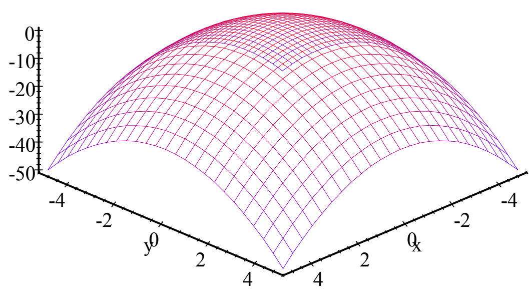
Functions of Several Variables: Partial Derivatives

Consider the function, $z = f(x, y)$. Now there is no such thing as "the" derivative of z. It depends which direction you are moving in.



The *partial derivative of z with respect to x*, $\frac{\partial z}{\partial x}$, is the slope of the function in the direction of the x-axis. In other words, it is the rate at which z changes in response to a small increase in x, *holding y constant.* This is sometimes written as $f_x(x, y)$ or $\frac{\partial f(x,y)}{\partial x}$. We can talk about the partial derivative of z with respect to y in exactly the same way.

To compute partial derivatives, treat all the other variables as constants.

example: $z = x^2y^2 + 3xy$

$\frac{\partial z}{\partial x} = 2xy^2 + 3y$ and $\frac{\partial z}{\partial y} = 2x^2y + 3x$.

Second partials: $\frac{\partial^2 z}{\partial x^2} = 2y^2$, $\frac{\partial^2 z}{\partial y^2} = 2x^2$, and $\frac{\partial^2 z}{\partial x \partial y} = 4xy + 3$.

To use the *chain rule*, the total effect on a function of several variables is the sum of the effects from each variable separately. Thus, if we have $z(\alpha) = f(x(\alpha), y(\alpha))$, then

$$\frac{dz}{d\alpha} = \frac{\partial f}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f}{\partial y}\frac{dy}{d\alpha} \tag{1}$$

example: $z(\alpha) = [x(\alpha)]^2 y(\alpha)$

$$\frac{dz}{d\alpha} = 2x(\alpha)y(\alpha)\frac{dx}{d\alpha} + [x(\alpha)]^2\frac{dy}{d\alpha} \tag{2}$$

You can plug in functions for $x(\alpha)$ and $y(\alpha)$ to verify equation (2).

Unconstrained Optimization

Functions of One Variable: $f(x)$

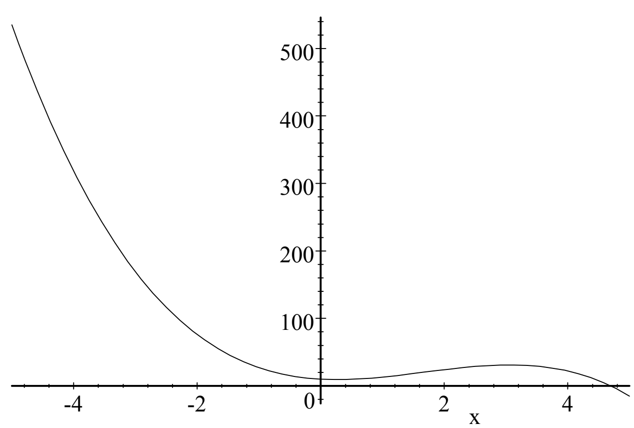
–local maximum

–local minimum

–global maximum

–global minimum

We find these maxima and minima by examining the *critical points*, where $f'(x) = 0$ and on the boundary.



$10 - 5x + 10x^2 - 2x^3$

If x^* is an *interior point* and we have $f'(x) = 0$, then

(1) $f''(x^*) < 0$ implies $f(x^*)$ is a local maximum

(2) $f''(x^*) > 0$ implies $f(x^*)$ is a local minimum

(3) $f''(x) < 0$ for all x implies $f(x^*)$ is the unique global maximum

(4) $f''(x) > 0$ for all x implies $f(x^*)$ is the unique global minimum

These are called the second order conditions.

Unconstrained Optimization: Functions of Several Variables

$z = f(x, y)$

For an interior critical point, we need:

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0. \tag{3}$$

These second order conditions are sufficient for condition (3) to determine a maximum:

$$f_{xx} < 0, f_{yy} < 0, f_{xx}f_{yy} - (f_{xy})^2 > 0 \text{ (concave function).} \tag{4}$$

These second order conditions are sufficient for condition (3) to determine a minimum:

$$f_{xx} > 0, f_{yy} > 0, f_{xx}f_{yy} - (f_{xy})^2 > 0 \text{ (convex function).} \tag{5}$$