Department of Economics
The Ohio State University
Final Exam Answers–Econ 805

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1. (30 points)

Consider the following situation involving two players. Each player receives an envelope containing a number of dollar bills between 1 and 100. Denote the number of dollars in player i's envelope as θ_i, and assume that θ_1 and θ_2 are independently distributed, where all integer amounts of money are equally likely:

$$pr(\theta_i = x) = \frac{1}{100} \quad \text{for } x = 1, 2, ..., 100.$$

After observing the amount of money in their own envelope, each player must simultaneously choose either "trade" or "no trade." If both players choose "trade," then they exchange envelopes; if at least one player chooses "no trade," then each player keeps his/her original envelope. The payoff for each player is the amount of money in the envelope he or she winds up with. Assume players seek to maximize their expected payoff.

(a) (10 points) Model this situation as a Bayesian game, by writing down the set of types, joint distribution of types, and payoff functions. Explain your notation.

(b) (20 points) Find **all** *pure-strategy Bayesian-Nash equilibria of this game.*

Answer:

(a) For each player, the set of types is $Z = \{1, 2, ..., 100\}$ and the joint distribution of types is given by

$$F(\theta_1, \theta_2) = \frac{1}{10{,}000} \text{ for all } (\theta_1, \theta_2) \in Z \times Z.$$

The strategy set for player i is $S_i = \{T, NT\}$, and the payoff function for player i is given by

$$\begin{aligned} \pi_i(s_1, s_2, \theta_1, \theta_2) &= \theta_{-i} \text{ if } s_1 = s_2 = T \\ &= \theta_i \text{ otherwise.} \end{aligned}$$

Note that in this game player i's payoff can depend on the strategy profile, player i's type, and player $-i$'s type.

(b) Let t_i be the highest type of player i that chooses to trade, where t_i is defined to be zero if all types refuse to trade. If $t_1 > t_2 \geq 1$ held at the

Bayesian NE, then player 1 could increase his expected payoff by not trading when his type is t_1. A symmetric argument for player 2 establishes that either $t_1 \leq 1$ and $t_2 \leq 1$ holds or $t_1 = t_2$ holds. If we have $t_1 = t_2 > 1$, then player 1 with type t_1 is receiving the same prize with positive probability and receiving a lower prize with positive probability; no trade results in higher expected payoff. Thus, a necessary condition for a Bayesian Nash equilibrium is $t_1 \leq 1$ and $t_2 \leq 1$. This condition is also sufficient. There are four possible equilibria, all of which involve each player consuming the amount of money in his/her original envelope. Any combination of the following two strategies for each player will form a BNE. The first strategy, denoted by σ_i^0, is to never trade:

$$\sigma_i^0(\theta_i) = NT \text{ for all } \theta_i$$

The second strategy, denoted by σ_i^1, is to offer to trade when the envelope contains one dollar, but not to trade otherwise:

$$\begin{aligned} \sigma_i^1(\theta_i) &= T \text{ for } \theta_i = 1 \\ \sigma_i^0(\theta_i) &= NT \text{ for } \theta_i > 1. \end{aligned}$$

2. (35 points)

Consider the following game with two players, a plaintiff and a defendant engaged in a civil law suit. Nature moves first, selecting which side would win if the case goes to trial. The (prior) probability that the plaintiff would win a trial is $\frac{1}{2}$, and the probability that the defendant would win a trial is $\frac{1}{2}$. The defendant (player 1) observes whether or not he would win, but the plaintiff (player 2) does not. Player 1 makes a "take-it-or-leave-it" offer to player 2, **which must be either 3 or 5**. Then player 2 must either accept or reject the offer. If player 2 accepts an offer of m (3 or 5) the parties do not go to court, in which case player 1 receives a payoff of $-m$ and player 2 receives a payoff of m. If player 2 rejects the offer, then the parties go to court. If the plaintiff wins, then player 1's payoff is -11 and player 2's payoff is 5. If the defendant wins, then player 1's payoff is -6 and player 2's payoff is 0.

