A Dynamic Oligopoly Game

Consider the following Stackelberg game with two firms and two states of nature. In the high state, the inverse demand function is given by

$$p^H = 120 - q_1 - q_2$$

where p^H is the price and q_i is the quantity supplied by firm i ($i = 1, 2$). In the low state, the inverse demand function is given by

$$p^L = 80 - q_1 - q_2$$

The probability of each state is $\frac{1}{2}$. Assume that production costs are zero.

The timing of moves is the following. Firm 1 moves first and selects its quantity. Then firm 2 chooses its quantity after observing firm 1's quantity. That is, q_2 can be a function of q_1.

(a) Suppose firm 2 can observe the true state of nature before choosing its quantity, but firm 1 cannot. Calculate the unique subgame perfect Bayesian Nash equilibrium. Be careful to fully specify firm 2's strategy.

(b) Suppose firm 1 can observe the true state of nature before choosing its quantity, but firm 2 cannot. Calculate one weak perfect Bayesian equilibrium (WPBE) of this game. Remember to specify beliefs as well as strategies.

(c) Without doing any calculations, carefully explain the intuition for why there are many WPBE of the game in part (b).

Answers

(a) Since firm 1 does not know the state, we have a subgame starting when firm 2 observes the state and firm 1's quantity. Given q_1, firm 2's profit in the high state is given by

$$\pi_2^H = (120 - q_1 - q_2^H)q_2^H. \tag{1}$$

Differentiating (1) with respect to q_2^H, and setting the expression equal to zero, we have firm 2's optimal strategy for the subgame,

$$q_2^H = 60 - \frac{q_1}{2}. \tag{2}$$

A similar calculation for the low state yields

$$q_2^L = 40 - \frac{q_1}{2}. \tag{3}$$

Thus, (2) and (3) constitute firm 2's subgame perfect equilibrium strategy.

To find firm 1's strategy, write the expected profit as

$$\pi_1 = \frac{1}{2}(120 - q_1 - (60 - \frac{q_1}{2}))q_1 + \frac{1}{2}(80 - q_1 - (40 - \frac{q_1}{2}))q_1. \tag{4}$$

Notice that firm 1 takes into account the impact of its output choice on firm 2's output choice. Differentiating (4) and setting the expression equal to zero, we have $q_1 = 50$. Thus, the subgame perfect Nash equilibrium is

$$\begin{aligned} q_1 &= 50 \\ q_2^H &= 60 - \frac{q_1}{2} \\ q_2^L &= 40 - \frac{q_1}{2} \end{aligned}$$

On the equilibrium path, we have $q_2^H = 35$ and $q_2^L = 15$, but a strategy for firm 2 must specify its output for every possible output it observes from firm 1.

(b) When firm 1 observes the state and firm 2 does not, then every value of q_1 defines an information set for firm 2, containing one node corresponding to the high state and one node corresponding to the low state. To construct a WPBE, start with firm 1's strategy, (q_1^H, q_1^L). We will construct a separating equilibrium, where we have $q_1^H \neq q_1^L$. It follows that, *on the equilibrium path*, firm 2 will be able to infer the state of nature based upon whether it observes q_1^H or q_1^L.

Firm 2's strategy can be written as $q_2(q_1)$. If firm 2 observes q_1^H, then it infers that the state is high with probability one, and sequential rationality yields

$$q_2(q_1^H) = 60 - \frac{q_1^H}{2}. \tag{5}$$

Similarly, if firm 2 observes q_1^L, then if infers that the state is low with probability one, and we have

$$q_2(q_1^L) = 40 - \frac{q_1^L}{2}. \tag{6}$$

Information sets following quantities other than q_1^H or q_1^L are off the equilibrium path, and the sequentially rational quantity for firm 2 depends on its beliefs. Because Bayes' rule does not apply, any beliefs are consistent. To help our construction of a WPBE, lets specify beliefs that makes firm 1 least likely to want to deviate from the equilibrium: Firm 2 believes that any deviation from firm 1 signals that the state is high with probability one (inducing firm 2 to produce more output, which is bad for firm 1). Thus, we have

$$\begin{aligned} q_2(q_1) &= 60 - \frac{q_1}{2} \quad \text{for } q_1 \neq q_1^H, q_1^L \qquad (7) \\ \mu(q_1, H) &= 1 \text{ and } \mu(q_1, L) = 0 \quad \text{for } q_1 \neq q_1^H, q_1^L. \end{aligned}$$

