Department of Economics
The Ohio State University
Midterm Answers–Econ 805

Prof. James Peck
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1. (40 points)

Consider the following pure-exchange economy with two consumers and two goods. Consumer 1 has the utility function

$$u_1(x_1^1, x_1^2) = \log(x_1^1) + \log(x_1^2)$$

and the initial endowment vector, $(A, 0)$, where A is a positive number. Consumer 2 has the quasi-linear utility function

$$u_2(x_2^1, x_2^2) = \frac{(x_2^1)^{1-\alpha}}{1-\alpha} + x_2^2$$

and the initial endowment vector, $(0, B)$, where B is a positive number and α is a parameter of the utility function satisfying $\alpha > 2$. (It is related to the demand elasticity for good 1.)

(a) (10 points) Define a competitive equilibrium for this economy.
(b) (15 points) Compute the competitive equilibrium price and allocation, which will depend on the parameters, A, B, and α. You can assume that B is large enough so that consumer 2's utility maximization problem has an interior solution with $x_2^2 > 0$.
(c) (10 points) Show that consumer 1 would receive higher utility by destroying some of her endowment before trading occurs. (You can substitute her consumption bundle from (b) into the utility function, and show that the resulting expression is decreasing in A.)
(d) (5 points) Briefly explain why the result in part (c) holds. You can provide the intuition even if you are not able to solve part (c).

Answer:
(a) A C.E. is a price vector, (p^1, p^2), and an allocation, $(x_1^1, x_1^2, x_2^1, x_2^2)$, such that (equalities are due to the fact that utility functions are strictly monotonic)
(i) (x_1^1, x_1^2) solves

$$\begin{aligned} & \max \log(x_1^1) + \log(x_1^2) \\ & \text{subject to} \\ p^1 x_1^1 + p^2 x_1^2 &= p^1 A \\ x_1 &\geq 0 \end{aligned}$$

(ii) (x_2^1, x_2^2) solves

$$\begin{aligned} & \max \frac{(x_2^1)^{1-\alpha}}{1-\alpha} + x_2^2 \\ & \text{subject to} \\ p^1 x_2^1 + p^2 x_2^2 &= p^2 B \\ x_2 &\geq 0 \end{aligned}$$

(iii) markets clear

$$\begin{aligned} x_1^1 + x_2^1 &= A \\ x_1^2 + x_2^2 &= B. \end{aligned}$$

(b) Normalize prices to be $(p, 1)$. The first-order conditions for consumer 1's UMP are the budget equation and the MRS condition

$$\frac{x_1^2}{x_1^1} = p,$$

yielding the demand functions

$$x_1^1 = \frac{A}{2} \quad \text{and} \quad x_1^2 = \frac{pA}{2}.$$

The first-order conditions for consumer 2's UMP are the budget equation and the MRS condition

$$(x_2^1)^{-\alpha} = p,$$

yielding the demand functions

$$x_2^1 = p^{-1/\alpha} \quad \text{and} \quad x_2^2 = B - p^{1-1/\alpha}.$$

Using market clearing for good 1, we have

$$\frac{A}{2} + p^{-1/\alpha} = A,$$

which can be solved to yield

$$p = (\frac{A}{2})^{-\alpha}.$$

Substituting the price into the demand functions yields the C.E. allocation,

$$x_1 = (\frac{A}{2}, (\frac{A}{2})^{1-\alpha}) \quad \text{and} \quad x_2 = (\frac{A}{2}, B - (\frac{A}{2})^{1-\alpha}).$$

(c) In the C.E., consumer 1 receives utility of

$$\begin{aligned} & \log(\frac{A}{2}) + \log((\frac{A}{2})^{1-\alpha}) \\ = \; & (2-\alpha)\log(\frac{A}{2}) = (2-\alpha)\log(A) - (2-\alpha)\log(2). \end{aligned}$$

Differentiating with respect to A, we have

$$\frac{\partial u_1}{\partial A} = \frac{(2-\alpha)}{A},$$

which is negative since $\alpha > 2$.

(d) The result holds because consumer 1 is manipulating the price by destroying some of her endowment. The destruction makes good 1 more scarce, thereby increasing the price of good 1. Because of the inelastic demand that consumer 2 has for good 1, the increase in the price of good 1 more than compensates for the reduction in her endowment of good 1, thereby increasing the value of her endowment by enough to reach a higher utility. (She consumes less of good 1 but much more of good 2.)

2. (20 points)

Consider a pure-exchange economy with n consumers and k goods. Consider a strongly Pareto optimal allocation x^*, and assume that x^* is the initial endowment allocation, so $\omega_i = x_i^*$ for $i = 1, ..., n$. Without assuming anything about the utility functions, prove that if this economy has a competitive equilibrium, $(\widehat{p}, \widehat{x})$, then $(\widehat{p}, x^*)$ is a competitive equilibrium.

Note: This proof can be done in 3 or 4 lines.

Answer: We are given that $(\widehat{p}, \widehat{x})$ is a CE, so $\widehat{x}_i$ solves consumer i's UMP when prices are $\widehat{p}$. Suppose that $(\widehat{p}, x^*)$ is not a CE, in which case there is a consumer i such that x_i^* does not solve consumer i's UMP when prices are $\widehat{p}$. Then we have

$$u_i(\widehat{x}_i) > u_i(x_i^*).$$

But for all consumers h, $\widehat{x}_h$ solves consumer h's UMP and x_h^* was affordable, so we must have

$$u_h(\widehat{x}_h) \geq u_h(x_h^*).$$

This contradicts the fact that x^* is SPO.

