Conditional Probabilities and Bayes' Rule

In order to evaluate whether a strategy profile satisfies sequential rationality, we will need to compute a player's expected payoff, which depends on the probability that she assigns to the types of the other players.

Typically, one can learn something about the types of the other players by observing their actions. This involves updating one's initial beliefs, *conditional* on observed behavior.

There is a mathematical formula that characterizes how rational individuals compute conditional probabilities, which is called *Bayes' Rule*.

Let A and B be two random events. For example, A can be the event that a defendant is guilty (that is, committed the crime) and B can be the event that the defendant's fingerprints are found at the crime scene.

Then the probability of the defendant being guilty, conditional on his fingerprints being found at the scene, denoted as $pr(A|B)$, is given by

$$pr(A|B) = \frac{pr(A \text{ and } B)}{pr(B)}.$$

This is Bayes' Rule. Another way of writing it is

$$pr(A|B) = \frac{pr(B|A)pr(A)}{pr(B)}.$$

Suppose that if the defendant is guilty, it is likely that his fingerprints will be found, $pr(B|A) = 0.8$, and that if the defendent is innocent (event $\tilde{}A$), his fingerprints are unlikely to be found, $pr(B|\tilde{}A) = 0.1$. Also suppose that the prior probability of being guilty before any evidence is collected is $pr(A) = 0.4$.

Then using Bayes' Rule, we have

$$pr(A|B) = \frac{pr(B|A)pr(A)}{pr(B)} = \frac{0.8(0.4)}{0.8(0.4) + 0.1(0.6)} = \frac{32}{38}.$$

Notice that in the denominator, we computed the probability of observing the defendant's fingerprints by adding: (i) the probability of the defendent being guilty and finding his fingerprints, and (ii) the probability of the defendant being innocent and finding his fingerprints:

$$pr(B) = pr(B|A)pr(A) + pr(B|\tilde{}A)pr(\tilde{}A)$$

We can use Bayes' Rule to solve decision problems that are otherwise very tricky, such as the "Let's Make a Deal" problem, based on the old TV show.

Suppose the contestant is given an opportunity to select one of three doors. Behind one of the doors is a great prize (say, a car), and there is nothing behind the other two doors. The host, Monty Hall, knows which door contains the car, but the contestant does not.

Before the show, the car is randomly placed behind one of the doors, so the contestant's prior probability beliefs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Since all doors are equally likely, we will simplify the discussion and assume that the contestant picks door 3.

Before her door is opened, Monty opens an empty door that the contestant did not pick. Thus, if the car is behind door 1, door 2 is opened; if the car is behind door 2, door 1 is opened. If the car is behind door 3, Monty is assumed to open door 1 with probability $\frac{1}{2}$ and door 2 with probability $\frac{1}{2}$.

After opening an empty door, Monty *always* offers the contestant the opportunity to switch doors.

If the contestant sticks with door 3, then she wins the car if the car is behind door 3, and receives nothing otherwise.

If the contestant switches to the other remaining unopened door, she wins the car if the car is behind her new door, and receives nothing if the car is behind door 3.

What would you do if you were the contestant?

Suppose door 1 is opened and shown to be empty. Let us use Bayes' Rule to determine the probability that the car is behind door 2 and the probability that the car is behind door 3, conditional on door 1 being opened.

Let $A1, A2$, or $A3$ be the events that the car is behind door 1, 2, or 3, and let B be the event that door 1 is opened. We want to compute $pr(A3|B)$.

When the car is behind door 3, the probability that door 1 is opened is $\frac{1}{2}$. Thus, $pr(B|A3) = \frac{1}{2}$.

The prior probability that the car is behind door 3 is $\frac{1}{3}$.

