1. The following economy has two commodities, $n$ consumers, and $F$ firms. For $i = 1, \ldots, n$, consumer $i$ has the endowment vector, $\omega_i = (1, 0)$, and the utility function,

$$u_i(x^1_i, x^2_i) = \log(x^1_i) + \log(x^2_i).$$

Each consumer owns an equal share, $1/n$, of each firm. For $f = 1, \ldots, F$, firm $f$ has a production set whose frontier (production function) is given by

$$y^2_f = -\sqrt{-y^1_f},$$

where we have $y^1_f \leq 0$ (reflecting the convention that inputs are negative outputs).

(a) Define a competitive equilibrium for this economy.

(b) Calculate the competitive equilibrium price vector and allocation, as a function of the parameters, $n$ and $F$.

(c) Is the utility received by consumers at the competitive equilibrium increasing or decreasing in $F$? Give the economic intuition for this result.

Answer:

(a) A competitive equilibrium is a price vector, $(p^1, p^2)$, and an allocation, $\{(x^1_i, x^2_i)\}_{i=1}^n$ and $\{(y^1_f, y^2_f)\}_{f=1}^F$, such that

(i) For $i = 1, \ldots, n$, $(x^1_i, x^2_i)$ solves

$$\max\{\log(x^1_i) + \log(x^2_i)\}$$

subject to

$$p^1 x^1_i + p^2 x^2_i \leq p^1 + \frac{1}{n} \sum_{f=1}^F \pi_f$$

$$x_i \geq 0.$$

(ii) For $f = 1, \ldots, F$, $(y^1_f, y^2_f)$ solves

$$\max \pi_f = p^1 y^1_f + p^2 y^2_f$$

subject to

$$y^2_f = -\sqrt{-y^1_f}$$

$$y^1_f \leq 0.$$
(iii) Markets clear

\[ \sum_{i=1}^{n} x^1_i \leq n + \sum_{f=1}^{F} y^1_f \]

\[ \sum_{i=1}^{n} x^2_i \leq \sum_{f=1}^{F} y^2_f. \]

(b) Because of monotonicity, the budget and market clearing inequalities hold as equalities. Let us normalize \( p^2 = 1 \) and \( p^1 = p \). Solving the profit maximization problem first, substitute (2) into (1), and differentiate with respect to \( y^1_f \). Setting the derivative equal to zero yields

\[ y^1_f = -\frac{1}{4p^2}. \]  (3)

Substituting (3) into (2) and (1), we have

\[ y^2_f = \frac{1}{2p}, \]  (4)

\[ \pi_f = \frac{1}{4p}. \]  (5)

Now we solve the utility maximization problem for consumer \( i \). Demand is calculated by solving the marginal rate of substitution equation and the budget equation (with (5) substituted for profit of each firm):

\[ \frac{x^2_i}{x^1_i} = p, \]

\[ px^1_i + x^2_i = p + F \frac{1}{n 4p}. \]

yielding

\[ x^1_i = \frac{1}{2} + \frac{F}{8np^2}, \]  (6)

\[ x^2_i = \frac{p}{2} + \frac{F}{8np}. \]  (7)

Market clearing of good 1 is given by

\[ n[\frac{1}{2} + \frac{F}{8np^2}] = n - \frac{F}{4p^2}, \]

from which we solve for

\[ p = \sqrt{\frac{3F}{4n}}. \]  (8)
Substituting (8) into (6), (7), (1), and (2) yields the equilibrium allocation

\[
x_1^i = \frac{2}{3} \\
x_i^2 = \sqrt{\frac{F}{3n}} \\
y_i^1 = \frac{n}{3F} \\
y_i^2 = \sqrt{\frac{n}{3F}}
\]

(c) From the final allocation, it is clear that consumption of good 1 is independent of \( F \), and consumption of good 2 is increasing in \( F \), so overall utility must be increasing in \( F \). This is not because more firms reduces the market power of any one firm, because firms are assumed to be price takers here. Rather, the answer is purely technological. Because we have a decreasing returns to scale technology, dividing the input of good 1 among more firms causes the production possibilities frontier to shift outward, leading to more aggregate production for a given aggregate input.

2. Consider the following economy with two consumers, two equally likely states of nature, and one good per state. For \( i = 1, 2 \), consumer \( i \) has the utility function

\[
V_i(x_1^i, x_2^i) = \frac{1}{2} \log(x_1^i) + \frac{1}{2} \log(x_2^i).
\]

The endowment vectors are \( \omega_1 = (2, 0) \) and \( \omega_2 = (0, 1) \). Suppose that before the state of nature is revealed, the consumers trade state-1 contingent consumption for state-2 contingent consumption.

