The Ohio State University
Department of Economics
Econ 805 Final Exam Answers

Professor Peck
Winter 2003

Here is the distribution of scores on the final exam:

100	6
90-99	5
80-89	6
70-79	4
60-69	6
50-59	6
40-49	7
30-39	8

1. Consider a pure exchange economy, which we will call the n-consumer economy, with k goods and n consumers. Each consumer's utility function and endowment vector satisfies assumption A:

Assumption A: The utility function is continuous, strictly monotonic, and strictly quasi-concave. The endowment vector is strictly positive.

Now consider the pure exchange economy in which an $n+1^{st}$ consumer, whose utility function and endowment also satisfies assumption A, joins the original n consumers. We call this economy the n+1-consumer economy.

For the statements in (a) and (b), either sketch the argument for why the statement is true, or give a counterexample if the statement is false.

(a) (15 points) For ***all*** $i = 1, ..., n$, *consumer i receives at least as high a utility level at the competitive equilibrium of the n+1-consumer economy than at the competitive equilibrium of the n-consumer economy.*

(b) (15 points) Suppose that the initial endowment allocation of the n-consumer economy is Pareto optimal. Then for ***all*** $i = 1, ..., n$, *consumer i receives at least as high a utility level at the competitive equilibrium of the n+1-consumer economy than at the competitive equilibrium of the n-consumer economy.*

Answer: (a) This statement is false. While it is possible for all of the first n consumers to be better off at the C.E. of the n+1-consumer economy, we

can find a counterexample. For example, let k=2 and n=2 hold, and consider the following utility functions and endowments:

$$\begin{aligned} u_1(x_1) &= \log(x_1^1) + \log(x_1^2), \qquad \omega_1 = (2,1) \\ u_2(x_2) &= \log(x_2^1) + \log(x_2^2), \qquad \omega_2 = (1,2) \\ u_3(x_3) &= \log(x_3^1) + \log(x_3^2), \qquad \omega_3 = (2,1). \end{aligned}$$

It is easy to check that consumer 1 receives a higher utility, in the C.E. to the economy with consumers 1 and 2, than in the C.E. to the economy with all three consumers. Intuitively, the introduction of consumer 3 creates an unfavorable price shift, making the good that consumer 1 wants to buy more expensive relative to the good that consumer 1 wants to sell.

(b) This statement is true. Because of strict quasiconcavity and the other assumptions, we can apply the SFTWE to conclude that the initial endowment allocation is a C.E. allocation for the n-consumer economy, and that this is the unique C.E. allocation. Thus, at the C.E. for the n-consumer economy, consumers can do no better than consume their endowments. At the C.E. for the n+1-consumer economy, consumers can always afford their endowments, so they must receive utility equal to, or higher than, the utility they receive at the C.E. for the n-consumer economy.

2. The following economy has two consumers, two goods, and one firm. For $i = 1, 2$, consumer i has the utility function,

$$u_i(x_i) = \log(x_i^1) + \log(x_i^2),$$

and the endowment vector, $\omega_i = (1, 1)$. The firm is owned by consumer 1, and has a production function (the frontier of its production set) given by

$$y^1 = \sqrt{-2y^2},$$

where y^1 is the nonnegative output of good 1 and y^2 is the negative net output of good 2, reflecting the convention that inputs are negative outputs.

(a) (15 points) Define a competitive equilibrium for this economy. If you decide to change the notational convention to make inputs positive instead of negative, explain your change in notation after you define the equilibrium.

(b) (20 points) Calculate the competitive equilibrium price vector and allocation.

Answer: (a)

A C.E. is a price vector, (p^1, p^2), and an allocation, $(x_1^1, x_1^2, x_2^1, x_2^2, y^1, y^2)$, such that:

1. (x_1^1, x_1^2) solves

$$\begin{aligned} & \max \log(x_1^1) + \log(x_1^2) \\ & \text{subject to} \\ p^1 x_1^1 + p^2 x_1^2 &= p^1 + p^2 + \pi \\ x_1 &\geq 0, \end{aligned}$$

where π denotes the profits of the firm,

2. (x_2^1, x_2^2) solves

$$\begin{aligned} & \max \log(x_2^1) + \log(x_2^2) \\ & \text{subject to} \\ p^1 x_2^1 + p^2 x_2^2 &= p^1 + p^2 \\ x_2 &\geq 0, \end{aligned}$$

