Dominance and Best Response

Consider the following game, Figure 6.1(a) from the text.

		player 2	
		L	R
player 1	U	2, 3	5, 0
	D	1, 0	4, 3

Suppose you are player 1. The strategy U yields higher payoff than any other strategy (i.e., D), no matter what strategy the other players (i.e., player 2) choose.

If player 2 chooses L, $2 > 1$. If player 2 chooses R, $5 > 4$.

We say that the strategy D is dominated by strategy U. No rational player should play a dominated strategy.

Notice that neither of player 2's strategies is dominated.

Here is another example, Figure 6.1(b).

		player 2 L	C	R
	U	$8, 3$	$0, 4$	$4, 4$
player 1	M	$4, 2$	$1, 5$	$5, 3$
	D	$3, 7$	$0, 1$	$2, 0$

Player 1's strategy D is dominated by M. However, D is not dominated by U, because if player 2 chooses C, player 1's payoff is 0 when she plays U and when she plays D. (We sometimes say that D is "weakly" dominated by U. Weak dominance is a less important concept than dominance, sometimes called "strict" dominance.)

None of player 2's strategies is dominated.

Here is a more difficult example, Figure 6.1(c).

		player 2	
		L	R
	U	$4, 1$	$0, 2$
player 1	M	$0, 0$	$4, 0$
	D	$1, 3$	$1, 2$

For player 1, clearly U and M cannot be dominated, and D cannot be dominated by any pure strategy. However, the mixed strategy, $\sigma_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ dominates D. To see this, if player 2 selects L, player 1's payoff is

$$\frac{1}{2}(4) + \frac{1}{2}(0) = 2 > 1$$

and if player 2 selects R, player 1's payoff is

$$\frac{1}{2}(0) + \frac{1}{2}(4) = 2 > 1.$$

Here is the general definition of dominance:

A pure strategy of player i, s_i, is dominated if there is a strategy (pure or mixed), σ_i, such that

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$$

for all strategy profiles $s_{-i} \in S_{-i}$ of the other players. If so, we say that σ_i dominates s_i.

A pure strategy that dominates every other pure strategy is called a *dominant* strategy.

To determine whether a strategy s_i is dominated in a matrix game:

1. Check whether it is dominated by another pure strategy. (For player 1, compare her payoff in the row corresponding to s_i with her payoff in the row corresponding to another strategy, column by column.)

2. If s_i is not dominated by another pure strategy, check whether it is dominated by a mixture of two of the other strategies. (This can be tedious, but it amounts to seeing if the mixing probability satisfies an inequality for each s_{-i}.)

For strategy D in Figure 6.1(c), the inequalities are:

$$\begin{aligned} L &: 4p + 0(1-p) > 1 \\ R &: 0p + 4(1-p) > 1. \end{aligned}$$

Tension between individual incentives and efficiency: consider the Prisoner's Dilemma.

		player 2	
		cooperate	defect
player 1	cooperate	$2, 2$	$0, 3$
	defect	$3, 0$	$1, 1$

For each player, cooperate is dominated by defect. Thus, rational players maximizing their individual payoffs leads to the payoff profile $(1, 1)$. However, this outcome is not *efficient*, because *both* players could be better off by cooperating.

Definition: A strategy profile, s, is (Pareto) efficient if there is no other strategy profile s' such that

$$u_i(s') \geq u_i(s) \text{ for all } i,$$

with strict inequality for some player.

Example:

		player 2		
		L	C	R
player 1	U	8, 3	0, 4	4, 4
	M	4, 2	1, 5	5, 3
	D	3, 7	0, 1	2, 0

Here, the efficient strategy profiles are (U, L), (U, R), and (D, L). For any other profile, there is an alternative that does not reduce anyone's payoff, and increases the payoff of at least one player.

Best Response

In many games, players have several undominated strategies, so rationality does not pin down their behavior.

Then a player's optimal strategy depends crucially on her beliefs about what strategies the other players are choosing.

Once a player has determined her beliefs, however, she should choose a strategy that is a best response to those beliefs.

Definition: Suppose player i has a belief, $\theta_{-i} \in \triangle S_{-i}$. Then player i's strategy $s_i \in S_i$ is a best response if for every $s_i' \in S_i$ we have

$$u_i(s_i, \theta_{-i}) \geq u_i(s_i', \theta_{-i}).$$

We denote the set of best responses to the belief θ_{-i} as $BR_i(\theta_{-i})$.

Example: In the Matching Pennies game,

		player 2	
		heads	tails
player 1	heads	$1, -1$	$-1, 1$
	tails	$-1, 1$	$1, -1$

for player 1, the best response to $(\frac{3}{5}, \frac{2}{5})$ is heads, and the best response to $(\frac{1}{5}, \frac{4}{5})$ is tails.

If player 1 has the belief $(\frac{1}{2}, \frac{1}{2})$, then both heads and tails are best responses.

Comparing Dominance and Best Response

Let UD_i (for undominated) denote the set of pure strategies for player i that are not dominated.

Let B_i denote the set of pure strategies for player i that are best responses for **some** belief of player i. The mathematical statement is

$$B_i = \{s_i \in S_i | \text{ there is a belief } \theta_{-i} \in \triangle S_{-i}$$

$$\text{such that } s_i \in BR_i(\theta_{-i})\}.$$

Going through all of the possible beliefs for a player can be difficult, but the following result is useful.

Result: $B_i = UD_i$.

This result says that a strategy is a best response to some belief if and only if it is undominated.

The flip side is that every dominated strategy is never a best response to any belief (obvious), and that if a strategy is never a best response, it is dominated.

For games with 3 or more players, we must make a qualification: beliefs can allow the other players' strategy choices to be correlated (e.g., with probability $\frac{1}{2}$, players 2 and 3 each choose U, and with probability $\frac{1}{2}$, players 2 and 3 each choose D). Explanation is difficult–don't ask!