

Bayesian Games

How do we incorporate into game theory the uncertainty players face about the characteristics of other players (their payoffs, what they know about our payoffs, etc.)?

We can model incomplete information by letting nature select the players' characteristics. This framework allows a player to know her own "type" but not the other players' types.

The payoff function is now $\pi_i(s_i, s_{-i}, \theta_i)$, where θ_i is player i 's type. The joint probability distribution, which allows correlation, is given by $F(\theta_1, \dots, \theta_n)$. It is assumed that this distribution is common knowledge, even though the realization of the θ 's is not.

A full specification of player i 's strategy must take into account that different types will make different choices. A strategy is a decision rule, $s_i(\theta_i)$.

To solve the game, we think of the players planning their decision rules at a stage before they observe their types. A Bayesian Nash equilibrium is a N.E. in the game of choosing decision rules.

Def: A Bayesian Nash equilibrium is a profile of decision rules, $(s_1(\cdot), \dots, s_n(\cdot))$, such that for all i , we have

$$E[\pi_i(s_i(\cdot), s_{-i}(\cdot), \theta_i)] \geq E[\pi_i(s'_i(\cdot), s_{-i}(\cdot), \theta_i)] \quad (1)$$

for all decision rules $s'_i(\cdot)$.

Expectations are taken based on the distribution, F .

An equivalent condition for Bayesian Nash equilibrium is that player i could not wake up with a type that wants to deviate from the decision rule.

Proposition: A profile of decision rules, $(s_1(\cdot), \dots, s_n(\cdot))$, is a Bayesian Nash equilibrium if and only if for all i and realized type $\bar{\theta}_i$, we have

$$E[\pi_i(s_i(\bar{\theta}_i), s_{-i}(\cdot), \bar{\theta}_i) \mid \bar{\theta}_i] \geq E[\pi_i(s'_i(\bar{\theta}_i), s_{-i}(\cdot), \bar{\theta}_i) \mid \bar{\theta}_i]. \quad (2)$$

Suppose $(s_1(\cdot), \dots, s_n(\cdot))$ is a Bayesian Nash equilibrium. If there was a beneficial deviation for some type, $\bar{\theta}_i$, then we can modify the decision rule accordingly and increase the ex ante expected payoff, contradicting the fact that $(s_1(\cdot), \dots, s_n(\cdot))$ is a Bayesian Nash equilibrium.

Similarly, we can show that if condition (2) holds, then condition (1) must hold as well.

Because the other players must react to all of your possible types, figuring out the optimal play given type $\bar{\theta}_i$ requires you to think about what you would have done with all other types.

Example: The strategy sets are $S_1 = \{top, bottom\}$ and $S_2 = \{left, right\}$. Player 1 knows his own payoffs for each combination of choices, but does not know player 2's payoffs. Player 2 knows her own payoffs and player 1's payoffs.

We can model this as nature choosing the payoff matrix, and only player 2 observing nature's choice before moving. Player 1's payoffs in the two matrices are the same, but player 2's payoffs are different.

A pure strategy for player 1 is $s_1 \in S_1$. A pure strategy for player 2 is $(s_2(1), s_2(2)) \in S_2 \times S_2$.

With probability $p > \frac{2}{3}$, player 2 is type 1, $\theta_2 = 1$, and the payoff matrix is:

		player 2 is type 1	
		left	right
player 1	top	1, 1	0, 0
	bottom	0, 1	2, 0

With probability $1 - p$, player 2 is type 2, $\theta_2 = 2$, and the payoff matrix is:

		player 2 is type 2	
		left	right
player 1	top	1, 1	0, 10
	bottom	0, 1	2, -10

Solving for the Bayesian Nash equilibrium:

If player 2 is of type 1, then she has a dominant strategy to play *left*, so $s_2(1) = \textit{left}$.

Notice that if player 2 is of type 2, then any N.E. of the actual game being played requires player 2 to randomize. However, because player 1 believes that type 1 is possible, that affects player 2's choice when she is type 2.

If we have $p > \frac{2}{3}$, then player 1's best strategy is to play *top*, no matter what he thinks player 2 will do when she is type 2. Therefore, we have $s_1 = \textit{top}$.

Since player 1 plays *top*, player 2 will play *right* when she is type 2.

We have shown that the unique Bayesian Nash equilibrium for this game is $s_1 = \textit{top}$, $(s_2(1), s_2(2)) = (\textit{left}, \textit{right})$.