\alpha)/\alpha$, where the definition of the now obsolete $K(\alpha)$ does not concern us here. Substituting this into the first line of his Eq. (I.19) on his p. 12 yields our (8). Substituting it into the third line of his (I.19) and setting $\lambda_B = 1$ yields

$$\lambda_A = (1 + \beta_A^2 \tan^2(\pi\alpha/2)^{-1/2}.$$

But Zolotarev's λ_A is equivalent to our c^α (cp. our (3) and his (A), p. 9). This implies that his formulas for $\alpha \neq 1$ are for an x^* which in the Cartesian notation has scale c^* as given by (7).

For $\alpha = 1$, his integral representations are based on the log cf

$$\log E\mathrm{e}^{\mathrm{i}Xt} = -|t|(\pi/2 + \mathrm{i}\beta \log|t|\, sign(t))$$

given in his (B) on p. 12. This is equivalent to our (3) with $c = \pi/2$ and $\delta = \beta \log(\pi/2)$. His formulas are therefore for an x^* as given in (10). Dividing Zolotarev's density representation by this c replaces his leading 1/2 by $1/\pi$.

In (1986), Zolotarev modifies the definition of his $K(\alpha)$ in such a way that β_B always has the same sign as β_A. Buckle (1995) employs an older version of the polar B representation, in which β_B has the sign opposite that of β_A when $\alpha > 1$. Thus his finding of a negative "β" for the stock he studies (1995, Fig. 5b) indicates that its returns are positively skewed, as is apparent from his Fig. 6. Furthermore, since his "β" is polar, it is quite different, even after negating, from the Cartesian β_A employed here. See Samorodnitsky and Taqqu (1994, 8–9) for further discussion of this distinction.

Hall (1981) and Samorodnitsky and Taqqu (1994, 5) use the standardized Cartesian form (1), but take the general log cf to be $\mathrm{i}\mu t + c^\alpha \psi_{\alpha\beta}(t)$. For $\alpha = 1$, this μ equals $\delta - (2/\pi)\beta c \log c$, and does not have the linearity property (2). For this location parameter, the shift that achieves continuity in the general case is $\beta c^\alpha \tan(\pi\alpha/2)$, rather than $\beta c \tan(\pi\alpha/2)$. See McCulloch (1996b) for details. Zolotarev's A representation (1986, 9) uses the same form, but with λ in place of c^α and $\lambda\gamma$ in place of μ. When $c = 1$, as in the standard case tabulated here, these distinctions are fortunately moot.

Panton (1992), following Worsdale (1975), standardizes the symmetric stable distributions in terms of an alternative scale parameter that may be written $\sigma = c\alpha^{1/\alpha}$. This scale parameter has the virtue that it equals the standard deviation in the normal case, without disturbing the standard Cauchy parameterization. However, the present study uses the more commonly encountered "standard scale" c.

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ERRATUM

Humphrey Tung and David Benson have caught two errors in our paper, "Precise tabulation of the maximally-skewed stable distributions and densities," *CSDA* **23** #3 (Jan. 1997): 307-320. The term in front of the integral in our Equation (11) should have c* in the numerator instead of the denominator. Also, the first term on the right hand side of our Equation (12) should be $\pi/(4\,|\beta|)$, rather than $1/(\pi\,|\beta|)$. The computations and figures were nevertheless done with the correct coefficients. The full tabulation is now available by World Wide Web in addition to FTP, at http://www.econ.ohio-state.edu/jhm.jhm.html.

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