[Note: If you are wondering about the story behind these payoff numbers, think of player 1 as having to pay player 2 damages of 5 if player 1 loses the trial, and player 1 paying court costs of 6 whether he wins or not.]

(a) (15 points) Give the extensive form representation of this game, by drawing the game tree and labeling things properly. Try to be neat! (If I can't read it, I will not assume that it is right.)

(b) (20 points) Find a pure-strategy weak perfect Bayesian equilibrium (WPBE) for this game. Remember to fully specify the strategies and the plaintiff's beliefs. Explain why the conditions for a WPBE are satisfied.

Answer:

(a) See the game tree on the last page.

(b) It turns out that there are three classes of WPBE, all involving pooling. You only needed to find one equilibrium, but here they all are.

WPBE 1. Player 1 offers 5, whether he is guilty or innocent, and player 2 accepts an offer of 5 and rejects an offer of 3. The beliefs are that the probability that the defendent is guilty is $\frac{1}{2}$ if he offers 5 and μ if he offers 3, for any value of μ such that $\frac{3}{5} \leq \mu \leq 1$.

To see that the above strategy profile and belief system is a WPBE, first note that player 2's strategy is sequentially rational, given her beliefs. She optimally accepts an offer of 5 and rejects an offer of 3. Player 1's strategy is sequentially rational given player 2's strategy, since offering 5 is better than offering 3 (whether he is guilty or not). The beliefs are consistent, because they satisfy Bayes' rule following an offer of 5, and Bayes' rule does not apply following an offer of 3.

WPBE 2. Player 1 offers 3, whether he is guilty or innocent, and player 2 accepts an offer of 5 and accepts an offer of 3. The beliefs are that the probability that the defendent is guilty is $\frac{1}{2}$ if he offers 3 and μ if he offers 5, for any value of μ such that $0 \leq \mu \leq 1$.

To see that the above strategy profile and belief system is a WPBE, first note that player 2's strategy is sequentially rational, given her beliefs. She optimally accepts an offer of 5 and accepts an offer of 3 (since the expected payoff from rejecting is only 2.5). Player 1's strategy is sequentially rational given player 2's strategy, since offering 3 is better than offering 5 (whether he is guilty or not, since both offers are accepted). The beliefs are consistent, because they satisfy Bayes' rule following an offer of 3, and Bayes' rule does not apply following an offer of 5.

WPBE 3. Player 1 offers 3, whether he is guilty or innocent, and player 2 rejects an offer of 5 and accepts an offer of 3. The beliefs are that the probability that the defendent is guilty is $\frac{1}{2}$ if he offers 3 and $\mu = 1$ if he offers 5.

To see that the above strategy profile and belief system is a WPBE, first note that player 2's strategy is sequentially rational, given her beliefs. She optimally rejects an offer of 5 (in fact, she is indifferent given her beliefs) and accepts an offer of 3 (since the expected payoff from rejecting is only 2.5). Player 1's strategy is sequentially rational given player 2's strategy, since offering 3 is better than offering 5 (whether he is guilty or not, since an offer of 5 leads to -6 or -11). The beliefs are consistent, because they satisfy Bayes' rule following an offer of 3, and Bayes' rule does not apply following an offer of 5.

3. (35 points)

Consider a market that is served by a monopolist with constant average and marginal costs equal to c. There are two consumers, and for $i = 1, 2$, consumer i has a quasi-linear utility function over consumption of the monopolist's produced good, x, and consumption of the numeraire good, M, given by

$$u_i(x_i, M_i) = (x_i - \frac{x_i^2}{2\theta_i}) + M_i \quad \text{for } x_i \in [0, \theta_i].$$

Also, assume that the consumer has a zero endowment of good x, and a large enough endowment of the numeraire good so that we do not have to worry about nonnegativity constraints on numeraire consumption. For the questions below, your answers can be functions of the parameters, θ_1, θ_2, and c.

