Production

\[ \mathcal{Y}_f \] prod. poss. set for firm \( f \)

\[ y_f \in \mathcal{Y}_f \] a net output vector, where negative components are inputs.

\[ p \cdot y_f \] firm \( f \)'s profits

\text{firm's problem}: \quad \max_{y_f} \quad p \cdot y_f \\
\text{s.t.} \quad y_f \in \mathcal{Y}_f

If \( \mathcal{Y}_f \) is a strictly convex set, then the firm's optimal supply is a function \( y_f(p) \).

Aggregate prod. poss. set:

\[ \mathcal{Y} = \{ y \in \mathbb{R}^k : y = \sum_{f=1}^{F} y_f \text{ and } y_f \in \mathcal{Y}_f \} \]
prop. \( y \) maximizes aggregate profit iff each \( y_f \) maximizes firm \( f \) profit.

What about duopoly and higher profits of forming a cartel?

**Modeling labor supply**

Workers have endowments of leisure. Supply of labor is \((w_i^L - \bar{x}_i)\).

- Must require \( x_i^L \leq w_i \) for consumer \( i \) in definition of consumption set

**Accounting for Profits**

\( 0 \leq T_i f \leq 1 \) consumer \( i \)'s share of firm \( f \)\n
\[ \sum_{i=1}^{n} T_i f = 1 \] for all \( f \)
**Consumer's Problem**

\[
\begin{align*}
\text{max} & \quad U_i (X_i) \\
\text{s.t.} & \quad p \cdot X_i \leq p \cdot \omega_i + \sum_{f=1}^{F} T_{if} p \cdot Y_{f}(p) \\
& \quad x_i \in X_i
\end{align*}
\]

\[
\zeta(p) = \sum_{i=1}^{n} X_i(p) - \sum_{f=1}^{F} Y_f(p) - \sum_{i=1}^{n} \omega_i
\]

**Walras' Law**  \( \text{If utility satisfies local non-satiation,} \)

\[
p \cdot \zeta(p) = 0 \quad \text{for all} \quad p.
\]

**Proof**

\[
p \cdot \zeta(p) = \sum_{i=1}^{n} p \cdot X_i(p) - \sum_{f=1}^{F} p \cdot Y_f(p) - \sum_{i=1}^{n} p \cdot \omega_i
\]

local m.s. \( \| \sum_{i=1}^{n} p \cdot \omega_i + \sum_{i=1}^{n} \sum_{f=1}^{F} T_{if} p \cdot Y_f(p) \]

\[
p \cdot \zeta(p) = \sum_{f=1}^{F} (p \cdot Y_f(p) \sum_{i=1}^{n} T_{if}) - \sum_{f=1}^{F} p \cdot Y_f(p)
\]
Def. A C.E. is a price vector $p \in S^{K-1}$ and an allocation $\mathbf{(X,Y)}$ satisfying

1. each consumer maximizes utility subject to his/her budget constraint
2. each firm maximizes profits s.t. technology
3. markets clear: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i + \sum_{f=1}^{F} y_{if}$

Existence of C.E.
- Assumption 1.
- Consumption sets are closed, convex, and bounded from below. Endowments are in the interior of consumption sets
- $Y_f$ contains 0 is closed and convex
- Irreversibility $Y_f \cap (-Y_{f'}) = \emptyset$ if $f \neq f'$
- Free disposal

Since we are allowing for constant returns, $Z(p)$ might be a correspondence, requiring Kakutani's thm.
(x, y) is feasible if

A feasible allocation is P.O. if there is no other feasible allocation (x', y') s.t.

\( U_i(x_i) \geq U_i(x_i') \) for all \( i \).

\( U_h(x_h') > U_h(x_h) \) for some \( h \)

PFTWE: Under local nonsatiation, any C.E. allocation \((x, y)\) is P.O.

SFTWE: Suppose \((x^*, y^*)\) is P.O. \( x^* \gg y^* \). Preferences are convex, continuous, strictly monotonic, if convex. Then for some endowments and shares, \((x^*, y^*)\) is a C.E. allocation

Sketch: separate the aggregate preferred bundles \( UC \), from the feasible aggregate bundles

\[ F = \sum \omega_i + \sum_{f=1}^{F} Y_f \mid Y_f + Y_f \geq 0 \]
Note: When you are solving for a C.E.,
(i) If technologies are constant returns, for
profit max to be well defined, this gives
an equation restricting prices. You will not
be able to solve for supply functions.