3. (y^1, y^2) solves

$$\begin{aligned} & \max p^1 y^1 + p^2 y^2 \\ & \text{subject to} \\ y^1 &= \sqrt{-2y^2}, \\ y^1 &\geq 0, \quad y^2 \leq 0, \end{aligned}$$

4. Markets clear:

$$\begin{aligned} x_1^1 + x_2^1 &= y^1 + 2 \\ x_1^2 + x_2^2 &= y^2 + 2. \end{aligned}$$

In the above definition, equalities in the budget constraints, profit max problem, and market clearing conditions are due to the fact that utility is strictly monotonic, so all prices will be positive.

(b) We will normalize the price of good 2 to be one and denote the price of good 2 as p, $(p^1, p^2) = (p, 1)$. Consumer 1's demand function is found by solving the budget equation and the marginal rate of substitution equation,

$$\begin{aligned} p x_1^1 + x_1^2 &= p + 1 + \pi \\ \frac{x_1^2}{x_1^1} &= p, \end{aligned}$$

yielding the demand functions

$$x_1^1 = \frac{p + 1 + \pi}{2p}, \tag{1}$$

$$x_1^2 = \frac{p + 1 + \pi}{2}. \tag{2}$$

Similarly, we can solve for consumer 2's demand functions. [Notice that consumer 2's demand functions are identical to consumer 1's demand functions when $\pi = 0$, so no new calculations are needed.]

$$x_2^1 = \frac{p+1}{2p}, \tag{3}$$

$$x_2^2 = \frac{p+1}{2}. \tag{4}$$

To solve for the firm's supply function, substitute the production function into the expression for profits

$$\max p\sqrt{-2y^2} + y^2,$$

yielding the first order condition

$$p\frac{1}{2}(-2y^2)^{-1/2}(-2) + 1 = 0. \tag{5}$$

Equation (3) can be simplified to

$$y^2 = -\frac{p^2}{2}, \tag{6}$$

which implies

$$y^1 = p \quad \text{and} \quad \pi = \frac{p^2}{2}. \tag{7}$$

Now we use market clearing to solve for the price. Market 2 clearing yields

$$\frac{p+1+\frac{p^2}{2}}{2} + \frac{p+1}{2} = -\frac{p^2}{2} + 2,$$

which can be simplified to

$$3p^2 + 4p - 4 = 0. \tag{8}$$

Equation (6) gives rise to two roots, $\frac{2}{3}$ and -2. Obviously, the positive root is the correct one, so we have $p = \frac{2}{3}$, which implies $\pi = \frac{2}{9}$. Plugging the price and profit into (1)-(4), (6) and (7) yields the allocation

$$\begin{aligned}
x_1^1 &= \frac{17}{12}, \\
x_1^2 &= \frac{17}{18}, \\
x_2^1 &= \frac{5}{4}, \\
x_2^2 &= \frac{5}{6}, \\
y^1 &= \frac{2}{3}, \\
y^2 &= -\frac{2}{9}.
\end{aligned}$$

3. Consider an exchange economy with 3 consumers and one consumption good per state. All consumers have the (Bernoulli) utility function over certain consumption,

$$u_i(x_i) = \log(x_i),$$

and are von Neumann-Morgenstern expected utility maximizers. Each consumer has an initial wealth of 1, which is his/her endowment in all states where he/she does not win the lottery. Each consumer is also endowed with a lottery ticket, which can be interpreted as additional state-contingent consumption. Exactly one of the tickets is the winning ticket, paying the winner 2 additional units of consumption. Assume that consumer 1's lottery ticket is the winning ticket with probability $\frac{1}{2}$, that consumer 2's lottery ticket is the winning ticket with probability $\frac{1}{4}$, and that consumer 3's lottery ticket is the winning ticket with probability $\frac{1}{4}$.

(a) (15 points) Define a competitive equilibrium for this economy, where the three consumers participate on a complete state-contingent commodities market before the winning lottery ticket is announced.