(a) (10 points) Assume that the monopolist must set a single price, p, of the good in terms of the numeraire, at which all transactions must occur. Find the monopoly price, p.

(b) (10 points) Assume that the monopolist knows the utility function of each consumer, and can set a different price for each consumer (third degree price discrimination). Find the optimal price charged to each consumer. Compare the price you found in part (a) with the prices you found in part (b). Briefly explain the reason for this relationship.

(c) (15 points) Assume that the monopolist knows the utility function of each consumer, and can offer a nonlinear pricing schedule for each consumer (first degree price discrimination). Find an optimal schedule to offer each consumer (there are many such optimal schedules), and the profit the monopolist receives from each consumer.

Answer:

(a) First we need to solve the utility maximization problem for the demand function of each type of consumer. Setting the marginal rate of substitution equal to the price ratio, we have

$$\begin{aligned} 1 - \frac{x_i}{\theta_i} &= p, \text{ or} \\ x_i &= \theta_i(1-p). \end{aligned} \tag{1}$$

Notice that the range of prices that puts the consumer in his/her consumption set is $0 \leq p \leq 1$. One alert student pointed out that we also need to assume that $c < 1$, or else the efficient quantity is zero. The total market demand is then given by $X = (\theta_1 + \theta_2)(1-p)$.

The monopoly solves the following problem

$$\max_p (p-c)(\theta_1+\theta_2)(1-p). \tag{2}$$

Setting the derivative equal to zero and solving for p, we have

$$p = \frac{1+c}{2}.$$

(b) If the monopolist can offer a different price to each consumer, the fact that marginal cost is constant allows us to separately consider each consumer. For consumer i, the problem becomes

$$\max_{p_i} (p_i - c)(\theta_i)(1-p_i),$$

yielding the solution

$$p_i = \frac{1+c}{2}.$$

Each consumer is offered the same price as in part (a). The reason is that monopoly profits are proportional to θ_i in part (b) and the sum $(\theta_1 + \theta_2)$ in part (a), so that these parameters multiply the profit level but do not affect the solution. Consumer 1's demand function is equivalent to demand from θ_1 identical consumers with demand function $(1 - p)$, and consumer 2's demand function is equivalent to demand from θ_2 identical consumers with demand function $(1 - p)$, so clearly each submarket and the entire market should be charged the same price. Another way to see this is to note that the price elasticity of demand is the same for the two consumers (at any price), so the optimal markup is the same.

(c) If the monopolist can engage is first degree price discrimination, the simplest schedule is to offer a take-it-or-leave-it price for a pre-specified package of output, x_i. The price for the package holds the consumer to his/her reservation utility level, and is therefore equal to