(ii) With strictly convex production sets, you
can solve for supply functions, but you
must account for profits in the
consumer opt. problems.
Uncertainty

\( X_i^{j,s} \quad s \in \text{States of nature} = \{1, 2, \ldots, S\} \)
\( P_{is} \quad \text{probability of state } s \)

**Definition.** A C.E. is a price vector

\[ P = (p_1^{1,s}, \ldots, p_j^{1,s}, p_2^{1,s}, \ldots, p_j^{2,s}, \ldots, p_k^{1,s}, \ldots, p_k^{K,S}) \]

and an allocation

\[ \sum_{i=1}^{n} \sum_{s=1}^{K} X_i^{j,s} = 1, \quad j = 1, \ldots, n, \quad k = 1, \ldots, K, \quad s = 1, \ldots, S \]

such that

1. Utility maximization, \( X_i \) solves

\[ \max \sum_{s=1}^{S} \Pi_s U_i(X_i^{1,s}, \ldots, X_i^{K,s}) = V_i(X_i) \]

subject to \( p \cdot X_i \leq p \cdot c_i \)
\( X_i \geq 0 \)

2. Market clearing

for all \( j \) and \( s \)

\[ \sum_{i=1}^{n} X_i^{j,s} \leq \sum_{i=1}^{n} c_i^{j,s} \]

**Note:**

1. \( X_i^{j,s} \) is a state-contingent commodity.
2. Good \( j \) is delivered iff the state is \( S \)
3. Specification is not completely general.

Von Neumann-Morgenstern does not allow utility of consuming an umbrella to depend on the state (except through \( P_{is} \)).
\[ W(\mathbf{x}, \mathbf{\theta}) = \Pi^x(\mathbf{x})^2 + \Pi^\theta(\mathbf{\theta})^2 \]

Each \( U(x) = x^2 \) is strictly quasi-concave

\[ x > y \implies (\theta x + (1-\theta)y)^2 > y^2 \]

\[ V \] is not strictly quasi-concave

Trade-offs are being made across states — the level of utility within a state matters. Risk aversion

\[ U(x) = \frac{x^{1-\alpha}}{1-\alpha} \quad \text{if } a > 0 \]

\[ U(x) = \log(x) \text{ if } a = 1 \]

\[ -\frac{U''(x)}{U'(x)} \times x = \alpha \quad \text{coefficient of relative risk aversion} \]
If each $u_i$ is strictly increasing, continuous, strictly concave, $C \in \mathbb{R}_{++}^{n \times k}$

1. Consumers are risk averse
2. Equilibrium exists and is Pareto optimal
3. Any P.A. allocation is a C.E. allocation for some redistribution of state-contingent endowments

$u_i$ differentiable, monotonicity, concavity

Special Case: $k=1$, no aggregate uncertainty

F.O.C.:

$$\Pi_s \frac{\partial u_i(X^s_i)}{\partial X^s_i} = \frac{P^s}{p^s} \quad \text{for all } s, s'$$

$$\sum_{s=1}^{S'} p^s X^s_i = \sum_{s=1}^{S'} p^s \omega^s_i$$

No aggregate uncertainty:

$$\sum_{i=1}^{n} \omega^s_i = \sum_{i=1}^{n} \omega^{s'}_i$$
Claim (i) Any Pareto Optimal point satisfies

\[ X_i^s = X_i^{s'} \quad \text{for all } s, s' \quad \text{and for all } i \]

(ii) Any point satisfying (i) is Pareto optimal

Proof of (i) Since \( x \) is P.O., each consumer has the same MRS at \( x \).

\[
\frac{\partial U_i(X_i^s)}{\partial x_i^s} = \frac{\partial U_h(X_h^{s'})}{\partial x_h^{s'}} \quad \forall i, h, s, s'
\]

Suppose \( X_i^s > X_i^{s'} \) for some \( i, s, s' \). Since we have no aggregate uncertainty,

\( X_h^s < X_h^{s'} \) for some consumer \( h \).

Since \( U_i \) and \( U_h \) are concave,

\[
\frac{\partial U_i(X_i^s)}{\partial x_i^s} < \frac{\partial U_i(X_i^{s'})}{\partial x_i^{s'}} \quad \frac{\partial U_h(X_h^s)}{\partial x_h^s} > \frac{\partial U_h(X_h^{s'})}{\partial x_h^{s'}}
\]

Contradicts Pareto optimality. Therefore, \( X_i^s = X_i^{s'} \) \( \forall i, s, s' \).
Proof of (ii) Any point satisfying (*) has
\[
\frac{\partial u_i(x^s_i)}{\partial x^s_i} = \frac{\Pi^s}{\Pi^s_i} \text{ for all } i, s, s' \text{ and}
\]
is therefore Pareto optimal.

\[N = 2, \quad S' = 2, \quad K = 1\]

Contrast curve is 45° line

Note: unique C.E. \( (p^1_s, \ldots, p^s_s, \ldots, p^{1S}_s) = (\Pi^1_s, \ldots, \Pi^s_s, \ldots, \Pi^{1S}_s) \)
(only relies on \( K = 1 \))
Complete Markets (?)

