Department of Economics
The Ohio State University
Final Exam Answers–Econ 805

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1. (40 points)

Consider the following two player, zero sum game between an attacker and a defender. The attacker must choose which one of three targets to attack, and the defender must choose which one of the three targets to defend. If the defender defends the target that the attacker attacks, both sides receive a payoff of zero. If the defender does not defend the target that the attacker attacks, then the payoffs depend on which target was attacked, as follows: if target 1 is successfully attacked, the payoffs to the attacker and defender are $(2, -2)$; if target 2 is successfully attacked, the payoffs to the attacker and defender are $(1, -1)$; and if target 3 is successfully attacked, the payoffs to the attacker and defender are $(v, -v)$, where we have $v < 1$.

(a) (10 points) Prove that this game does not have a Nash equilibrium in pure strategies.

(b) (20 points) Assuming that there is a mixed strategy Nash equilibrium in which all three targets are attacked with positive probability, find the equilibrium mixing probabilities for the attacker (p_1, p_2, p_3) and the equilibrium mixing probabilities for the defender (q_1, q_2, q_3). Your answer should be a function of the parameter, v.

(c) (10 points) For what values of v will there be a mixed strategy Nash equilibrium in which all three targets are attacked with positive probability?

Answer: (a) Suppose there is a NE in pure strategies. If the attacker is choosing the same target that the defender is defending, then the attacker can deviate to a different target and increase his payoff. If the attacker is choosing a different target from the one that the defender is defending, then the defender can deviate to the attacker's target and increase her payoff. Thus, the strategy profile cannot be a NE.

(b) If the attacker is attacking each target with positive probability, he must be indifferent between all three actions. Therefore, we have

$$\begin{aligned} 2(1-q_1) &= 1-q_2 = v(1-q_3) \text{ and} \\ q_1+q_2+q_3 &= 1. \end{aligned}$$

Solving these three equations (I am leaving out the algebra) yields the mixing

probabilities for the defender,

$$\begin{aligned} q_1 &= \frac{2+v}{2+3v} \\ q_2 &= \frac{2-v}{2+3v} \\ q_3 &= \frac{3v-2}{2+3v}. \end{aligned}$$

If the defender is defending each target with positive probability, she must be indifferent between all three actions. Therefore, we have

$$\begin{aligned} -p_2 - vp_3 &= -2p_1 - vp_3 = -p_2 - 2p_1 \text{ and} \\ p_1 + p_2 + p_3 &= 1. \end{aligned}$$

Solving these three equations (I am leaving out the algebra) yields the mixing probabilities for the attacker,

$$\begin{aligned} p_1 &= \frac{v}{2+3v} \\ p_2 &= \frac{2v}{2+3v} \\ p_3 &= \frac{2}{2+3v}. \end{aligned}$$

(c) According to the candidate equilibrium in part (b), the attacker's probabilities are always positive. However, the whole calculation relied on the defender's mixing probabilities also being positive for all three targets. From the expression for q_3, we see that $v > \frac{2}{3}$ must hold. [Intuitively, when all three targets are sufficiently valuable, the defender puts more weight on a more valuable target (to make the attacker willing to mix) and the attacker puts more weight on a less valuable target (to make the defender willing to mix). When v falls below two thirds, target 3 is no longer worth attacking, so the probability of attacking that target discontinuously drops from $\frac{2}{2+3v}$ to zero.]

2. (30 points)

Consider the following version of Spence's signaling model with two potential signals. Type 1 workers have productivity 1, and type 2 workers have productivity 2. Signal y has signaling costs for the two types given by

$$\begin{aligned} c_1(y) &= a_1 y, \\ c_2(y) &= a_2 y, \end{aligned}$$

where $0 < a_2 < a_1$ holds. Signal z has signaling costs for the two types given by

$$\begin{aligned} c_1(z) &= b_1 z, \\ c_2(z) &= b_2 z, \end{aligned}$$

where $0 < b_2 < b_1$ holds. Adopt the usual signaling-game timing where first workers choose their signal, y or z, and then firms competitively offer wages.

Define signal y to be **more efficient than** signal z if the best separating equilibrium using signal y (and not signal z) yields higher utility to type 2 workers than the best separating equilibrium using signal z (and not signal y).

Based on the parameters, a_1, a_2, b_1, b_2, find a necessary and sufficient condition for signal y to be more efficient than signal z.

