Department of Economics
The Ohio State University
Final Exam Questions and Answers–Econ 805

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Directions: *Answer all questions, carefully label all diagrams, and show all work.*

1. (30 points)

Consider an exchange economy with three consumers and two goods, $n = 3$ and $k = 2$. For $i = 1, 2, 3$, the utility function of consumer i, $u_i(x_i)$ is differentiable, strictly concave, and strictly monotonic. For each of the following statements, either prove the statement (if the statement is true) or find a counterexample (if the statement is false).

(a) If we have a feasible allocation, $x >> 0$, such that consumer 1's marginal rate of substitution is equal to consumer 2's marginal rate of substitution, then there cannot exist another feasible allocation, x', such that

$$\begin{aligned} u_1(x_1') &> u_1(x_1) \\ u_2(x_2') &> u_2(x_2) \\ u_3(x_3') &\geq u_3(x_3). \end{aligned}$$

(b) Given initial endowments, $\omega = (\omega_1, \omega_2, \omega_3)$, it is impossible for any two of the consumers to strictly be better off excluding the third consumer from trade. That is, two of the consumers cannot each receive strictly higher utility by trading among themselves, intead of having the three consumers trade to a competitive equilibrium.

(c) If the initial endowments, $\omega = (\omega_1, \omega_2, \omega_3)$, are strongly Pareto optimal, then there is a competitive equilibrium in which no trade takes place.

Answer:

(a) False. Although consumers 1 and 2 have the same marginal rate of substitution, consumer 3's MRS can be different, so there can be potential gains from trade. For example, suppose that, for $i = 1, 2, 3$, we have $u_i(x_i^1, x_i^2) = \log(x_i^1) + \log(x_i^2)$, and let $x_1 = (100, 1), x_2 = (100, 1)$, and $x_3 = (1, 150)$. Now let $x_1' = (50, 50), x_2 = (50, 50)$, and $x_3 = (101, 52)$. This is feasible and provides everyone with higher utility. A few of you noticed that we never required x to be nonwasteful, so counterexamples could be constructed based on x being wasteful.

(b) True. Although one of the consumers might want to exclude one of the others, it is impossible for two of the consumers each to prefer to exclude the third. To prove it, let (p^*, x^*) be the competitive equilibrium with 3 consumers, and suppose without loss of generality that consumers 1 and 2 could each receive strictly higher utility by trading among themselves. Then there is (x_1, x_2) such that

$$u_i(x_i) > u_i(x_i^*) \text{ for } i = 1, 2 \quad (1)$$
$$x_1 + x_2 \leq \omega_1 + \omega_2. \quad (2)$$

From (1), we conclude that for $i = 1, 2$, consumer i cannot afford x_i at prices p^*, so we have

$$p^* \cdot \sum_{i=1}^{2} [x_i - \omega_i] > 0 \ . \quad (3)$$

Inequality (3) implies that $x_1^j + x_2^j > \omega_1^j + \omega_2^j$ must hold for either commodity $j = 1$ or commodity $j = 2$, contradicting (2).

Another way of showing this is to think about the competitive equilibrium price that supports trade between consumers 1 and 2 (no loss in assuming that they trade to a Pareto optimum, and use the SFTWE). If adding a third consumer changes the price, at least one of the first two consumers must be able to afford the bundle they chose with 2-way trade.

[Note: I neglected to specify that endowments are strictly interior, to guarantee that a competitive equilibrium exists, but luckily this was not a source of confusion.]

(c) I intended to specify that the initial endowments are strictly interior, but apparently you all remembered me saying how difficult it was to find a counterexample to the SFTWE, dropping the assumption of interior endowments but keeping monotonicity and the other assumptions. You must have all figured out that I just made an oversight, and didn't bother asking me at the exam, right? Anyway, I decided to give full credit to those who argued that the assumptions of the SFTWE are satisfied, so that we can conclude that there is a competitive equilibrium price, p^*, such that the strongly Pareto optimal allocation ω is a competitive equilibrium allocation, and no trade takes place. If you noticed that the assumption of strictly interior endowments was not specified, that was an acceptable answer as well.

