

Department of Economics
The Ohio State University
Econ 805–Homework #1 Answers

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1. A consumer's *offer curve* is the locus of tangencies between the budget lines and indifference curves as relative prices change. (See Varian, pp. 316-317.)

(i) For a two-person, Edgeworth Box economy, explain why any intersection of the two offer curves is a competitive equilibrium.

(ii) What is the equation for the following consumer's offer curve?

$$u(x^1, x^2) = \log(x^1) + \log(x^2),$$

$$(\omega^1, \omega^2) = (1, 1).$$

Answer: (i) The absolute value of the slope of the line connecting the endowment point and the offer curve is the price ratio for which the bundle demanded is the point on the offer curve. If two offer curves intersect in an Edgeworth Box diagram, then the intersection point represents a feasible allocation that is utility maximizing for the same price ratio. Thus, the allocation, along with the price ratio equal to the absolute value of the slope of the segment from the endowment to the offer curve, together forms a competitive equilibrium.

(ii) Normalizing the price of good 2 to be one and the price of good one to be p , the (necessary and sufficient) first order conditions of the utility maximization problem are

$$\frac{x^2}{x^1} = p \quad \text{and} \quad (1)$$

$$px^1 + x^2 = p + 1. \quad (2)$$

We want to determine from (1) and (2) an equation in consumption space, so we must eliminate p . Substituting (1) into (2), we have

$$\frac{x^2}{x^1}x^1 + x^2 = \frac{x^2}{x^1} + 1,$$

which simplifies to

$$x^2 = \frac{x^1}{2x^1 - 1}.$$

Of course, since we are restricting ourselves to positive consumption, we must have $x^1 > \frac{1}{2}$. In other words, this is a hyperbola, and we restrict ourselves to the curve contained in the nonnegative orthant.

2. Using an Edgeworth box diagram, provide a counterexample to the second welfare theorem when utility functions are continuous and strictly monotonic, but not necessarily quasi-concave. That is, find an example of a Pareto optimal allocation that cannot be supported as a competitive equilibrium allocation (for any redistribution of initial endowments).

Answer: The easiest way to construct a counterexample is to first draw a "typical" Pareto optimal allocation x^* in the Edgeworth box, with the utility functions nicely behaved, and a tangent line to x^* . In order to support this PO allocation as a CE, the price line must be this tangent line; any other slope will cut through the indifference curves and not be consistent with utility maximization at that allocation. Next, redraw consumer 1's indifference curve, so that it now cuts through the budget line somewhere away from x^* , but does not intersect consumer 2's indifference curve a second time. This will ensure that x^* remains Pareto optimal, but cannot be supported as a CE allocation. (We will show you an example in recitation.)

3. Suppose there is no market for some commodity (say commodity K). Since consumers cannot trade commodity K , then everyone consumes his/her endowment of that good. A competitive equilibrium is a price vector for the $K - 1$ tradable goods, $p = (p^1, \dots, p^{K-1})$, and an allocation for all K commodities, satisfying market clearing and the following consumer optimization problem:

$$\begin{aligned} & \max u_i(x_i) \\ & \text{subject to} \\ & \sum_{j=1}^{K-1} p^j x_i^j \leq \sum_{j=1}^{K-1} p^j \omega_i^j \\ & x_i^K = \omega_i^K \\ & x_i \geq 0. \end{aligned}$$

Assume that all utility functions are continuous and strictly monotonic.

a) Will the competitive equilibrium be fully Pareto optimal, where dominating allocations can involve reallocations of commodity K as well as the other commodities?

b) Will the C.E. be constrained Pareto optimal, where only the first $K - 1$ commodities can be reallocated?

Answer: (a) In general, the missing-market competitive equilibrium will not be Pareto optimal in the traditional sense, where dominating allocations can involve reallocations of commodity K as well as the other commodities, because reallocating all K goods can generally create gains from trade. For example, if all utility functions are differentiable, the competitive equilibrium will equate

the marginal rates of substitution for all pairs of goods not including good K . However, it would only be by coincidence that

$$\frac{\frac{\partial u_i}{\partial x_i^j}}{\frac{\partial u_i}{\partial x_i^K}} = \frac{\frac{\partial u_h}{\partial x_h^j}}{\frac{\partial u_h}{\partial x_h^K}}$$

holds for two consumers, i and h .

