Department of Economics
The Ohio State University
Midterm Exam Questions and Answers–Econ 805

Prof. James Peck
Winter 2012

1. (30 points)

A pure-exchange economy has n consumers and two goods. The aggregate excess demand functions for goods 1 and 2, defined for all strictly positive price vectors $p = (p^1, p^2)$, are given by

$$\begin{aligned} Z^1(p) &= \frac{p^1 + 3p^2}{2p^1} - A \\ Z^2(p) &= \frac{p^1 + 3p^2}{2p^2} - B, \end{aligned}$$

where A and B are real numbers. Assume that these excess demand functions are derived from each consumer i maximizing a strictly monotonic utility function subject to the budget constraint $p \cdot x_i \leq p \cdot \omega_i$.

(a) (20 points) Find all values of A and B that are consistent with the aggregate excess demand functions being derived from utility maximization subject to a budget constraint.

(b) (10 points) Normalizing prices to lie in the simplex (i.e., $p^1 + p^2 = 1$) and using your answer from part (a), find the competitive equilibrium price vector.

Answer: (a) If demand is based on utility maximization subject to a budget constraint, Walras' Law implies that for all price vectors, we have:

$$\begin{aligned} p^1 Z^1(p) + p^2 Z^2(p) &= 0, \text{ or} \\ p^1\left[\frac{p^1 + 3p^2}{2p^1} - A\right] + p^2\left[\frac{p^1 + 3p^2}{2p^2} - B\right] &= 0 \\ \frac{p^1 + 3p^2}{2} - Ap^1 + \frac{p^1 + 3p^2}{2} - Bp^2 &= 0 \\ p^1[1 - A] + p^2[3 - B] &= 0. \qquad (1) \end{aligned}$$

Since (1) must hold for all (p^1, p^2), the numbers A and B must satisfy $A = 1$ and $B = 3$.

(b) Because utility is strictly monotonic, a necessary and sufficient condition for a competitive equilibrium price vector is that excess demand is zero. Substituting $A = 1$ into the equation $Z^1(p) = 0$, we have

$$\frac{p^1 + 3p^2}{2p^1} - 1 = 0.$$

Solving yields $\frac{p^1}{p^2} = 3$. Equating excess demand for good 2 equal to zero yields the same answer. Thus, the normalized price vector is $(\frac{3}{4}, \frac{1}{4})$.

2. (35 points)

The following economy has two goods, food and labor/leisure, one consumer, and two firms. Normalize the price of labor/leisure to be 1 and denote the price of food as p. Denoting the consumer's consumption of food as x and her consumption of leisure as ℓ, her utility function is given by

$$u(x, \ell) = \log(x) + \log(\ell).$$

The consumer (we can call her Robin Crusoe) has an initial endowment of 0 units of food and 1 unit of labor/leisure, and she owns both firms.

Firm 1 produces food using labor as an input, and seeks to maximize profits. Denoting firm 1's (positive) labor input as L_1 and its output as y_1, its technology is characterized by the production function

$$y_1 = (L_1)^{1/2}.$$

Firm 2 also produces food using labor as an input, and seeks to maximize profits. Denoting firm 2's (positive) labor input as L_2 and its output as y_2, its technology is characterized by the production function

$$y_2 = (2L_2)^{1/2}.$$

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (20 points) Compute the competitive equilibrium price and allocation.

(c) (5 points) Is there any way Robin could achieve higher utility by changing the amount of labor she supplies to the two firms and consuming the output without going through the market process? Very briefly explain why.

Answer: (a) A competitive equilibrium is a (normalized) price vector, $(p, 1)$, and an allocation, $(x, \ell, y_1, y_2, L_1, L_2)$, such that:

(i) (y_1, L_1) solves

$$\begin{aligned} \max \pi_1 &\equiv py_1 - L_1 \\ &\text{subject to} \\ y_1 &= (L_1)^{1/2} \end{aligned}$$

(ii) (y_2, L_2) solves

$$\begin{aligned} &\max \pi_2 \equiv py_2 - L_2 \\ &\text{subject to} \\ y_2 &= (2L_2)^{1/2} \end{aligned}$$

(iii) (x, ℓ) solves

$$\begin{aligned} & \max \log(x) + \log(\ell) \\ & \text{subject to} \\ px + \ell &= 1 + \pi_1 + \pi_2 \qquad (2) \\ x &\geq 0, \ell \geq 0 \end{aligned}$$

(iv) markets clear:

$$\begin{aligned} x &= y_1 + y_2 \\ \ell + L_1 + L_2 &= 1 \end{aligned}$$

Note: market clearing and budget conditions are written as equalities because utility is strictly monotonic.