2. (40 points)

The following economy has two commodities, n consumers, and F firms. For $i = 1, ...n$, consumer i has the endowment vector, $\omega_i = (1, 0)$, and the utility function,

$$u_i(x_i^1, x_i^2) = \log(x_i^1) + \log(x_i^2).$$

For $f = 1, ..., F$, firm f has a production set whose frontier (production function) is given by

$$y_f^2 = \sqrt{-y_f^1},$$

where we have $y_f^1 \leq 0$ (reflecting the convention that inputs are negative outputs).

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (20 points) Calculate the competitive equilibrium price vector and allocation, as a function of the parameters, n and F.

(c) (10 points) Is the utility received by consumers at the competitive equilibrium increasing or decreasing in F? Give the economic intuition for this result.

Answer:

Note: As I mentioned at the exam, you need to know the additional condition that each consumer owns a share, $1/n$, of each firm.

(a) A competitive equilibrium is a price vector, (p^1, p^2), and an allocation, $\{(x_i^1, x_i^2)\}_{i=1}^n$ and $\{(y_f^1, y_f^2)\}_{f=1}^F$, such that

(i) For $i = 1, ..., n$, (x_i^1, x_i^2) solves

$$\begin{aligned} & \max[\log(x_i^1) + \log(x_i^2)] \\ & \text{subject to} \\ p^1 x_i^1 + p^2 x_i^2 & \leq p^1 + \frac{1}{n}\sum_{f=1}^F \pi_f \\ x_i & \geq 0. \end{aligned}$$

(ii) For $f = 1, ..., F$, (y_f^1, y_f^2) solves

$$\max \pi_f = p^1 y_f^1 + p^2 y_f^2 \tag{4}$$

subject to

$$y_f^2 = \sqrt{-y_f^1} \tag{5}$$

$$y_f^1 \leq 0.$$

(iii) Markets clear

$$\begin{aligned} \sum_{i=1}^n x_i^1 & \leq n + \sum_{f=1}^F y_f^1 \\ \sum_{i=1}^n x_i^2 & \leq \sum_{f=1}^F y_f^2. \end{aligned}$$

(b) Because of monotonicity, the budget and market clearing inequalities hold as equalities. Let us normalize $p^2 = 1$ and $p^1 = p$. Solving the profit maximization problem first, substitute (5) into (4), and differentiate with respect to y_f^1. Setting the derivative equal to zero yields

$$y_f^1 = -\frac{1}{4p^2}. \tag{6}$$

Substituting (6) into (5) and (4), we have

$$y_f^2 = \frac{1}{2p}, \tag{7}$$

$$\pi_f = \frac{1}{4p}. \tag{8}$$

Now we solve the utility maximization problem for consumer i. Demand is calculated by solving the marginal rate of substitution equation and the budget equation (with (8) substituted for profit of each firm):

$$\frac{x_i^2}{x_i^1} = p,$$

$$px_i^1 + x_i^2 = p + \frac{F}{n}\frac{1}{4p},$$

yielding

$$x_i^1 = \frac{1}{2} + \frac{F}{8np^2}, \tag{9}$$

$$x_i^2 = \frac{p}{2} + \frac{F}{8np}. \tag{10}$$

Market clearing of good 1 is given by

$$n[\frac{1}{2} + \frac{F}{8np^2}] = n - \frac{F}{4p^2},$$

from which we solve for

$$p = \sqrt{\frac{3F}{4n}}. \tag{11}$$

Substituting (11) into (9), (10), (4), and (5) yields the equilibrium allocation

$$x_i^1 = \frac{2}{3}$$

$$x_i^2 = \sqrt{\frac{F}{3n}}$$

$$y_f^1 = -\frac{n}{3F}$$

$$y_f^2 = \sqrt{\frac{n}{3F}}.$$

(c) From the final allocation, it is clear that consumption of good 1 is independent of F, and consumption of good 2 is increasing in F, so overall utility must be increasing in F. This is not because more firms reduces the market power of any one firm, because firms are assumed to be price takers here. Rather, the answer is purely technological. Because we have a decreasing returns to scale technology, dividing the input of good 1 among more firms causes the production possibilities frontier to shift outward, leading to more aggregate production for a given aggregate input.