(a) Define a competitive equilibrium.
(b) Calculate the competitive equilibrium price vector and allocation.

**Answer:**

(a) A competitive equilibrium is a price vector, \((p^1, p^2)\), and an allocation, \((x_1^1, x_2^1, x_1^2, x_2^2)\), such that

(i) \((x_1^1, x_2^1)\) solves

\[
\max \frac{1}{2} \log(x_1^1) + \frac{1}{2} \log(x_2^1)
\]

subject to

\[
p^1 x_1^1 + p^2 x_2^1 = 2p^1 \\
x_1^1 \geq 0
\]

\(x_1^1\)
(ii) $(x^1_2, x^3_2)$ solves
\[
\max \frac{1}{2} \log(x^1_2) + \frac{1}{2} \log(x^3_2)
\]
subject to
\[
p^1 x^1_2 + p^2 x^3_2 = p^2
\]
\[
x^2_2 \geq 0
\]

(iii) markets clear
\[
x^1_1 + x^1_2 = 2
\]
\[
x^2_1 + x^2_2 = 1
\]

Note: In the definition, equalities were imposed because utility functions are strictly monotonic.

(b) Normalize the price vector to $(p, 1)$. The first order conditions for consumer 1 are
\[
\frac{x^2_1}{x^1_1} = p
\]
\[
x^1_1 + x^1_2 = 2p
\]
from which we derive the demand functions:
\[
x^1_1 = 1
\]
\[
x^2_1 = p.
\]

The first order conditions for consumer 2 are
\[
\frac{x^2_2}{x^1_2} = p
\]
\[
x^1_2 + x^1_2 = 1
\]
from which we derive the demand functions:
\[
x^1_2 = \frac{1}{2p}
\]
\[
x^2_2 = \frac{1}{2}.
\]

To find the equilibrium price, use market clearing for good 2:
\[
p + \frac{1}{2} = 1
\]
\[
p = \frac{1}{2}.
\]

4
The equilibrium allocation is found by substituting the price into the demand functions:

\[(x^1_1, x^1_2, x^2_1, x^2_2) = \left(1, \frac{1}{2}, 1, \frac{1}{2}\right).\]

3. For the economy of problem 2, suppose that before the state of nature is revealed, the consumers trade state-1 and state-2 Arrow securities. Then the state is revealed, securities are redeemed, and we have a spot market.

(a) Define a competitive equilibrium.

(b) Calculate the competitive equilibrium price vector and allocation.

Answer:

(a) A competitive equilibrium is a price vector, \((q^1, q^2, p(1), p(2))\), and an allocation, \(\{(x_i(1), x_i(2))\}_{i=1,2}\) and \(\{(b^1_i, b^2_i)\}_{i=1,2}\), such that:

(i) \((x_1(1), x_1(2), b^1_1, b^2_1)\) solves

\[
\max \frac{1}{2} \log(x_1(1)) + \frac{1}{2} \log(x_1(2))
\]

subject to

\[
q^1b^1_1 + q^2b^2_1 = 0
\]

\[
p(1)x_1(1) = 2p(1) + b^1_1,
\]

\[
p(2)x_1(2) = b^2_1,
\]

\[
x_1 \geq 0,
\]

(ii) \((x_2(1), x_2(2), b^1_2, b^2_2)\) solves

\[
\max \frac{1}{2} \log(x_2(1)) + \frac{1}{2} \log(x_2(2))
\]

subject to

\[
q^1b^1_2 + q^2b^2_2 = 0
\]

\[
p(1)x_2(1) = b^1_2,
\]

\[
p(2)x_2(2) = p(2) + b^2_2,
\]

\[
x_2 \geq 0,
\]

(iii) markets clear:

\[
b^1_1 + b^1_2 = 0
\]

\[
b^2_1 + b^2_2 = 0
\]

\[
x_1(1) + x_2(1) = 2
\]

\[
x_1(2) + x_2(2) = 1
\]

Note: The definition imposed equalities because utility is strictly monotonic.
(b) Normalize the price vector to be \( (q, 1, 1, 1) \). To find consumer 1’s demand functions, solve the spot market budget constraints for the securities holdings, and substitute into the securities constraint to get the equivalent problem

\[
\begin{align*}
\text{max } & \frac{1}{2} \log(x_1(1)) + \frac{1}{2} \log(x_1(2)) \\
\text{subject to } & q(x_1(1) - 2) + x_1(2) = 0.
\end{align*}
\]

Solving for the demand functions, we have

\[
\begin{align*}
x_1(1) &= 1 \\
x_1(2) &= q.
\end{align*}
\]

(note: when in doubt about the correct first order conditions, just set it up as a Lagrangean problem.)