(b) (20 points) Calculate the competitive equilibrium price vector and allocation. [Time-saving hint: what is the aggregate endowment in each state?]

Answer: The key step is recognizing that there are three states of nature. For example, consumer 3 might think that the only relevant uncertainty to her is whether she wins the lottery or not, but that is not true. When consumer 3 does not win the lottery, the markets must draw a distinction between consumer 1 winning and consumer 2 winning. Who will take the other side of consumer 3's trade? Consumer 1 does not want to trade contingent only on whether consumer 3 wins, because that does not distinguish between the state in which consumer 1 wins and the state in which consumer 2 wins (and consumer 1 loses). That is why **the state of nature must fully resolve all of the uncertainty anyone faces**.

Here is the answer:

(a) A C.E. is a price vector, (p^1, p^2, p^3), and an allocation, $(x_1^1, x_1^2, x_1^3, x_2^1, x_2^2, x_2^3, x_3^1, x_3^2, x_3^3)$, such that

1. (x_1^1, x_1^2, x_1^3) solves

$$\begin{aligned} & \max \frac{1}{2}\log(x_1^1) + \frac{1}{4}\log(x_1^2) + \frac{1}{4}\log(x_1^3) \\ & \text{subject to} \\ p^1x_1^1 + p^2x_1^2 + p^3x_1^3 &= 3p^1 + p^2 + p^3 \\ x_1 &\geq 0, \end{aligned}$$

2. (x_2^1, x_2^2, x_2^3) solves

$$\begin{aligned} & \max \frac{1}{2}\log(x_2^1) + \frac{1}{4}\log(x_2^2) + \frac{1}{4}\log(x_2^3) \\ & \text{subject to} \\ p^1 x_2^1 + p^2 x_2^2 + p^3 x_2^3 &= p^1 + 3p^2 + p^3 \\ x_2 &\geq 0, \end{aligned}$$

3. (x_3^1, x_3^2, x_3^3) solves

$$\begin{aligned} & \max \frac{1}{2}\log(x_3^1) + \frac{1}{4}\log(x_3^2) + \frac{1}{4}\log(x_3^3) \\ & \text{subject to} \\ p^1 x_3^1 + p^2 x_3^2 + p^3 x_3^3 &= p^1 + p^2 + 3p^3 \\ x_3 &\geq 0, \end{aligned}$$

4. Markets clear

$$\begin{aligned} x_1^1 + x_2^1 + x_3^1 &= 5 \\ x_1^2 + x_2^2 + x_3^2 &= 5 \\ x_1^3 + x_2^3 + x_3^3 &= 5. \end{aligned}$$

In the above definition, equalities in the budget constraints and market clearing conditions follow from strict monotonicity.

(b) The long way to solve this problem is to first find the demand functions, by solving the budget equation and two marginal rate of substitution equations for the demands for the three goods, and then use two of the market clearing equations to solve for the two price variables remaining after normalization. The easy way to solve for the C.E. is to notice that there is no aggregate uncertainty. Therefore, prices are proportional to probabilities,

$$(p^1, p^2, p^3) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}). \tag{9}$$

Also, consumption for a given consumer is the same in each state. Using this fact, and equation (9), consumer 1's budget equation is

$$\frac{1}{2}x_1^1 + \frac{1}{4}x_1^1 + \frac{1}{4}x_1^1 = \frac{3}{2} + \frac{1}{4} + \frac{1}{4}. \tag{10}$$

From (10), we have

$$x_1^1 = x_1^2 = x_1^3 = 2.$$

For consumer 2, we have

$$\frac{1}{2}x_2^1 + \frac{1}{4}x_2^1 + \frac{1}{4}x_2^1 = \frac{1}{2} + \frac{3}{4} + \frac{1}{4}. \tag{11}$$

From (11), we have

$$x_2^1 = x_2^2 = x_2^3 = \frac{3}{2}.$$

For consumer 3, we have

$$\frac{1}{2}x_3^1 + \frac{1}{4}x_3^1 + \frac{1}{4}x_3^1 = \frac{1}{2} + \frac{1}{4} + \frac{3}{4}. \tag{12}$$

From (11), we have

$$x_3^1 = x_3^2 = x_3^3 = \frac{3}{2}.$$