Department of Economics
The Ohio State University
Econ 805–Homework #2 Answers

Prof James Peck
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1. Consider the following economy with two consumers, two equally likely states of nature, and one good per state. For $i = 1, 2$, consumer i has the utility function

$$V_i(x_i^1, x_i^2) = \frac{1}{2}\log(x_i^1) + \frac{1}{2}\log(x_i^2).$$

The endowment vectors are $\omega_1 = (2, 0)$ and $\omega_2 = (0, 1)$. Suppose that before the state of nature is revealed, the consumers trade state-1 contingent consumption for state-2 contingent consumption.

(a) Define a competitive equilibrium.

(b) Calculate the competitive equilibrium price vector and allocation.

Answer:

(a) A competitive equilibrium is a price vector, (p^1, p^2), and an allocation, $(x_1^1, x_1^2, x_2^1, x_2^2)$, such that

(i) (x_1^1, x_1^2) solves

$$\begin{aligned} & \max \frac{1}{2}\log(x_1^1) + \frac{1}{2}\log(x_1^2) \\ & \text{subject to} \\ p^1x_1^1 + p^2x_1^2 &= 2p^1 \\ x_1 &\geq 0 \end{aligned}$$

(ii) (x_2^1, x_2^2) solves

$$\begin{aligned} & \max \frac{1}{2}\log(x_2^1) + \frac{1}{2}\log(x_2^2) \\ & \text{subject to} \\ p^1x_2^1 + p^2x_2^2 &= p^2 \\ x_2 &\geq 0 \end{aligned}$$

(iii) markets clear

$$\begin{aligned} x_1^1 + x_2^1 &= 2 \\ x_1^2 + x_2^2 &= 1 \end{aligned}$$

Note: In the definition, equalities were imposed because utility functions are strictly monotonic.

(b) Normalize the price vector to $(p,1)$. The first order conditions for consumer 1 are

$$\begin{aligned} \frac{x_1^2}{x_1^1} &= p \\ px_1^1 + x_1^2 &= 2p \end{aligned}$$

from which we derive the demand functions:

$$\begin{aligned} x_1^1 &= 1 \\ x_1^2 &= p. \end{aligned}$$

The first order conditions for consumer 2 are

$$\begin{aligned} \frac{x_2^2}{x_2^1} &= p \\ px_2^1 + x_2^2 &= 1 \end{aligned}$$

from which we derive the demand functions:

$$\begin{aligned} x_2^1 &= \frac{1}{2p} \\ x_2^2 &= \frac{1}{2}. \end{aligned}$$

To find the equilibrium price, use market clearing for good 2:

$$\begin{aligned} p + \frac{1}{2} &= 1 \\ p &= \frac{1}{2}. \end{aligned}$$

The equilibrium allocation is found by substituting the price into the demand functions:

$$(x_1^1, x_1^2, x_2^1, x_2^2) = (1, \frac{1}{2}, 1, \frac{1}{2}).$$

2. For the economy of problem 1, suppose that before the state of nature is revealed, the consumers trade state-1 and state-2 Arrow securities. Then the state is revealed, securities are redeemed, and we have a spot market.

(a) Define a competitive equilibrium.

(b) Calculate the competitive equilibrium price vector and allocation.

Answer:

(a) A competitive equilibrium is a price vector, $(q^1, q^2, p(1), p(2))$, and an allocation, $\{(x_i(1), x_i(2))\}_{i=1,2}$ and $\{(b_i^1, b_i^2)\}_{i=1,2}$, such that:

(i) $(x_1(1), x_1(2), b_1^1, b_1^2)$ solves

$$\begin{aligned}
& \max \frac{1}{2}\log(x_1(1)) + \frac{1}{2}\log(x_1(2)) \\
& \text{subject to} \\
q^1 b_1^1 + q^2 b_1^2 &= 0 \\
p(1)x_1(1) &= 2p(1) + b_1^1, \\
p(2)x_1(2) &= b_1^2, \\
x_1 &\geq 0,
\end{aligned}$$

(ii) $(x_2(1), x_2(2), b_2^1, b_2^2)$ solves

$$\begin{aligned}
& \max \frac{1}{2}\log(x_2(1)) + \frac{1}{2}\log(x_2(2)) \\
& \text{subject to} \\
q^1 b_2^1 + q^2 b_2^2 &= 0 \\
p(1)x_2(1) &= b_2^1, \\
p(2)x_2(2) &= p(2) + b_2^2, \\
x_2 &\geq 0,
\end{aligned}$$

(iii) markets clear:

$$\begin{aligned}
b_1^1 + b_2^1 &= 0 \\
b_1^2 + b_2^2 &= 0 \\
x_1(1) + x_2(1) &= 2 \\
x_1(2) + x_2(2) &= 1
\end{aligned}$$

Note: The definition imposed equalities because utility is strictly monotonic.

