Game Theoretic Solutions

Def: A strategy $s_i \in S_i$ is strictly dominated for player i if there exists another strategy, $s_i' \in S_i$ such that, for all $s_{-i} \in S_{-i}$, we have

$$\pi_i(s_i', s_{-i}) > \pi_i(s_i, s_{-i}).$$

In this case, we say that s_i' strictly dominates s_i.

Def: A strategy $s_i \in S_i$ is a *strictly dominant strategy* for player i if it strictly dominates every other strategy in S_i.

Note: A player has, at most, one strictly dominant strategy. When a player has a dominant strategy, that strategy is a compelling choice.

Def: A (strict) *dominant strategy equilibrium* is a vector of strategies, $s = (s_1, ..., s_n) \in S$, such that for each player i, s_i is a strictly dominant strategy.

Example: Prisoner's dilemma. (defect, defect) is a dominant strategy equilibrium.

		player 2	
		cooperate	defect
player 1	cooperate	$3, 3$	$0, 4$
	defect	$4, 0$	$1, 1$

Def: A strategy $s_i \in S_i$ is *weakly* dominated for player i if there exists another strategy, $s_i' \in S_i$ such that, for all $s_{-i} \in S_{-i}$, we have

$$\pi_i(s_i', s_{-i}) \geq \pi_i(s_i, s_{-i}),$$

with strict inequality for some s_{-i}. In this case, we say that s_i' weakly dominates s_i.

Def: A strategy $s_i \in S_i$ is a *weakly dominant strategy* for player i if it weakly dominates every other strategy in S_i.

Note: If you are unsure what your opponent will play, you should not play a weakly dominated strategy, but if you are sure, then playing a weakly dominated strategy can be rational.

Example: For player 2, playing right is weakly dominated by left, but playing right is rational if she thinks that player 1 will play bottom.

		player 2	
		left	right
player 1	top	$2, 1$	$0, 0$
	bottom	$1, 2$	$1, 2$

Iterated elimination of strictly dominated strategies

Sometimes the players do not have dominant strategies, but we can still provide a compelling solution to the game, based on *common knowledge of rationality*.

		player 2	
		L	R
	A	11,6	5,5
player 1	B	10,12	10,0
	C	6,0	6,10
	D	5,6	11,5

Neither player has a strictly dominant strategy. However, strategy C is strictly dominated by B, and can be eliminated.

		player 2	
		L	R
	A	11,6	5,5
player 1	B	10,12	10,0
	D	5,6	11,5

Once player 2 eliminates the possibility of player 1 choosing C, we have the reduced game above. Now strategy L strictly dominates R, so R can be eliminated from consideration.

		player 2
		L
	A	11,6
player 1	B	10,12
	D	5,6

In the reduced game above, A is a strictly dominant strategy for player 1. Therefore, the solution to the game is (A, L).

Note: the order of elimination does not affect the strategy or strategies we end up with.

Nash Equilibrium

Sometimes a game has no compelling solution based on strictly dominant strategies or iterative elimination of strictly dominated strategies. Put another way, a player's optimal strategy generally depends on what strategies the other players will choose. Nash equilibrium is a weaker concept, more likely to exist.

Def: A strategy profile, $s = (s_1, ..., s_n) \in S$, is a Nash equilibrium in pure strategies if for each player i, we have

$$\pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i}), \tag{1}$$

for all $s_i' \in S$.

Holding the other players' strategies fixed at their Nash equilibrium levels, no player can receive a higher payoff with a unilateral deviation to a different strategy. Each player is choosing a best response to the other players' strategies.

For the matching pennies game, there is no Nash equilibrium in pure strategies. (more on that later)

For the Battle of the Sexes game, there are two Nash equilibria in pure strategies.

		player 2	
		wrestling match	ballet
player 1	wrestling match	$2, 1$	$0, 0$
	ballet	$0, 0$	$1, 2$

Interpretations of Nash equilibrium

1. If there is to be a unique outcome that rational people could come to, it must be a Nash equilibrium. Otherwise, someone is not being rational. But which Nash equilibrium?

2. Self-enforcing agreement.

3. Because of common culture, social convention, or obvious focal point argument, a combination of strategies enters people's minds. It must be a Nash equilibrium to be stable.

4. Each player has *rational expectations* about how the others will play, and optimizes accordingly. (Form beliefs, which turn out to be correct.) Thus, N.E. does is not a prediction of how the game will be played, but it is a consistent theory of how the game might be played.

Mixed-strategy Nash equilibrium

When there is no N.E. in pure strategies, any combination of strategies that the players come to anticipate will not be played. *Allow the players to randomize.*

Def: A *mixed strategy* for player i is a function, $\sigma_i : S_i \longrightarrow [0,1]$, where we have

$$\begin{aligned} \sigma_i(s_i) &\geq 0 \text{ for all } s_i \in S_i \\ \sum_{s_i \in S_i} \sigma_i(s_i) &= 1. \end{aligned}$$

We interpret $\sigma_i(s_i)$ as the probability with which player i chooses s_i.

Def: A *mixed strategy Nash equilibrium* is a profile of mixed strategies, $\sigma = (\sigma_1, ..., \sigma_n)$, such that no player can benefit by deviating to a different mixed strategy. That is, for all i and all probability distributions $\sigma_i' \in \Delta(S_i)$ we have

$$\begin{aligned} & \sum_{s \in S} [\sigma_1(s_1) \cdots \sigma_i(s_i) \cdots \sigma_n(s_n)] \pi_i(s) \\ \geq \; & \sum_{s \in S} [\sigma_1(s_1) \cdots \sigma_i'(s_i) \cdots \sigma_n(s_n)] \pi_i(s). \end{aligned}$$

–A mixed strategy N.E. to a game, Γ, is a pure strategy N.E. to an augmented game, $\widehat{\Gamma}$, where the strategy set is expanded to have players choose probability distributions, $\widehat{S}_i = \Delta(S_i)$.

–A pure strategy N.E. is also a (degenerate) mixed strategy N.E.

–In a mixed strategy N.E., any s_i played with positive probability must yield at least as high an expected payoff as any other strategy s_i'.

Calculating a mixed strategy N.E.

		player 2	
		wrestling match	ballet
player 1	wrestling match	$2, 1$	$0, 0$
	ballet	$0, 0$	$1, 2$

Suppose player 1 chooses wrestling with probability σ_1 and ballet with probability $1 - \sigma_1$; player 2 chooses wrestling with probability σ_2 and ballet with probability $1 - \sigma_2$.

Player 1's expected payoff for choosing wrestling is $2\sigma_2$, and his payoff for choosing ballet is $1 - \sigma_2$. For player 1 to be indifferent, the other player's probabilities are determined: $\sigma_2 = \frac{1}{3}$.

Player 2's expected payoff for choosing wrestling is σ_1, and her payoff for choosing ballet is $2(1 - \sigma_1)$. For player 2 to be indifferent, the other player's probabilities are determined: $\sigma_1 = \frac{2}{3}$.

For the matching pennies game, it is easy to calculate that each player must choose heads or tails with probability $\frac{1}{2}$. Even though there is no pure strategy N.E., there is a mixed strategy N.E.

When, in general, is there at least one N.E.?

Theorem (Nash): A pure strategy N.E. exists if for each i,

(i) S_i is a nonempty, convex, and compact subset of $\Re^M$, and

(ii) the payoff function is continuous in $(s_1, ..., s_n)$ and quasiconcave in s_i.

With a finite number of possible strategies, S_i cannot be a convex set, but the set of probability distributions is convex. As a corollary to Nash's theorem, any game in which each S_i is finite has a mixed strategy N.E.