(1) A country experiencing an earthquake is disproportionately hit financially.
(2) State of nature must specify whose hard disk crashes, who has an accident, indigestion ....
(3) Moral hazard problems
(4) Adverse selection can destroy insurance
(5) It is costly to set up markets — is there a more economical way? Arrow Securities
Arrow Securities Market

Stage 1: consumers buy and sell securities, where a security $s$ pays 1 unit of account on the state-$s$ spot market.

Stage 2: the state of nature is observed, securities are redeemed, and commodities are traded on a spot market.

- $b_i^s$: consumer $i$'s holdings of security $s$. $b_i^s > 0$ implies $i$ receives $b_i^s$ "dollars" in state $s$.
- $q^s$: the price of security $s$.
- $X_i(s)$: consumption of commodity $j$ received from the state-$s$ spot market.
- $p_j^s(s)$: price of commodity $j$ on the state-$s$ spot market.

Utility Max Problem

$$\max_{X_i(s), b_i^s} \sum_{s=1}^{S} \pi_i \ u_i(X_i(s))$$

s.t. $\sum_{s=1}^{S} q^s b_i^s \leq 0$

$$\sum_{j=1}^{K} p_j^s(s) x_i^j(s) \leq \sum_{j=1}^{K} p_j^s(s) w_j^s(s) + b_i^s \quad \text{for all } s$$
Market Clearing

\[
\sum_{i=1}^{n} b_i^s \leq 0 \quad \text{for all } s
\]
\[
\sum_{i=1}^{n} x_i^j(s) \leq \sum_{i=1}^{n} \omega_i^j(s) \quad \text{for all } j \text{ and } s
\]

Notice: 1. \( \Pi_s > 0 \) and monotonicity implies \( q^s > 0 \)
2. We can normalize \( q^s = 1 \) without affecting the set of affordable bundles.

\[
F(q, P) = F(2q, P)
\]

3. We can normalize \( p^s(s) = 1 \) without loss of generality, for each \( s \).

\[
F(q^s, \ldots, q^s, p^s, \ldots, p(s), \ldots, p(s')) = F(q^s, \ldots, q^s, \ldots, p(s), \ldots, p(s'))
\]

Def. A C.E. is a set of prices \( \{q, P(s)\} \), base security holdings \( \sum b^s_i s_{i=1}^s \), and consumption \( \sum x_i(s) s_{i=1}^s \), satisfying market clearing and utility maximization.
Thm (Arrow) The contingent commodities model and the Arrow Securities model are equivalent.

\[ \exists X_i^{j,s} \sum \in B_i \Rightarrow \exists X_i^{j} \sum \in B_i \text{ for some } \sum \in B_i \]

\[ \exists X_i^{j,s} (s) \sum \text{ is a C.E. allocation, where } X_i^{j,s} (s) = X_i^{j,s} \forall i, j, s \]

AND \[ \exists X_i^{j,s} (s) \sum \text{ is a C.E. allocation } \Rightarrow \exists X_i^{j,s} \sum \text{ is a C.E. allocation, where } X_i^{j,s} = X_i^{j} (s) \).

"proof" Construct the Arrow Security Prices from the A-D prices and vice versa.
Relative price of consumption is 

- give up 1 unit of \( X^1(s) \)
- receive \( p(s) \) units of account
- allows you to demand \( p(s) \) fewer state-5 securities
- allows you to have \( p(s) \cdot 2^{s} \) more income on the securities market.
- buy \( \frac{p(s)}{2^{s}} \) more state-5 securities

allows you to buy \( \frac{p^1(s) \cdot 2^5}{p^2(s) \cdot 2^5} \) units of \( X^2(s') \)

price of \( X^1(s) \) [in terms of \( X^1(s') \)] is \( \frac{p^1(s) \cdot 2^5}{p^2(s') \cdot 2^5} \)
example: \( u_i(x_i^s) = \log(x_i^s) \)

\( \pi_\alpha = \frac{1}{4} \quad \pi_\beta = \frac{3}{4} \)