$$(x_i - \frac{x_i^2}{2\theta_i}). \tag{3}$$

The optimal quantity solves

$$\max_{x_i} x_i - \frac{x_i^2}{2\theta_i} - cx_i.$$

Setting the derivative equal to zero and solving for x_i, we have

$$x_i = (1-c)\theta_i. \tag{4}$$

Substituting (4) into (3), we have the total payment from consumer i equal to

$$\frac{(1-c^2)\theta_i}{2}$$

and a total profit from consumer i equal to

$$\frac{(1-c)^2\theta_i}{2}.$$

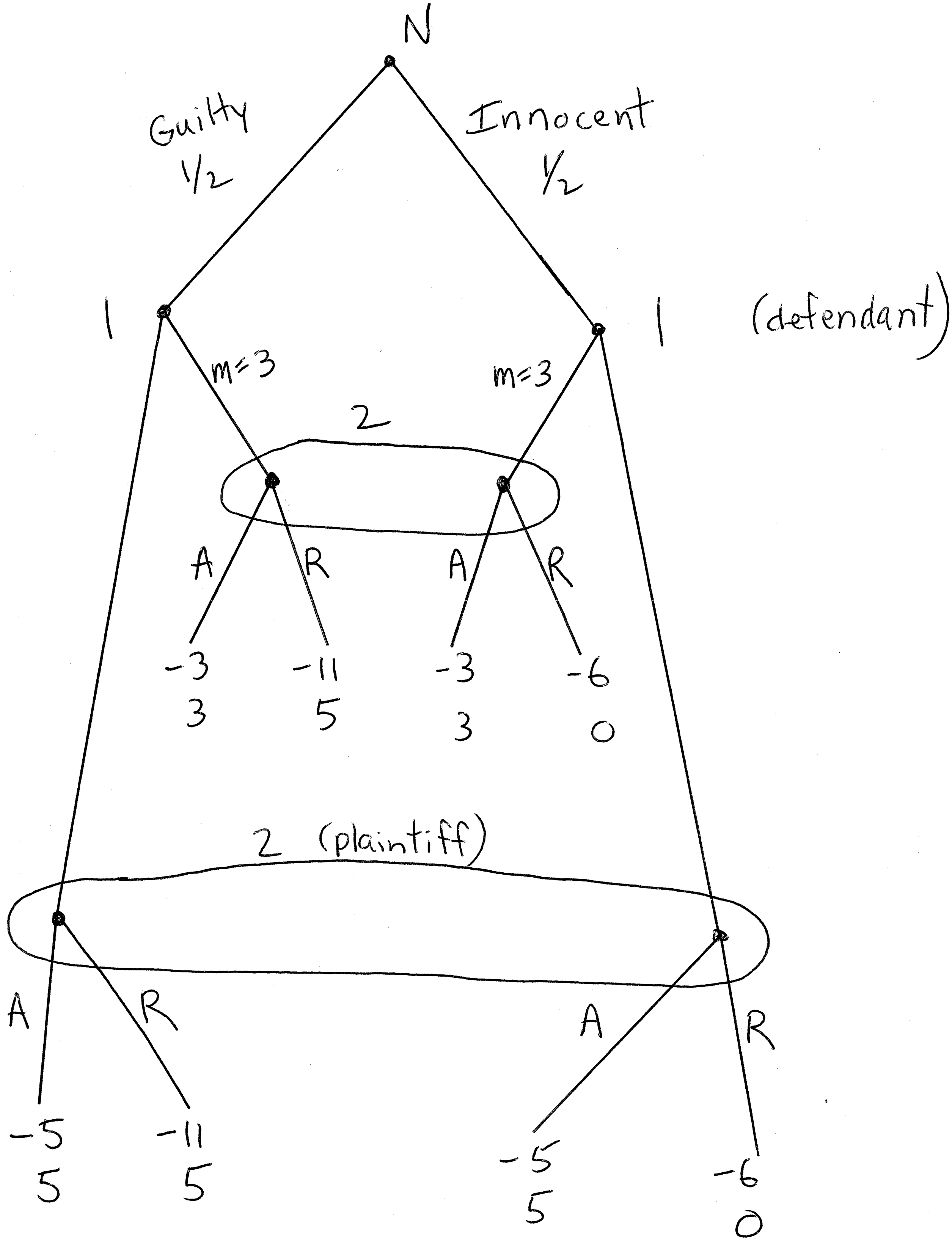
Notice that under first degree price discrimination, the two consumers are treated differently. An alternative optimal schedule is to charge each consumer his/her marginal willingness to pay for each unit, where the price of the x-th unit is given by

$$p(x) = 1 - \frac{x}{\theta_i}.$$

A third optimal schedule is to charge a fixed fee plus an additional charge of c per unit. The optimal fixed fee would then be the full surplus generated from trade at the efficient quantity (4),

$$Fee = \frac{(1-c)^2\theta_i}{2}$$

The second and third optimal schedules above allow each consumer to choose the quantity he/she wants, knowing that he/she will choose the efficient quantity.



N
Guilty
1/2
Innocent
1/2
1
1
(defendant)
m=3
m=3
2
A
R
A
R
-3
3
-11
5
-3
3
-6
0
2 (plaintiff)
A
R
A
R
-5
5
-11
5
-5
5
-6
0