(b) Consumer demand functions are found by simultaneously solving (2) and the marginal rate of substitution condition,

$$\frac{\ell}{x} = p, \tag{3}$$

yielding

$$x = \frac{1 + \pi_1 + \pi_2}{2p} \tag{4}$$

$$\ell = \frac{1 + \pi_1 + \pi_2}{2} \tag{5}$$

To solve firm 1's profit maximization problem, substitute the constraint into the objective to write profits as a function of labor input, $p(L_1)^{1/2} - L_1$. Then differentiate, set the expression equal to zero, and solve:

$$\begin{aligned} \frac{1}{2}p(L_1)^{-1/2} &= 1 \\ L_1 &= \frac{p^2}{4}. \qquad (6) \end{aligned}$$

Substituting (6) into the production function and into the profit function, we have

$$\begin{aligned} y_1 &= \frac{p}{2} \\ \pi_1 &= \frac{p^2}{4}. \end{aligned}$$

Going through the same steps to solve firm 2's profit maximization problem, we have

$$\begin{aligned} L_2 &= \frac{p^2}{2} \\ y_2 &= p \\ \pi_2 &= \frac{p^2}{2}. \end{aligned}$$

Using the computed demand and supply functions, market clearing for labor/leisure becomes

$$\begin{aligned}
\frac{1+\pi_1+\pi_2}{2}+\frac{p^2}{4}+\frac{p^2}{2} &= 1\\
[\frac{1+\frac{p^2}{4}+\frac{p^2}{2}}{2}]+\frac{p^2}{4}+\frac{p^2}{2} &= 1\\
[\frac{1}{2}+\frac{p^2}{8}+\frac{p^2}{4}]+\frac{p^2}{4}+\frac{p^2}{2} &= 1\\
\frac{9p^2}{8} &= \frac{1}{2}\\
p &= \frac{2}{3}.
\end{aligned}$$

Substituting the equilibrium price $p=\frac{2}{3}$ into the demand and supply functions gives us the equilibrium allocation,

$$x=1, \ell=\frac{2}{3}, y_1=\frac{1}{3}, y_2=\frac{2}{3}, L_1=\frac{1}{9}, L_2=\frac{2}{9}.$$

(c) The answer is NO. The FFTWE applies, so the CE allocation is Pareto optimal. Therefore, there does not exist any other feasible allocation that can provide Robin with higher utility.

3. (35 points)

The following economy has 2 consumers, 2 states of nature, and one commodity per state. The probability of state 1 is $\frac{1}{3}$, and the probability of state 2 is $\frac{2}{3}$. The endowment vector of consumer 1 is $\omega_1=(\omega_1(1),\omega_1(2))=(3,0)$, and the endowment vector of consumer 2 is $\omega_2=(\omega_2(1),\omega_2(2))=(0,3)$. For $i=1,2$, consumer i is a von Neumann-Morgenstern expected utility maximizer, with Bernoulli utility function $u_i(x_i)=\log(x_i)$.

Before observing the state, the two consumers trade two securities. Security 1 pays 50 units of account in state 1 and 0 units of account in state 2. Security 2 pays 0 units of account in state 1 and 100 units of account in state 2. After the securities market, the state of nature is observed, securities are redeemed, and consumption is traded on a spot market. Denote the security holdings as b_i^1 and b_i^2 and denote security prices as q^1 and q^2.

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (20 points) Calculate the competitive equilibrium prices, consumptions, and security holdings. Make clear how you are normalizing prices.

(c) (5 points) Without performing any new calculations, explain how the equilibrium prices and allocation would be affected if consumer 2's Bernoulli utility function was instead given by $u_2(x_2)=\arctan(x_2)$.

Hint: Use the fact that arctan is an increasing and strictly concave function. And don't spend too much time on this–it is only worth 5 points.

Answer: (a) A competitive equilibrium is a vector of security and spot market prices, $(q^1, q^2, p(1), p(2))$, and an allocation of security holdings and consumptions, $(b_1^1, b_1^2, b_2^1, b_2^2)$ and $(x_1(1), x_1(2), x_2(1), x_2(2))$, such that

(i) $(b_1^1, b_1^2, x_1(1), x_1(2))$ solves

$$\begin{aligned} & \max \frac{1}{3}\log(x_1(1)) + \frac{2}{3}\log(x_1(2)) \\ & \text{subject to} \\ q^1 b_1^1 + q^2 b_1^2 &= 0 \\ p(1)x_1(1) &= 3p(1) + 50b_1^1 \\ p(2)x_1(2) &= 100b_1^2 \\ x_1(1) &\geq 0, x_1(2) \geq 0 \end{aligned}$$