Answer: In a separating equilibrium using signal y, there is a y^* such that type 2 workers choose $y = y^*$ and receive a wage of 2, and type 1 workers choose $y = 0$ and receive a wage of 1. For the widest possible range of y^* consistent with a separating equilibrium, firms believe that a worker choosing $y < y^*$ is type 1 with probability 1, and a worker choosing $y \geq y^*$ is type 2 with probability 1. We have an equilibrium if the following sequential rationality (sometimes called incentive compatibility) conditions are satisfied

$$\begin{aligned} 1 &\geq 2 - a_1 y^* \quad \text{(type 1)} \\ 2 - a_2 y^* &\geq 1 \quad \text{(type 2)} \end{aligned}$$

which is equivalent to

$$\frac{1}{a_1} \leq y^* \leq \frac{1}{a_2}.$$

For the best separating equilibrium using signal y, set $y^* = \frac{1}{a_1}$. The resulting utility for type 2 workers is

$$2 - \frac{a_2}{a_1}.$$

The same exercise yields the best separating equilibrium using signal z, characterized by $z^* = \frac{1}{b_1}$. The resulting utility for type 2 workers is

$$2 - \frac{b_2}{b_1}.$$

Thus, signal y is more efficient than signal z if and only if the signaling cost is less (so utility is higher), yielding the condition,

$$\frac{a_2}{a_1} < \frac{b_2}{b_1}.$$

3. (30 points)

Two firms are engaging in Stackelberg (quantity) competition. The market demand curve is given by

$$D(p) = 360 - 3p,$$

where p denotes the market price and $D(p)$ denotes the total quantity demanded at that price. Assume that each firm has constant marginal cost equal to 40.

The timing of the (extensive form) game is that firm 1 chooses its output, q_1, and then firm 2 observes q_1 before choosing its output, q_2. That is, firm 2's output can be a function of firm 1's output.

(a) (20 points) Compute the unique subgame perfect Nash equilibrium (SPNE) of this game, and explain why it is unique. Remember to specify completely both firms' strategies.

(b) (10 points) Find a Nash equilibrium of this game that is ***NOT*** *a SPNE. Remember to specify completely both firms' strategies.*

Answer: (a) By inverting the demand function, we can derive an expression for the market clearing price, as a function of the two outputs,

$$p = 120 - \frac{q_1 + q_2}{3}.$$

This is a game with perfect information, so to find the SPNE, we solve the game by backwards induction. Given q_1, firm 2's payoff function is given by

$$\pi_2 = [120 - \frac{q_1 + q_2}{3}]q_2 - 40q_2.$$

Differentiating with respect to q_2, setting the expression equal to zero, and solving for q_2, we derive firm 2's equilibrium strategy. [Note–this is the reaction function for firm 2 in the simultaneous move game, where firm 2 reacts to expectations of firm 1's output. In the sequential move game of this problem, this calculation yields sequentially rational output choices for firm 2, or in other words, firm 2's equilibrium strategy.] Thus, we have

$$\begin{aligned} 120 - \frac{q_1}{3} - \frac{2q_2}{3} - 40 &= 0 \text{ or} \\ q_2 &= 120 - \frac{q_1}{2}. \end{aligned}$$

[Note: If $q_1 > 240$, firm 2's optimal output is zero.]

Firm 1 takes into account how firm 2's output depends on its own output, so its payoff function can be written as

$$\pi_1 = [120 - \frac{q_1 + 120 - \frac{q_1}{2}}{3}]q_1 - 40q_1.$$

Simplifying, differentiating, setting the expression equal to zero, and solving yields

$$\begin{aligned} 80 - \frac{q_1}{3} - 40 &= 0 \text{ or} \\ q_1 &= 120. \end{aligned}$$

The SPNE is given by the strategy profile (an output for firm 1 and a function for firm 2),

$$q_1 = 120, \quad q_2 = 120 - \frac{q_1}{2}.$$

The SPNE is unique because given firm 1's output, firm 2's optimal output is uniquely determined. Since every subgame must be in equilibrium, firm 2's SPNE strategy function is uniquely determined. Given firm 2's strategy, firm 1 has the unique best response, $q_1 = 120$.

(b) There are many Nash equilibria that are not subgame perfect. However, we must find a NE of this game, where a strategy profile is an output for firm 1 and a function for firm 2. Firm 1 must be best responding to firm 2's output function, and firm 2 must be best responding to firm 1's output.

Here are some examples.

1. $q_1 = 80, \quad q_2 = 80$ for all q_1. In this NE, the equilibrium path is the same as in the (Cournot) NE of the simultaneous move game. It is a NE because given $q_1 = 80$, firm 2 cannot do better than choose a quantity of 80. Firm 2's quantity responding to outputs other than 80 do not affect firm 2's payoff. It is easy to see that firm 1's quantity of 80 is a best response to firm 2's strategy function. This is not a SPNE because the subgames following $q_1 \neq 80$ are not in equilibrium.

2. $q_1 = 10, \quad q_2 = 115$ if $q_1 = 10$ and $q_2 = 240$ if $q_1 \neq 10$. Due to firm 2's strategy, which floods the market and brings the price below marginal cost unless firm 1 chooses a quantity of 10, firm 1's strategy is a best response. Firm 2's strategy is a best response, because it is optimizing on the equilibrium path. It is not a SPNE, because firm 2 is not optimizing off the equilibrium path.

3. $q_1 = 120, \quad q_2 = 120 - \frac{q_1}{2}$ if $0 \leq q_1 \leq 240$, and $q_2 = 1000$ if $q_1 > 240$.