3. (30 points)

Consider the following economy, with an equal number of two types of consumers, two states of nature based on the outcome of an election, and one good per state of nature. Type 1 consumers are endowed with 2 units of pretax consumption, and are expected utility maximizers with the Bernoulli utility function, $u_1(x_1) = \log(x_1)$. Type 2 consumers are endowed with 4 units of pretax consumption, and are expected utility maximizers with the Bernoulli utility function, $u_2(x_2) = \log(x_2)$.

In state 1, which occurs with probability α, the conservative party wins the election and maintains the "status quo," with no taxes or transfers. In state 2, which occurs with probability $1-\alpha$, the liberal party wins the election and "soaks the rich," taxing type 2 consumers and transfering to type 1 consumers, so that the after-tax endowment of each consumer is 3 units of consumption. Before the election outcome is known, all consumers trade on a contingent commodity market.

(a) (10 points) Define a competitive equilibrium for this economy, with a complete contingent commodity market.

(b) (15 points) Calculate the competitive equilibrium price vector and allocation.

(c) (5 points) In the competitive equilibrium, do the type 1 consumers receive higher utility when the liberal party wins the election than when the conservative party wins the election? Briefly explain.

Answer:

(a) Suppose there are $\frac{n}{2}$ consumers of each type. A competitive equilibrium is a price vector, (p^1, p^2), and an allocation, $\{(x_i^1, x_i^2)\}_{i=1}^n$, such that

(i) For type 1 consumers, $i = 1, ..., \frac{n}{2}$, (x_i^1, x_i^2) solves

$$\begin{aligned} & \max[\alpha \log(x_i^1) + (1-\alpha)\log(x_i^2)] \\ & \text{subject to} \\ p^1 x_i^1 + p^2 x_i^2 \quad & \leq \quad 2p^1 + 3p^2 \\ x_i \quad & \geq \quad 0. \end{aligned}$$

(ii) For type 2 consumers, $i = \frac{n}{2} + 1, ..., n$, (x_i^1, x_i^2) solves

$$\begin{aligned} & \max[\alpha \log(x_i^1) + (1-\alpha)\log(x_i^2)] \\ & \text{subject to} \\ p^1 x_i^1 + p^2 x_i^2 \quad & \leq \quad 4p^1 + 3p^2 \\ x_i \quad & \geq \quad 0. \end{aligned}$$

(iii) Markets clear.

$$\sum_{i=1}^{n} x_i^1 \leq 3n$$
$$\sum_{i=1}^{n} x_i^2 \leq 3n$$

Note: It is acceptable to define a C.E. as if there were one representative consumer of each type, so that $n = 2$ holds, and the total endowment is 6 in each state.

(b) The easiest way to solve for the competitive equilibrium is to notice that there is no aggregate uncertainty. The proposition proved in class can be applied, so the equilibrium price vector is proportional to the probability vector, $(p^1, p^2) = (\alpha, 1 - \alpha)$. Also, consumption for each consumer is independent of the state, $x_i^1 = x_i^2$. For a type 1 consumer, the budget constraint reduces to

$$\alpha x_i^1 + (1 - \alpha)x_i^1 = 2\alpha + 3(1 - \alpha),$$

or $x_i^1 = x_i^2 = (3 - \alpha)$. For a type 2 consumer, the budget constraint reduces to

$$\alpha x_i^1 + (1 - \alpha)x_i^1 = 4\alpha + 3(1 - \alpha),$$

or $x_i^1 = x_i^2 = (3 + \alpha)$.

(c) No, type 1 consumers are fully insured, because there is no aggregate uncertainty. Although the probability of the liberal party winning affects ex ante wealth and therefore consumption, the election **outcome** does not affect consumption. Type 1 consumers receive the same utility in each state.