1. (30 points)

Consider the following class of strategic form games, based on the parameter, $a$:

|       | player 2
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>left</td>
<td>top</td>
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<tr>
<td></td>
<td>1, 2</td>
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<tr>
<td>right</td>
<td>bottom</td>
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</tbody>
</table>
|       | $a, a$    

(a) (15 points) For what values of the parameter, $a$, is there a Nash equilibrium in pure strategies?

(b) (15 points) For what values of the parameter, $a$, is there a mixed-strategy Nash equilibrium in which the players choose each of their actions with strictly positive probability?

Answer: (a) (bottom, right) is a NE whenever we have $2 \geq a$ (player 1 is best-responding) and $1 \geq a$ (player 2 is best responding). This simplifies to $1 \geq a$.

(b) We have a nontrivial mixed-strategy NE, where player 1 chooses top with probability $p$ and player 2 chooses left with probability $q$, if and only if each player is indifferent between his/her choices given the other player’s mixing probability and $0 < p < 1, 0 < q < 1$. For player 1, the expected payoff from top is $q + (1 - q)a$ and the expected payoff from bottom is $qa + (1 - q)2$. Equating these expressions, we have

$$q = \frac{2 - a}{3 - 2a}.$$  

For player 2, the expected payoff from left is $p2 + (1 - p)a$ and the expected payoff from right is $pa + (1 - p)$. Equating these expressions, we have

$$p = \frac{1 - a}{3 - 2a}.$$  

The question is, when will these probabilities be strictly between zero and one? We will look at the fraction for $q$, since $q$ is between zero and one if and only if $p$ is between zero and one. The fraction is positive whenever we have
either \( a > 2 \) and \( a > 3/2 \) (numerator and denominator are negative), or we have \( a < 2 \) and \( a < 3/2 \) (numerator and denominator are positive). This condition simplifies to
\[
a > 2 \quad \text{or} \quad a < \frac{3}{2}
\]
(1)
Similarly, the fraction is less than one whenever we have either \( a < 3/2 \) and \( a < 1 \), or we have \( a > 3/2 \) and \( a > 1 \). This condition simplifies to
\[
a < 1 \quad \text{or} \quad a > \frac{3}{2}.
\]
(2)
Since we need (1) and (2) to be satisfied, this occurs if and only if either \( a < 1 \) or \( a > 2 \) hold. (Indeed, for \( 1 < a < 2 \), bottom and left are dominant strategies for the two players, so there cannot be mixing.)

2. (35 points)

In the following game, there is an asset that pays a random investment return with realization \( R \). The two potential investors do not observe \( R \), but they each observe a private signal that is correlated with \( R \). Specifically, player 1 observes \( z_1 \) and player 2 observes \( z_2 \), where \( z_1 \) and \( z_2 \) are independent random variables uniformly distributed on the unit interval \([0, 1]\). The investment return is given by
\[R = z_1 + z_2.\]
Each player also faces a cost of investing, \( c \). Assume that \( 1/2 < c < 3/2 \) holds.

The timing of the game is as follows. First, player 1 observes \( z_1 \) and decides whether or not to invest. Next, player 2 observes \( z_2 \) and observes whether or not player 1 invested; then player 2 decides whether or not to invest.

For \( i = 1, 2 \), player \( i \) receives a payoff of \( R - c \) if she invests, and receives a payoff of zero if she does not invest.

Find a weak perfect Bayesian equilibrium (WPBE) for this game, where the strategies and beliefs will depend on the parameter, \( c \). For this problem, it is enough to specify the strategy profile and player 2’s beliefs about player 1’s signal (if player 1 invests and if player 1 does not invest).

Hint: Since player 1 does not care about what player 2 does after him, you can solve for player 1’s strategy first. Then figure out player 2’s beliefs and strategy.

**Answer:** Player 1’s payoff does not depend on player 2’s action, so his sequentially rational strategy is quite simple. If he invests, his expected payoff is \( z_1 + E(z_2) - c \). Because of the uniform distribution of player 2’s signal, player 1 should invest if and only if \( z_1 > c - 1/2 \).