To find consumer 2’s demand functions, solve the spot market budget constraints for the securities holdings, and substitute into the securities constraint to get the equivalent problem

\[
\begin{align*}
\text{max } & \frac{1}{2} \log(x_2(1)) + \frac{1}{2} \log(x_2(2)) \\
\text{subject to } & q(x_2(1)) + (x_1(2) - 1) = 0.
\end{align*}
\]

Solving for the demand functions, we have

\[
\begin{align*}
x_1(1) &= \frac{1}{2q} \\
x_1(2) &= \frac{1}{2}.
\end{align*}
\]

Market clearing for the state 2 spot market gives

\[
\begin{align*}
q + \frac{1}{2} &= 1 \\
q &= \frac{1}{2}.
\end{align*}
\]

Substituting the price into the demand functions, and solving the spot market budget constraints for the securities holdings yields the equilibrium allocation:

\[
\begin{align*}
x_1 &= (1, \frac{1}{2}) \\
x_2 &= (1, \frac{1}{2}) \\
b_1 &= (-1, \frac{1}{2}) \\
b_2 &= (1, -\frac{1}{2}).
\end{align*}
\]
4. Now modify the previous economy to include a good that must be consumed before the state of nature is revealed. For \( i = 1, 2 \), consumer \( i \) has the utility function over consumption at date 1, consumption at date 2 in state 1, and consumption at date 2 in state 2 given by

\[
V_i(x_{i1}^1, x_{i2}^{2,1}, x_{i2}^{2,2}) = \log(x_{i1}^1) + \frac{1}{2}\log(x_{i2}^{2,1}) + \frac{1}{2}\log(x_{i2}^{2,2}).
\]

The endowment vectors are \( \omega_1 = (1, 2, 0) \) and \( \omega_2 = (1, 0, 1) \). Suppose that before the state of nature is revealed, the consumers trade date-1 consumption, state-1 Arrow securities, and state-2 Arrow securities. Consumer \( i \)'s income on this date-1 market results from selling her endowment \( \omega_{i1} \), and the consumption she purchases, \( x_{i1}^1 \), is consumed immediately. Then the state is revealed, securities are redeemed, and there is a spot market on which date-2 consumption is traded.

(a) Define a competitive equilibrium.
(b) Calculate the competitive equilibrium price vector and allocation.

Answer:
(a) A competitive equilibrium is a price vector, \( (q^1, q^2, p^1, p^{2,1}, p^{2,2}) \), and an allocation, \( \{(x_{i1}^1, x_{i2}^{2,1}, x_{i2}^{2,2})\}_{i=1,2} \) and \( \{(b_{i1}^1, b_{i2}^2)\}_{i=1,2} \), such that:

(i) \( (x_{i1}^1, x_{i1}^{2,1}, x_{i2}^{2,2}) \) solves

\[
\max \log(x_{i1}^1) + \frac{1}{2}\log(x_{i2}^{2,1}) + \frac{1}{2}\log(x_{i2}^{2,2})
\]
subject to

\[
q^1b_{i1}^1 + q^2b_{i2}^2 + p^1x_{i1}^1 = p^1
\]
\[
p^{2,1}x_{i1}^{2,1} = 2p^{2,1} + b_{i1}^1,
\]
\[
p^{2,2}x_{i1}^{2,2} = b_{i1}^2,
\]
\[
x_{i1} \geq 0,
\]

(ii) \( (x_{i2}^1, x_{i2}^{2,1}, x_{i2}^{2,2}) \) solves

\[
\max \log(x_{i2}^1) + \frac{1}{2}\log(x_{i2}^{2,1}) + \frac{1}{2}\log(x_{i2}^{2,2})
\]
subject to

\[
q^1b_{i2}^1 + q^2b_{i2}^2 + p^1x_{i2}^1 = p^1
\]
\[
p^{2,1}x_{i2}^{2,1} = b_{i2}^1,
\]
\[
p^{2,2}x_{i2}^{2,2} = p^{2,2} + b_{i2}^2,
\]
\[
x_{i2} \geq 0,
\]

(iii) markets clear:

\[
b_{i1}^1 + b_{i2}^1 = 0
\]
\[
b_{i2}^2 + b_{i2}^2 = 0
\]
\[
x_{i1}^1 + x_{i2}^{2,1} = 2
\]
\[
x_{i1}^{2,1} + x_{i2}^{2,1} = 2
\]
\[
x_{i1}^{2,2} + x_{i2}^{2,2} = 1
\]
Note: The definition imposed equalities because utility is strictly monotonic.