3. (40 points)

The following economy has 2 **equally likely** states of nature (call them α and β), 2 physical commodities per state (think of commodity 1 as labor/leisure and commodity 2 as food), one firm, and one consumer (who owns the firm). The consumer is a von Neumann-Morgenstern expected utility maximizer, with Bernoulli utility function

$$u(x^{1,s}, x^{2,s}) = \log(x^{1,s}) + \log(x^{2,s}).$$

The consumer is endowed with 1 unit of commodity 1 in each state, and zero units of commodity 2 in each state.

The firm's technology is such that the labor input (commodity 1) must be delivered before the state of nature is observed. Thus, rather than the usual 4 state-contingent commodities, we can think of the following 3 commodities being traded before the state of nature is observed: (i) commodity 1 to be delivered in both states, (ii) commodity 2 to be delivered contingent on the state being α, and (iii) commodity 2 to be delivered contingent on the state being β. Normalizing the price of commodity 1 to be 1, denote the price vector as $(1, p^{2,\alpha}, p^{2,\beta})$ and denote an allocation as $(x^1, x^{2,\alpha}, x^{2,\beta}, y^1, y^{2,\alpha}, y^{2,\beta})$. The frontier of the firm's production set is given by

$$\begin{aligned} y^{2,\alpha} &= (-3y^1)^{1/2} \\ y^{2,\beta} &= (-12y^1)^{1/2} \end{aligned}$$

(a) (10 points) Define a competitive equilibrium for this economy.
(b) (30 points) Compute the competitive equilibrium price and allocation.

Suggestions: You might want to work with the input of good 1 instead of a negative net output of good 1. You can define $L = -y^1$ and work with the L notation. Also, if you get bogged down in algebra, make sure you specify the equations that can be solved for $p^{2,\alpha}$ and $p^{2,\beta}$ to maximize your partial credit.

Answer:

(a) I will use the notation $L = -y^1$. A CE is a price vector $(1, p^{2,\alpha}, p^{2,\beta})$ and an allocation $(x^1, x^{2,\alpha}, x^{2,\beta}, L, y^{2,\alpha}, y^{2,\beta})$ such that

(i) $(x^1, x^{2,\alpha}, x^{2,\beta})$ solves

$$\begin{aligned} & \max \log(x^1) + \frac{1}{2}\log(x^{2,\alpha}) + \frac{1}{2}\log(x^{2,\beta}) \\ & \text{subject to} \\ x^1 + p^{2,\alpha}x^{2,\alpha} + p^{2,\beta}x^{2,\beta} &= 1 + \pi \end{aligned}$$

(ii) $(L, y^{2,\alpha}, y^{2,\beta})$ solves

$$\begin{aligned} \max \pi &\equiv p^{2,\alpha}y^{2,\alpha} + p^{2,\beta}y^{2,\beta} - L \\ & \text{subject to} \\ y^{2,\alpha} &= (3L)^{1/2} \\ y^{2,\beta} &= (12L)^{1/2} \\ L &\geq 0 \end{aligned}$$

(iii) markets clear

$$\begin{aligned} x^1 + L &= 1 \\ x^{2,\alpha} &= y^{2,\alpha} \\ x^{2,\beta} &= y^{2,\beta}. \end{aligned}$$

Note: In the definition, monotonicity implies that we can use equalities and ignore nonnegativity constraints on consumption.

(b) For utility maximization, the first order conditions are the following two MRS equations and the budget equation:

$$\frac{x^{2,\alpha}}{\frac{1}{2}x^1} = \frac{1}{p^{2,\alpha}}$$

$$\frac{x^{2,\beta}}{\frac{1}{2}x^1} = \frac{1}{p^{2,\beta}}.$$

Solving each MRS equation for state-contingent good 2 consumption and substituting into the budget equation, we can solve for the demand functions:

$$x^1 = \frac{1+\pi}{2}$$

$$x^{2,\alpha} = \frac{1+\pi}{4p^{2,\alpha}}$$

$$x^{2,\beta} = \frac{1+\pi}{4p^{2,\beta}}.$$

Substituting the constraints into the profit function, we have the unconstrained problem

$$\max p^{2,\alpha}(3L)^{1/2} + p^{2,\beta}(12L)^{1/2} - L.$$

Defining $Q = [p^{2,\alpha}(3)^{1/2} + p^{2,\beta}(12)^{1/2}]$, we can rewrite the problem as

$$\max_L Q(L)^{1/2} - L.$$

Setting the derivative equal to zero and solving for L, we have

$$L = \frac{Q^2}{4}$$

$$\pi = \frac{Q^2}{4}, y^{2,\alpha} = (3L)^{1/2} = (3)^{1/2}\frac{Q}{2}, y^{2,\beta} = (12L)^{1/2} = (3)^{1/2}Q.$$

Market clearing for good 1 yields

$$\frac{1+\frac{Q^2}{4}}{2} + \frac{Q^2}{4} = 1$$

which can be solved for

$$Q = \frac{2}{\sqrt{3}} \text{ and } \pi = \frac{1}{3}.$$

Market clearing for good $(2, \alpha)$ yields

$$\frac{\frac{4}{3}}{4p^{2,\alpha}} = (3L)^{1/2} = 1$$

$$p^{2,\alpha} = \frac{1}{3}.$$

We can use the definition of Q to solve for the other price,

$$p^{2,\beta} = \frac{1}{6}.$$

The equilibrium allocation is then given by

$$\begin{aligned} x &= (\frac{2}{3}, 1, 2) \\ y &= (\frac{1}{3}, 1, 2). \end{aligned}$$