(ii) $(b_2^1, b_2^2, x_2(1), x_2(2))$ solves

$$\begin{aligned} & \max \frac{1}{3}\log(x_2(1)) + \frac{2}{3}\log(x_2(2)) \\ & \text{subject to} \\ q^1 b_2^1 + q^2 b_2^2 &= 0 \\ p(1)x_2(1) &= 50b_2^1 \\ p(2)x_2(2) &= 3p(2) + 100b_2^2 \\ x_2(1) &\geq 0, x_2(2) \geq 0 \end{aligned}$$

(iii) markets clear:

$$\begin{aligned} b_1^1 + b_2^1 &= 0 \\ b_1^2 + b_2^2 &= 0 \\ x_1(1) + x_2(1) &= 3 \\ x_1(2) + x_2(2) &= 3. \end{aligned}$$

Note: equalities are due to monotonic utility functions.

(b) Let us adopt the following normalizations: $p(1) = p(2) = q^2 = 1$. We can solve consumer 1's utility maximization problem by eliminating the security holdings:

$$b_1^1 = \frac{x_1(1) - 3}{50} \tag{7}$$

$$b_1^2 = \frac{x_1(2)}{100}. \tag{8}$$

Substituting (7) and (8) into the securities constraint, we have the simplified problem

$$\begin{aligned} & \max \frac{1}{3}\log(x_1(1)) + \frac{2}{3}\log(x_1(2)) \\ & \text{subject to} \\ q^1\frac{x_1(1)-3}{50} + \frac{x_1(2)}{100} &= 0 \\ x_1(1) &\geq 0, x_1(2) \geq 0 \end{aligned}$$

The constraint can be rewritten as

$$2q^1x_1(1) + x_1(2) = 6q^1. \tag{9}$$

We can solve for the demand functions by simultaneously solving (9) and the marginal rate of substitution condition,

$$\begin{aligned} \frac{\frac{1}{3}x_1(2)}{\frac{2}{3}x_1(1)} &= 2q^1 \quad \text{or} \\ x_1(2) &= 4q^1x_1(1). \end{aligned} \tag{10}$$

Solving (9) and (10), we have

$$\begin{aligned} x_1(1) &= 1 \\ x_1(2) &= 4q^1. \end{aligned}$$

We can solve consumer 2's utility maximization problem by eliminating the security holdings:

$$b_2^1 = \frac{x_2(1)}{50} \tag{11}$$

$$b_2^2 = \frac{x_2(2)-3}{100}. \tag{12}$$

Substituting (11) and (12) into the securities constraint, we have the simplified problem

$$\begin{aligned} & \max \frac{1}{3}\log(x_2(1)) + \frac{2}{3}\log(x_2(2)) \\ & \text{subject to} \\ q^1\frac{x_2(1)}{50} + \frac{x_2(2)-3}{100} &= 0 \\ x_2(1) &\geq 0, x_2(2) \geq 0 \end{aligned}$$

The constraint can be rewritten as

$$2q^1x_2(1) + x_2(2) = 3. \tag{13}$$

We can solve for the demand functions by simultaneously solving (13) and the marginal rate of substitution condition,

$$\begin{aligned} \frac{\frac{1}{3}x_2(2)}{\frac{2}{3}x_2(1)} &= 2q^1 \quad \text{or} \\ x_2(2) &= 4q^1 x_2(1). \end{aligned} \tag{14}$$

Solving (13) and (14), we have

$$\begin{aligned} x_2(1) &= \frac{1}{2q^1} \\ x_2(2) &= 2. \end{aligned}$$

Now we can solve for the remaining price, q^1, by using market clearing on the state-2 spot market. We have

$$\begin{aligned} 4q^1 + 2 &= 3 \\ q^1 &= \frac{1}{4}. \end{aligned}$$

Substituting $q^1 = \frac{1}{4}$ into the demand functions, we have

$$\begin{aligned} x_1(1) &= 1, x_1(2) = 1 \\ x_2(1) &= 2, x_2(2) = 2. \end{aligned}$$

From (7), (8), (11), and (12), we have

$$\begin{aligned} b_1^1 &= -\frac{1}{25}, b_1^2 = \frac{1}{100} \\ b_1^1 &= \frac{1}{25}, b_1^2 = -\frac{1}{100}. \end{aligned}$$

(c) This market structure is equivalent to complete markets (you can verify that the securities payoff matrix is full rank), so the consumption in each state is the same as what would occur with complete contingent-commodity markets. Since **there is no aggregate uncertainty**, consumption does not depend on the state of nature, and equals the expected endowment (1 for consumer 1 and 2 for consumer 2). Moreover, this result holds for any strictly increasing and strictly concave Bernoulli utility functions, so the allocation would be the same if consumer 2's utility function was given by $u_2(x_2) = \arctan(x_2)$.