(b) Normalize $p^1 = p^{2,1} = p^{2,2} = 1$. To find consumer 1’s demand functions, solve the spot market budget constraints for the securities holdings, and substitute into the date-1 constraint to get the equivalent problem

$$\max \ log(x^1_1) + \frac{1}{2} \ log(x^{2,1}_1) + \frac{1}{2} \ log(x^{2,2}_1)$$

subject to

$$q^1(x^{2,1}_1 - 2) + q^2 x^{2,2}_1 + x^1_1 = 1.$$ 

The first order conditions are the constraint, and the two marginal rate of substitution conditions,

$$\frac{2x^{2,1}_1}{x^1_1} = \frac{1}{q^1}$$

$$\frac{2x^{2,2}_1}{x^1_1} = \frac{1}{q^2}$$

Substituting the MRS conditions into the budget constraint yields

$$q^1(x^{2,1}_1 - 2) + q^2(x^{2,2}_1) + x^1_1 = 1$$

$$x^1_1 = \frac{2q^1 + 1}{2}$$

Substitution into the MRS conditions yields

$$x^{2,1}_1 = \frac{2q^1 + 1}{4q^1}$$

$$x^{2,2}_1 = \frac{2q^1 + 1}{4q^2}$$

To find consumer 2’s demand functions, solve the spot market budget constraints for the securities holdings, and substitute into the date-1 constraint to get the equivalent problem

$$\max \ log(x^2_2) + \frac{1}{2} \ log(x^{2,1}_2) + \frac{1}{2} \ log(x^{2,2}_2)$$

subject to

$$q^1(x^{2,1}_2) + q^2(x^{2,2}_2 - 1) + x^1_2 = 1.$$ 

A similar computation for consumer 2 yields the demand functions

$$x^1_2 = \frac{q^2 + 1}{2}$$

$$x^{2,1}_2 = \frac{q^2 + 1}{4q^1}$$

$$x^{2,2}_2 = \frac{q^2 + 1}{4q^2}$$
We can solve for the two remaining prices by using market clearing on the two spot markets:

\[
\frac{2q^1 + 1}{4q^1} + \frac{q^2 + 1}{4q^1} = 2 \\
\frac{2q^1 + 1}{4q^2} + \frac{q^2 + 1}{4q^2} = 1
\]

which can be solved for \( q^1 = \frac{1}{2} \) and \( q^2 = 1 \). Substituting into the demand functions yields consumption, and substituting into the spot market budget constraints yields the securities holdings:

\[
\begin{align*}
x_1 & = (1, 1, \frac{1}{2}) \\
x_2 & = (1, 1, \frac{1}{2}) \\
b_1 & = (-1, \frac{1}{2}) \\
b_2 & = (1, -\frac{1}{2}).
\end{align*}
\]

5. (This is a sample R-S problem for preparing for the first midterm. It is not assigned, because we will not finish the R-S model until February 1.) In the following Rothschild-Stiglitz model, all consumers have initial wealth of 10 and a potential accident with damages of 5, so the state-contingent initial endowment is (10,5). All consumers have von Neumann-Morgenstern utility functions with a "Bernoulli" utility of certain consumption given by \( u(W) = \log(W) \). For high risk consumers, the probability of an accident (state 2) is given by \( \pi^H = \frac{1}{2} \), and for low risk consumers, the probability of an accident (state 2) is given by \( \pi^L = \frac{1}{4} \). The population fraction of high risk consumers is given by \( \lambda \).

(a) What is the unique "candidate" equilibrium contract offered to the high risk consumers?

(b) What is the unique "candidate" equilibrium contract offered to the low risk consumers?

Answer:

(a) The high risk consumers are offered full insurance at full odds. Since the slope of the fair odds line is \(-1\) for the high risk types and the endowment is (10,5), the contract \( \alpha_H = (\frac{15}{2}, \frac{15}{2}) \).

(b) We need to find the intersection of the indifference curve of the high-risks running through \( \alpha_H \), and the fair odds line of the low risk types. The equation of the indifference curve is given by

\[
\frac{1}{2} \log(W_1) + \frac{1}{2} \log(W_2) = \frac{1}{2} \log(\frac{15}{2}) + \frac{1}{2} \log(\frac{15}{2})
\]
which can be simplified to

\[ W_1W_2 = \frac{225}{4}. \]

Since the fair odds line has slope of \(-3\) and goes through the endowment point, the equation is given by

\[ \frac{W_2 - 5}{W_1 - 10} = -3. \]

Simultaneously solving these equations yields a quadratic equation, and the correct solution for \(\alpha_L\) is the one below the 45 degree line:

\[
\begin{align*}
W_1 &= \frac{35}{6} + \frac{5\sqrt{22}}{6} \approx 9.742 \\
W_2 &= \frac{35}{2} - \frac{5\sqrt{22}}{2} \approx 5.774
\end{align*}
\]

You can see that not much insurance is provided.