Department of Economics
The Ohio State University
Second Midterm Exam Answers–Econ 805

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1. (30 points)

Consider the following class of strategic form games, based on the parameter, a:

		player 2	
		left	right
player 1	top	$1, 2$	a, a
	bottom	a, a	$2, 1$

(a) (15 points) For what values of the parameter, a, is there a Nash equilibrium in pure strategies?
(b) (15 points) For what values of the parameter, a, is there a mixed-strategy Nash equilibrium in which the players choose each of their actions with strictly positive probability?

Answer: (a) (bottom,right) is a NE whenever we have $2 \geq a$ (player 1 is best-responding) and $1 \geq a$ (player 2 is best responding). This simplifies to $1 \geq a$.

(bottom,left) is a NE whenever we have $a \geq 1$ and $a \geq 1$. Therefore, there is a pure-strategy NE for all values of a.

(b) We have a nontrivial mixed-strategy NE, where player 1 chooses top with probability p and player 2 chooses left with probability q, if and only if each player is indifferent between his/her choices given the other player's mixing probability and $0 < p < 1, 0 < q < 1$. For player 1, the expected payoff from top is $q + (1 - q)a$ and the expected payoff from bottom is $qa + (1 - q)2$. Equating these expressions, we have

$$q = \frac{2 - a}{3 - 2a}.$$

For player 2, the expected payoff from left is $p2 + (1 - p)a$ and the expected payoff from right is $pa + (1 - p)$. Equating these expressions, we have

$$p = \frac{1 - a}{3 - 2a}.$$

The question is, when will these probabilities be strictly between zero and one? We will look at the fraction for q, since q is between zero and one if and only if p is between zero and one. The fraction is positive whenever we have either $a > 2$ **and** $a > 3/2$ (numerator and denominator are negative), or we have $a < 2$ **and** $a < 3/2$ (numerator and denominator are positive). This condition simplifies to

$$a > 2 \text{ or } a < \frac{3}{2}. \tag{1}$$

Similarly, the fraction is less than one whenever we have either $a < 3/2$ **and** $a < 1$, or we have $a > 3/2$ **and** $a > 1$. This condition simplifies to

$$a < 1 \text{ or } a > \frac{3}{2}. \tag{2}$$

Since we need (1) and (2) to be satisfied, this occurs if and only if either $a < 1$ or $a > 2$ hold. (Indeed, for $1 < a < 2$, bottom and left are dominant strategies for the two players, so there cannot be mixing.)

2. (35 points)

In the following game, there is an asset that pays a random investment return with realization R. The two potential investors do not observe R, but they each observe a private signal that is correlated with R. Specifically, player 1 observes z_1 and player 2 observes z_2, where z_1 and z_2 are independent random variables **uniformly** distributed on the unit interval $[0, 1]$. The investment return is given by

$$R = z_1 + z_2.$$

Each player also faces a cost of investing, c. Assume that $1/2 < c < 3/2$ holds.

The timing of the game is as follows. First, player 1 observes z_1 and decides whether or not to invest. Next, player 2 observes z_2 and observes whether or not player 1 invested; then player 2 decides whether or not to invest.

For $i = 1, 2$, player i receives a payoff of $R - c$ if she invests, and receives a payoff of zero if she does not invest.

Find a weak perfect Bayesian equilibrium (WPBE) for this game, where the strategies and beliefs will depend on the parameter, c. For this problem, it is enough to specify the strategy profile and player 2's beliefs about player 1's signal (if player 1 invests and if player 1 does not invest).

Hint: Since player 1 does not care about what player 2 does after him, you can solve for player 1's strategy first. Then figure out player 2's beliefs and strategy.

Answer: Player 1's payoff does not depend on player 2's action, so his sequentially rational strategy is quite simple. If he invests, his expected payoff is $z_1 + E(z_2) - c$. Because of the uniform distribution of player 2's signal, player 1 should invest if and only if $z_1 > c - 1/2$.