What values of q_1^H and q_1^L are consistent with equilibrium? Since firm 1 is admitting that the state is high when it chooses q_1^H, that quantity must be the Stackelberg equilibrium quantity of the game where demand is known to be high. Firm 1's profits can be written as

$$\pi_1^H = (120 - q_1^H - (60 - \frac{q_1^H}{2}))q_1^H, \tag{8}$$

which is maximized by choosing $q_1^H = 60$.

There are many values of q_1^L that are consistent with equilibrium. However, the Stackelberg equilibrium quantity of the game where demand is known to be low, $q_1 = 40$, will **not** work. The reason is that a high-type firm 1 could profitably deviate to $q_1 = 40$, taking advantage of the fact that it is fooling firm 2 into thinking that the state is low. Any value of q_1^L is consistent with equilibrium, as long as a low-type firm 1 does not want to deviate to a different quantity, and a high-type firm 1 does not want to pretend to be a low-type by choosing q_1^L. For example, suppose we set $q_1^L = 10$. Then firm 2 will produce 35 units (from (6)), the price will be 35, and a low-type firm 1 receives profits of 350. If firm 1 chooses any other quantity, firm 2 will believe that the state is high, and choose output according to (5). You can check that the most that firm 1 can receive is a profit of 200 (by producing 20 units). Thus, a low-type firm 1 does not want to deviate.

For a high-type firm 1, maximization of (8) implies that it does not want to deviate unless it can trick firm 2 into believing that the state is low. If firm 1 chooses an output of 10, then firm 2 produces 35 units (from (6)), the price is 75, and firm 1 receives profits of 750. By choosing output of 60, firm 2 produces 30 units, the price is 30, and firm 1 receives profits of 1800. Thus, firm 1 does not want to deviate.

Recapping, an equilibrium is

$$\begin{aligned}
q_1^H &= 60 \\
q_1^L &= 10 \\
q_2(q_1) &= 60 - \frac{q_1}{2} \qquad \text{for } q_1 \neq 10 \\
q_2(10) &= 35 \\
\mu(q_1, H) &= 1 \ \text{ and } \ \mu(q_1, L) = 0 \quad \text{for } q_1 \neq 10 \\
\mu(q_1, H) &= 0 \ \text{ and } \ \mu(q_1, L) = 1 \quad \text{for } q_1 = 10.
\end{aligned}$$

Besides there being many of these "separating" equilibria, there are also many pooling equilibria, where we have $q_1^H = q_1^L$. In a pooling equilibrium, then firm 2 must assign probability $\frac{1}{2}$ to each state if firm 1 produces its equilibrium output, and any beliefs are consistent if firm 1 deviates. To help construct the equilibrium, specify beliefs that provide the least incentive for firm 1 to deviate: have firm 2 believe that the state is high if firm 1 chooses an output that is off the equilibrium path. The key condition to check is that a high-type firm 1 does not want to deviate, given that firm 2 will change its conditional probability of state H from $\frac{1}{2}$ to 1. I will leave the details to you.

(c) The intuition is that many belief systems are consistent with equilibrium, because at most two quantities for firm 1 are on the equilibrium path. We can discourage deviations by believing that the deviation signals that the state is high. For example, the equilibrium specified above has firm 2 believing that the state is low if $q_1 = 10$, but the state is high if $q_1 = 9.99$ or 10.01. The discontinuous change in beliefs takes away the incentive to deviate to a quantity that is off the equilibrium path, but the point of discontinuity was chosen arbitrarily. Any beliefs off the equilibrium path are consistent, but these beliefs determine what is on and what is off the equilibrium path.