(b) Normalize the price vector to be $(q, 1, 1, 1)$. To find consumer 1's demand functions, solve the spot market budget constraints for the securities holdings, and substitute into the securities constraint to get the equivalent problem

$$\begin{aligned}
& \max \frac{1}{2}\log(x_1(1)) + \frac{1}{2}\log(x_1(2)) \\
& \text{subject to} \\
q(x_1(1) - 2) + (x_1(2)) &= 0.
\end{aligned}$$

Solving for the demand functions, we have

$$\begin{aligned}
x_1(1) &= 1 \\
x_1(2) &= q
\end{aligned}$$

(note: when in doubt about the correct first order conditions, just set it up as a Lagrangean problem.)

To find consumer 2's demand functions, solve the spot market budget constraints for the securities holdings, and substitute into the securities constraint to get the equivalent problem

$$\begin{aligned} & \max \frac{1}{2}\log(x_2(1)) + \frac{1}{2}\log(x_2(2)) \\ & \text{subject to} \\ q(x_2(1)) + (x_1(2) - 1) &= 0. \end{aligned}$$

Solving for the demand functions, we have

$$\begin{aligned} x_1(1) &= \frac{1}{2q} \\ x_1(2) &= \frac{1}{2}. \end{aligned}$$

Market clearing for the state 2 spot market gives

$$\begin{aligned} q + \frac{1}{2} &= 1 \\ q &= \frac{1}{2}. \end{aligned}$$

Substituting the price into the demand functions, and solving the spot market budget constraints for the securities holdings yields the equilibrium allocation:

$$\begin{aligned} x_1 &= (1, \frac{1}{2}) \\ x_2 &= (1, \frac{1}{2}) \\ b_1 &= (-1, \frac{1}{2}) \\ b_2 &= (1, -\frac{1}{2}). \end{aligned}$$

3. Now modify the previous economy to include a good that must be consumed before the state of nature is revealed. For $i = 1, 2$, consumer i has the utility function over consumption at date 1, consumption at date 2 in state 1, and consumption at date 2 in state 2 given by

$$V_i(x_i^1, x_i^{2,1}, x_i^{2,2}) = \log(x_i^1) + \frac{1}{2}\log(x_i^{2,1}) + \frac{1}{2}\log(x_i^{2,2}).$$

The endowment vectors are $\omega_1 = (1, 2, 0)$ and $\omega_2 = (1, 0, 1)$. Suppose that before the state of nature is revealed, the consumers trade date-1 consumption, state-1 Arrow securities, and state-2 Arrow securities. Consumer i's income on this date-1 market results from selling her endowment ω_i^1, and the consumption she purchases, x_i^1, is consumed immediately. Then the state is revealed, securities are redeemed, and there is a spot market on which date-2 consumption is traded.

(a) Define a competitive equilibrium.
(b) Calculate the competitive equilibrium price vector and allocation.

Answer:

(a) A competitive equilibrium is a price vector, $(q^1, q^2, p^1, p^{2,1}, p^{2,2})$, and an allocation, $\{(x_i^1, x_i^{2,1}, x_i^{2,2})\}_{i=1,2}$ and $\{(b_i^1, b_i^2)\}_{i=1,2}$, such that:

(i) $(x_1^1, x_1^{2,1}, x_1^{2,2}, b_1^1, b_1^2)$ solves

$$\begin{aligned} & \max \log(x_1^1) + \frac{1}{2}\log(x_1^{2,1}) + \frac{1}{2}\log(x_1^{2,2}) \\ & \text{subject to} \\ q^1 b_1^1 + q^2 b_1^2 + p^1 x_1^1 &= p^1 \\ p^{2,1} x_1^{2,1} &= 2p^{2,1} + b_1^1, \\ p^{2,2} x_1^{2,2} &= b_1^2, \\ x_1 &\geq 0, \end{aligned}$$

(ii) $(x_2^1, x_2^{2,1}, x_2^{2,2}, b_2^1, b_2^2)$ solves

$$\begin{aligned} & \max \log(x_2^1) + \frac{1}{2}\log(x_2^{2,1}) + \frac{1}{2}\log(x_2^{2,2}) \\ & \text{subject to} \\ q^1 b_2^1 + q^2 b_2^2 + p^1 x_2^1 &= p^1 \\ p^{2,1} x_2^{2,1} &= b_2^1, \\ p^{2,2} x_2^{2,2} &= p^{2,2} + b_2^2, \\ x_2 &\geq 0, \end{aligned}$$

(iii) markets clear:

$$\begin{aligned} b_1^1 + b_2^1 &= 0 \\ b_1^2 + b_2^2 &= 0 \\ x_1^1 + x_2^1 &= 2 \\ x_1^{2,1} + x_2^{2,1} &= 2 \\ x_1^{2,2} + x_2^{2,2} &= 1 \end{aligned}$$

Note: The definition imposed equalities because utility is strictly monotonic.