Now that we know player 1's equilibrium strategy, we can find player 2's beliefs and sequentially rational action as a function of z_2 and whether player

1 invests. Since all information sets are on the equilibrium path, we must use Bayes' rule to determine beliefs. Because signals are independent, for any z_2, when player 1 invests z_1 is uniformly distributed on the interval $[c - 1/2, 1]$, and when player 1 does not invest z_1 is uniformly distributed on the interval $[0, c - 1/2]$. Therefore, player 2's expected payoff from investing when player 1 invests is

$$\begin{aligned} & E(z_1 | \text{player 1 invests}) + z_2 - c \\ = & \frac{c - 1/2 + 1}{2} + z_2 - c \\ = & z_2 - \frac{c}{2} + \frac{1}{4}. \end{aligned}$$

Thus, if player 1 invests, player 2 invests if and only if

$$z_2 > \frac{c}{2} - \frac{1}{4}.$$

If player 1 does not invest, we have

$$E(z_1 | \text{player 1 does not invest}) = \frac{c - 1/2}{2},$$

so player 2 invests if and only if

$$z_2 > \frac{c}{2} + \frac{1}{4}.$$

3. (35 points)

Consider a market that is served by a monopolist who chooses both the *quality* of its product, θ, and the price per unit of output, p, in terms of the numeraire good. The total cost of producing x units of quality θ (in terms of the numeraire good) is given by $\theta^2 x$.

Assume that there is a single consumer, with the quasi-linear utility function over consumption of the monopolist's produced good, x, and consumption of the numeraire good, M, given by

$$u(x, M, \theta) = (x - \frac{x^2}{2})\theta + M \quad \text{for } x \in [0, 1].$$

Also, assume that the consumer has a zero endowment of good x, and a large enough endowment of the numeraire good so that we do not have to worry about nonnegativity constraints on numeraire consumption.

(a) (20 points) Assume that the monopolist must set a single price, p, of the good in terms of the numeraire, at which all transactions must occur. Find the monopoly price, p, and quality, θ.

(b) (15 points) What is the socially optimal quantity of output, x, and quality, θ, that would be chosen by a planner seeking to maximize total surplus?

Answer: I should have been explicit that the consumer observes the quality, but fortunately no one seemed confused about that. [If the consumer could not observe quality, then the firm should choose quality zero, and the consumer would demand zero output at any price.]

(a) Given quality, θ, the solution to the consumer's utility maximization problem involves equating the marginal rate of substitution to the price ratio, p. This condition determines the inverse demand,

$$\theta(1-x) = p. \tag{3}$$

From (3), we can express the monopolist's profit in terms of x and θ:

$$\pi^m = \theta(1-x)x - \theta^2 x.$$

Differentiating with respect to x, we have the first order condition

$$\begin{aligned} \theta(1-2x) - \theta^2 &= 0, \text{ or} \\ \theta &= 1-2x. \end{aligned}$$

(Clearly $\theta = 0$ cannot be optimal so we can divide by θ.) Differentiating with respect to θ, we have the first order condition

$$\begin{aligned} (1-x)x - 2\theta x &= 0, \text{ or} \\ \theta &= \frac{1-x}{2}. \end{aligned}$$

Simultaneously solving the two conditions and substituting into (3) to get the price, we have

$$x = \frac{1}{3}, \theta = \frac{1}{3}, p = \frac{2}{9}.$$

(b) Because utility is quasi-linear, we can equate social welfare to the total surplus of the produced good (benefit minus cost), given by

$$(x - \frac{x^2}{2})\theta - \theta^2 x.$$

To find the socially optimal x and θ, solve the two first order conditions. Differentiating welfare with respect to x, we have the first order condition

$$\begin{aligned} (1-x)\theta - \theta^2 &= 0, \text{ or} \\ \theta &= 1-x. \end{aligned}$$

Differentiating welfare with respect to θ, we have the first order condition

$$\begin{aligned} (x - \frac{x^2}{2}) - 2\theta x &= 0, \text{ or} \\ \theta &= \frac{1}{2} - \frac{x}{4}. \end{aligned}$$

Simultaneously solving the two conditions, we have

$$x = \frac{2}{3}, \theta = \frac{1}{3}.$$

Thus, the monopolist chooses the socially optimal quality but produces less than the socially optimal quantity.