The probability that door 1 is opened is

$$\begin{aligned} pr(B) &= pr(B|A1)pr(A1) + pr(B|A2)pr(A2) \\ &\quad + pr(B|A3)pr(A3) \\ &= (0 \cdot \frac{1}{3}) + (1 \cdot \frac{1}{3}) + (\frac{1}{2} \cdot \frac{1}{3}) = \frac{1}{2}. \end{aligned}$$

Using the facts from the previous slide and Bayes' Rule, we have

$$\begin{aligned} pr(A3|B) &= \frac{pr(B|A3)pr(A3)}{pr(B)} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}. \end{aligned}$$

Therefore, the probability that the correct door is door 3, given that door 1 is turned over, is $\frac{1}{3}$. A similar calculation shows that $pr(A2|B) = \frac{2}{3}$, so the contestant should switch to door 2.

A Game of Herding

Suppose a "blue urn" contains 7 blue balls and 3 red balls, and a "red urn" contains 7 red balls and 3 blue balls. One of the two urns is selected at random (equal likelihood for each) and put on a table.

Then player 1 draws a ball at random (equal likelihood for each ball) from the selected urn, looks at the ball, and puts the ball back in the urn. Player 1 then makes a guess about which urn was selected.

Then player 2 draws a ball at random. She also observes player 1's guess, but not the ball picked by player 1. Then player 2 makes a guess. Then player 3 draws a ball, observes all previous guesses, and makes her own guess. And so on.

Payoffs are 1 for a correct guess, 0 for an incorrect guess.

Solving the Game of Herding makes heavy use of Bayes' rule.

Suppose player 1 observes a blue ball as his "signal." Bayes' rule gives

$$pr(\text{blue urn}|\text{blue signal}) = \frac{\frac{1}{2}\frac{7}{10}}{\frac{1}{2}\frac{7}{10} + \frac{1}{2}\frac{3}{10}} = \frac{7}{10},$$

where the numerator is the probability of the blue urn multiplied by the probability of the blue signal from the blue urn.

Player 1 will guess "blue urn" when he receives a blue signal and "red urn" when he receives a red signal.

Now to player 2. If her signal matches the guess made by player 1, clearly she will guess according to her signal.

Suppose player 1 guesses "blue urn" (B) and player 2 receives the red signal. Bayes' rule gives

$$pr(\text{blue urn}|B, \text{red signal}) = \frac{\frac{1}{2}\frac{7}{10}\frac{3}{10}}{\frac{1}{2}\frac{7}{10}\frac{3}{10} + \frac{1}{2}\frac{3}{10}\frac{7}{10}} = \frac{1}{2},$$

where the numerator is the probability of the blue urn multiplied by the probability of the blue signal from the blue urn (since player 1 guesses B if and only if he has a blue signal) multiplied by the probability of the red signal from the blue urn (player 2's signal). Intuitively, player 1's inferred blue signal and player 2's blue signal cancel each other.

Player 2 is indifferent as to what to guess, but we will assume that she will guess according to her signal when indifferent (in this case, R).

Now to player 3. If players 1 and 2 make opposite guesses (BR or RB), it must be that they have opposite signals, since player 2 guesses according to her signal when indifferent. Then player 3 will guess according to her signal.

Player 3 will guess according to her signal when it matches the guess of players 1 and 2.

Now suppose the previous guesses are BB but player 3 has the red signal, so players 1 and 2 both have the blue signal. Using Bayes' rule gives

$$pr(\text{blue urn}|BB, \text{red signal}) = \frac{\frac{1}{2}\frac{7}{10}\frac{7}{10}\frac{3}{10}}{\frac{1}{2}\frac{7}{10}\frac{7}{10}\frac{3}{10} + \frac{1}{2}\frac{3}{10}\frac{3}{10}\frac{7}{10}} = \frac{7}{10},$$

where the numerator is the probability of the blue urn multiplied by the probability of the blue signal from the blue urn (player 1) multiplied by the probability of the blue signal from the blue urn (player 2) multiplied by the probability of the red signal from the blue urn (player 3).

Player 3, if her signal does not match the guess of players 1 and 2, will go against her signal and "join the herd" created by players 1 and 2!

A herd will start whenever the imbalance between red guesses and blue guesses reaches 2. Notice that an "incorrect herd" is very possible. When information is privately held, there is a tremendous amount of information out there, but signals are not revealed to the market once the herd starts and all player types take the same action.