\( \omega_1 = (1,2) \quad \omega_2 = (2,1) \)

normalized:

\( p(\alpha) = p(\beta) = 1 \)

Consumer 1:

\[
\max_{x_1, b_1} \frac{1}{4} \log(x_1^\alpha) + \frac{3}{4} \log(x_1^\beta) \\
\text{s.t.} \quad q_\alpha x_1^\alpha + q_\beta b_1^\beta = 0 \\
\quad x_1^\alpha = 1 + b_1^\alpha \\
\quad x_1^\beta = 2 + b_1^\beta
\]

substitute spot market budget constraints into sec. market, to get

\[
q_\alpha (x_1^\alpha - 1) + q_\beta (x_1^\beta - 2) = 0 \tag{eq 1}
\]

f.o.c. (multiplier \( \lambda_1 \)):

\[
\frac{1}{4x_1^\alpha} - \lambda_1 q_\alpha = 0 \tag{eq 2}
\]

\[
\frac{3}{4x_1^\beta} - \lambda_1 q_\beta = 0 \tag{eq 3}
\]

Solving, \( \lambda_1 = \frac{1}{4q_\alpha x_1^\alpha} = \frac{3}{4q_\beta x_1^\beta} \)

so \( x_1^\alpha = \frac{3q_\alpha x_1^\alpha}{q_\beta} \tag{eq 4} \)
Solving (eq.1) and (eq.4), we have
\[ q^\alpha x - q^\alpha + 3 q^\alpha x - 2 q^\beta = 0 \]

\[ \therefore x^\alpha_1 = \frac{q^\alpha + 2 q^\beta}{4 q^\alpha} \quad \text{and} \quad x^\beta_1 = \frac{3 (q^\alpha + 2 q^\beta)}{4 q^\beta} \]

Going through the same steps for consumer 2 yields
\[ x^\alpha_2 = \frac{2 q^\alpha + q^\beta}{4 q^\alpha} \quad \text{and} \quad x^\beta_2 = \frac{3 (2 q^\alpha + q^\beta)}{4 q^\beta} \]

Let us normalize, \( q^\beta = 1 \)

Market clearing on the \( P \)-spot market:
\[ \frac{3 (q^\alpha + 2)}{4} + \frac{3 (2 q^\alpha + 1)}{4} = 3 \]

\[ \therefore q^\alpha = \frac{1}{3} \]

\[ x_1 = \left( \frac{7}{4}, \frac{7}{4} \right) \quad x_2 = \left( \frac{5}{4}, \frac{5}{4} \right) \]

\[ b_1 = \left( \frac{3}{4}, -\frac{1}{4} \right) \quad b_2 = \left( -\frac{3}{4}, \frac{1}{4} \right) \]

Note: no aggregate uncertainty
Production and Uncertainty

\[ Y_f \]
\[ \text{is a set in } \mathbb{R}^{kS} \]

- Usually, output in state \( s \) only depends on inputs in state \( s \).

- Often, input of commodity \( (j, s) \) must be the same as input of \( (j, s') \). For example, a worker is hired to work, no matter what the state is.

- Setup allows for "technology shocks," since output depends on \( s_j \), even for the same inputs.

- Usually, there is joint production, since output is not only produced in one state.

- The firm's profits, \( P \cdot Y_f \), do not depend on the realized state! It buys and sells state-contingent contracts before the state is known.

- If firms are risk neutral, how can a C.E. be Pareto optimal, then?
More complicated securities:

firm's gross return \[ R(s) = \begin{cases} 0 & s = 1 \\ 1 & s = 2 \\ 5 & s = 3 \\ 5 & s = 4 \\ 10 & s = 5 \end{cases} \]

vs. \[ \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \]

debt level: \[ D(s) = \begin{cases} 0 & s = 1 \\ 1 & s = 2 \\ 3 & s = 3 \\ 3 & s = 4 \\ 3 & s = 5 \end{cases} \]

\[ \Delta(s) = \min \left[ R(s), 3 \right] \]

equity: \[ E(s) = R(s) - D(s) \]

\[ E(s) = \begin{cases} 0 & s = 1 \\ 0 & s = 2 \\ 2 & s = 3 \\ 2 & s = 4 \\ 7 & s = 5 \end{cases} \]

\* \* \* arbitrage: \[ P^E + P^D = P^R \]

\* \* \* \[ p(s) \cdot x_i(s) \leq p(s) \cdot c_i(s) + b \cdot R(s) + b \cdot \bar{D} + b \cdot \bar{E} \]

real vs. nominal