(b) Normalize $p^1 = p^{2,1} = p^{2,2} = 1$. To find consumer 1's demand functions, solve the spot market budget constraints for the securities holdings, and substitute into the date-1 constraint to get the equivalent problem

$$\begin{aligned} & \max \log(x_1^1) + \frac{1}{2}\log(x_1^{2,1}) + \frac{1}{2}\log(x_1^{2,2}) \\ & \text{subject to} \\ q^1(x_1^{2,1} - 2) + q^2 x_1^{2,2} + x_1^1 &= 1. \end{aligned}$$

The first order conditions are the constraint, and the two marginal rate of substitution conditions,

$$\begin{aligned}\frac{2x_1^{2,1}}{x_1^1} &= \frac{1}{q^1}\\ \frac{2x_1^{2,2}}{x_1^1} &= \frac{1}{q^2}.\end{aligned}$$

Substituting the MRS conditions into the budget constraint yields

$$\begin{aligned}q^1(\frac{x_1^1}{2q^1}-2)+q^2(\frac{x_1^1}{2q^2})+x_1^1 &= 1\\ x_1^1 &= \frac{2q^1+1}{2}\end{aligned}$$

Substitution into the MRS conditions yields

$$\begin{aligned}x_1^{2,1} &= \frac{2q^1+1}{4q^1}\\ x_1^{2,2} &= \frac{2q^1+1}{4q^2}.\end{aligned}$$

To find consumer 2's demand functions, solve the spot market budget constraints for the securities holdings, and substitute into the date-1 constraint to get the equivalent problem

$$\begin{aligned}& \max \log(x_2^1)+\frac{1}{2}\log(x_2^{2,1})+\frac{1}{2}\log(x_2^{2,2})\\ & \text{subject to}\\ q^1(x_2^{2,1})+q^2(x_2^{2,2}-1)+x_2^1 &= 1.\end{aligned}$$

A similar computation for consumer 2 yields the demand functions

$$\begin{aligned}x_2^1 &= \frac{q^2+1}{2}\\ x_2^{2,1} &= \frac{q^2+1}{4q^1}\\ x_2^{2,2} &= \frac{q^2+1}{4q^2}\end{aligned}$$

We can solve for the two remaining prices by using market clearing on the two spot markets:

$$\begin{aligned}\frac{2q^1+1}{4q^1}+\frac{q^2+1}{4q^1} &= 2\\ \frac{2q^1+1}{4q^2}+\frac{q^2+1}{4q^2} &= 1\end{aligned}$$

which can be solved for $q^1 = \frac{1}{2}$ and $q^2 = 1$. Substituting into the demand functions yields consumption, and substituting into the spot market budget constraints yields the securities holdings:

$$\begin{aligned} x_1 &= (1, 1, \frac{1}{2}) \\ x_2 &= (1, 1, \frac{1}{2}) \\ b_1 &= (-1, \frac{1}{2}) \\ b_2 &= (1, -\frac{1}{2}). \end{aligned}$$

4. In the following Rothschild-Stiglitz model, all consumers have initial wealth of 10 and a potential accident with damages of 5, so the state-contingent initial endowment is (10,5). All consumers have von Neumann-Morgenstern utility functions with a "Bernoulli" utility of certain consumption given by $u(W) = \log(W)$. For high risk consumers, the probability of an accident (state 2) is given by $\pi^H = \frac{1}{2}$, and for low risk consumers, the probability of an accident (state 2) is given by $\pi^L = \frac{1}{4}$. The population fraction of high risk consumers is given by λ.

(a) What is the unique "candidate" equilibrium contract offered to the high risk consumers?

(b) What is the unique "candidate" equilibrium contract offered to the low risk consumers?

Answer:

(a) The high risk consumers are offered full insurance at full odds. Since the slope of the fair odds line is -1 for the high risk types and the endowment is $(10, 5)$, the contract $\alpha_H = (\frac{15}{2}, \frac{15}{2})$.

(b) We need to find the intersection of the indifference curve of the high-risks running through α_H, and the fair odds line of the low risk types. The equation of the indifference curve is given by

$$\frac{1}{2}\log(W_1) + \frac{1}{2}\log(W_2) = \frac{1}{2}\log(\frac{15}{2}) + \frac{1}{2}\log(\frac{15}{2})$$

which can be simplified to

$$W_1 W_2 = \frac{225}{4}.$$

Since the fair odds line has slope of -3 and goes through the endowment point, the equation is given by

$$\frac{W_2 - 5}{W_1 - 10} = -3.$$

Simultaneously solving these equations yields a quadratic equation, and the correct solution for α_L is the one below the 45 degree line:

$$\begin{aligned} W_1 &= \frac{35}{6} + \frac{5\sqrt{22}}{6} \approx 9.742 \\ W_2 &= \frac{35}{2} - \frac{5\sqrt{22}}{2} \approx 5.774 \end{aligned}$$

You can see that not much insurance is provided.