Now that we know player 1’s equilibrium strategy, we can find player 2’s beliefs and sequentially rational action as a function of \( z_2 \) and whether player

2
1 invests. Since all information sets are on the equilibrium path, we must use Bayes’ rule to determine beliefs. Because signals are independent, for any \( z_2 \), when player 1 invests \( z_1 \) is uniformly distributed on the interval \([c - 1/2, 1]\), and when player 1 does not invest \( z_1 \) is uniformly distributed on the interval \([0, c - 1/2]\). Therefore, player 2’s expected payoff from investing when player 1 invests is

\[
E(z_1 | \text{player 1 invests}) + z_2 - c = \frac{c - 1/2 + 1}{2} + z_2 - c = z_2 - \frac{c}{2} + \frac{1}{4}.
\]

Thus, if player 1 invests, player 2 invests if and only if

\[
z_2 > \frac{c}{2} - \frac{1}{4}.
\]

If player 1 does not invest, we have

\[
E(z_1 | \text{player 1 does not invest}) = \frac{c - 1/2}{2},
\]

so player 2 invests if and only if

\[
z_2 > \frac{c}{2} + \frac{1}{4}.
\]

3. (35 points)

Consider a market that is served by a monopolist who chooses both the quality of its product, \( \theta \), and the price per unit of output, \( p \), in terms of the numeraire good. The total cost of producing \( x \) units of quality \( \theta \) (in terms of the numeraire good) is given by \( \theta^2 x \).

Assume that there is a single consumer, with the quasi-linear utility function over consumption of the monopolist’s produced good, \( x \), and consumption of the numeraire good, \( M \), given by

\[
u(x, M, \theta) = (x - \frac{x^2}{2})\theta + M \quad \text{for } x \in [0, 1].
\]

Also, assume that the consumer has a zero endowment of good \( x \), and a large enough endowment of the numeraire good so that we do not have to worry about nonnegativity constraints on numeraire consumption.

(a) (20 points) Assume that the monopolist must set a single price, \( p \), of the good in terms of the numeraire, at which all transactions must occur. Find the monopoly price, \( p \), and quality, \( \theta \).
(b) (15 points) What is the socially optimal quantity of output, $x$, and quality, $\theta$, that would be chosen by a planner seeking to maximize total surplus?

**Answer:** I should have been explicit that the consumer observes the quality, but fortunately no one seemed confused about that. [If the consumer could not observe quality, then the firm should choose quality zero, and the consumer would demand zero output at any price.]

(a) Given quality, $\theta$, the solution to the consumer’s utility maximization problem involves equating the marginal rate of substitution to the price ratio, $p$. This condition determines the inverse demand,

$$\theta(1 - x) = p. \tag{3}$$

From (3), we can express the monopolist’s profit in terms of $x$ and $\theta$:

$$\pi^m = \theta(1 - x)x - \theta^2x.$$

Differentiating with respect to $x$, we have the first order condition

$$\theta(1 - 2x) - \theta^2 = 0, \text{ or } \theta = 1 - 2x.$$

(Clearly $\theta = 0$ cannot be optimal so we can divide by $\theta$.) Differentiating with respect to $\theta$, we have the first order condition

$$(1 - x)x - 2\theta x = 0, \text{ or } \theta = 1 - x \frac{2}{2}.$$

Simultaneously solving the two conditions and substituting into (3) to get the price, we have

$$x = \frac{1}{3}, \theta = \frac{1}{3}, p = \frac{2}{9}.$$

(b) Because utility is quasi-linear, we can equate social welfare to the total surplus of the produced good (benefit minus cost), given by

$$(x - \frac{x^2}{2})\theta - \theta^2x.$$

To find the socially optimal $x$ and $\theta$, solve the two first order conditions. Differentiating welfare with respect to $x$, we have the first order condition

$$(1 - x)\theta - \theta^2 = 0, \text{ or } \theta = 1 - x.$$

Differentiating welfare with respect to $\theta$, we have the first order condition

$$(x - \frac{x^2}{2}) - 2\theta x = 0, \text{ or } \theta = \frac{1}{2} - \frac{x}{4}.$$
Simultaneously solving the two conditions, we have

\[ x = \frac{2}{3}, \theta = \frac{1}{3}. \]

Thus, the monopolist chooses the socially optimal quality but produces less than the socially optimal quantity.