(b) The missing-market competitive equilibrium is Pareto optimal in the constrained sense, where feasible dominating allocations must leave consumption of commodity K unchanged. To see this, consider a “hypothetical” economy with $K - 1$ goods where consumer i has the endowment vector, $\tilde{\omega}_i = (\omega_i^1, \dots, \omega_i^{K-1})$, and the utility function,

$$\tilde{u}_i = u_i(x_i^1, \dots, x_i^{K-1}; \omega_i^K).$$

In other words, we treat ω_i^K as a parameter of the utility function. A missing-market C.E. of this economy is a C.E. of the hypothetical economy. Under the assumption that the original utility function is strictly monotonic, then \tilde{u}_i satisfies local nonsatiation. Therefore, by the FFTWE, a C.E. allocation of the hypothetical economy is Pareto optimal for the hypothetical economy: there is no way to reallocate the $K - 1$ goods to make all consumers better off. Thus, the missing-market C.E. allocation is constrained Pareto optimal.

4. Consider a pure exchange economy with two consumers, $i = 1, 2$, and two goods. The consumers’ utility functions are given by

$$u_i(x_i^1, x_i^2) = (x_i^1)(x_i^2)$$

and endowments are given by

$$\omega_i = (a_i, b_i) \text{ for } i = 1, 2.$$

Let p denote the relative price of good 1 in terms of good 2.

- (i) Compute the excess demand function for this economy.
- (ii) Define a competitive equilibrium for this economy.
- (iii) Find the equilibrium price ratio.
- (iv) Determine an equation for the contract curve (the set of Pareto optimal points).

Answer: (i) Normalizing the price of good 2 to be one and the price of good one to be p , the first order conditions of consumer i ’s utility maximization problem are

$$\frac{x_i^2}{x_i^1} = p \text{ and} \tag{3}$$

$$px_i^1 + x_i^2 = pa_i + b_i. \tag{4}$$

Solving (3) and (4) for the demand functions yields

$$x_i^1 = \frac{pa_i + b_i}{2p} \quad \text{and} \quad x_i^2 = \frac{pa_i + b_i}{2}. \quad (5)$$

From (5), the excess demand function is

$$Z^1(p) = \frac{pa_1 + b_1}{2p} + \frac{pa_2 + b_2}{2p} - a_1 - a_2, \quad (6)$$

$$Z^2(p) = \frac{pa_1 + b_1}{2} + \frac{pa_2 + b_2}{2} - b_1 - b_2. \quad (7)$$

(ii) A competitive equilibrium is a (normalized) price vector, $(p, 1)$, and an allocation, $(x_1^1, x_1^2, x_2^1, x_2^2)$, such that:

1. for $i = 1, 2$, (x_i^1, x_i^2) solves

$$\begin{aligned} & \max x_i^1 x_i^2 \\ & \text{subject to} \\ & px_i^1 + x_i^2 \leq pa_i + b_i, \\ & x_i \geq 0, \end{aligned}$$

2. markets clear

$$\begin{aligned} x_1^1 + x_2^1 & \leq a_1 + a_2 \\ x_1^2 + x_2^2 & \leq b_1 + b_2. \end{aligned}$$

(iii) To find the C.E. price ratio, solve the market clearing equation, by setting either (6) or (7) equal to zero. Market clearing for good 2 requires

$$\frac{pa_1 + b_1}{2} + \frac{pa_2 + b_2}{2} - b_1 - b_2 = 0,$$

which can be solved for p ,

$$p = \frac{b_1 + b_2}{a_1 + a_2}. \quad (8)$$

(iv) Because the utility functions are nicely behaved, the set of Pareto optimal allocations are characterized by utilization of all resources and equal marginal rates of substitution. Thus, we have:

$$\frac{x_1^2}{x_1^1} = \frac{x_2^2}{x_2^1} \quad (9)$$

$$x_2^1 = a_1 + a_2 - x_1^1 \quad (10)$$

$$x_2^2 = b_1 + b_2 - x_1^2 \quad (11)$$

Substituting (10) and (11) into (9), we have

$$\frac{x_1^2}{x_1^1} = \frac{b_1 + b_2 - x_1^2}{a_1 + a_2 - x_1^1}. \quad (12)$$

Equation (12) implicitly relates x_1^2 to x_1^1 , so it can be considered an equation for the contract curve. We can simplify (12) to derive the functional relationship,

$$x_1^2 = \left[\frac{b_1 + b_2}{a_1 + a_2} \right] x_1^1. \quad (13)$$