The Ohio State University
Department of Economics
Econ 808–Problem Set #2 Questions and Answers

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Levin and Peck

1. Consider the following economy with one physical commodity per state of nature and three consumers, each of whom seek to maximize expected utility. For i = 1,2, consumer i is risk averse, with utility of certain consumption given by $u_i(x_i) = \log(x_i)$. For i = 1,2, consumer i is endowed with 1 unit of consumption when she does not have an accident, 0 units of consumption when she has an accident.

Consumer 1 is a "low risk" consumer, with a probability of an accident equal to 1/3. Consumer 2 is a "high risk" consumer, with a probability of an accident equal to $\frac{1}{2}$. Consumer 1 having an accident and consumer 2 having an accident are independent events.

Consumer 3 is risk neutral, with utility of certain consumption given by $u_3(x_3) = x_3$, and has an endowment of 2 units of consumption in all states of nature. For parts (i) and (ii), assume that consumer 3 knows that consumer 1 is low risk and that consumer 2 is high risk, so information is symmetric.

(i) Define a competitive equilibrium for the economy with complete state-contingent commodity markets. Specify how many states of nature there are and the probability of each state.

(ii) Calculate the competitive equilibrium price vector and allocation for the economy with complete state-contingent commodity markets.

For parts (iii) and (iv), suppose that consumers 1 and 2 are in a Rothschild-Stiglitz world. That is, instead of consumer 3, there are many risk-neutral firms who cannot observe which consumer is low risk and which consumer is high risk. Firms compete by offering contracts, specifying consumption a policyholder receives when she has an accident and consumption she receives when she does not have an accident.

(iii) Find the pooling contract providing full insurance (consumption is independent of whether the policyholder has an accident) and yielding zero expected profits when both consumers accept the contract.

(iv) Show that the pooling contract of part (iii), call it α, is not an equilibrium. That is, find another contract, β, that would be chosen by the low risk consumer, would not be chosen by the high risk consumer, and yields expected profits for the firm offering the contract. Be as explicit as you can.

Answer:

(i) There are 4 states of nature, corresponding to the four possible outcomes: consumer 1 does not have an accident and consumer 2 does not have an accident, and so on. Call the four states gg, gb, bg, and bb, where the first letter (g for good, b for bad) describes whether consumer 1 had an accident and the second letter describes whether consumer 2 had an accident. The probabilities of the four states are: (1/3, 1/3, 1/6, 1/6).

A competitive equilibrium is a price vector, $(p^{gg}, p^{gb}, p^{bg}, p^{bb})$, and an allocation, $\{x_i^s\}_{i=1,2,3;s=gg,gb,bg,bb}$ such that

(1) x_1 solves:

$$\begin{aligned} & \max \frac{1}{3}\log(x_1^{gg}) + \frac{1}{3}\log(x_1^{gb}) + \frac{1}{6}\log(x_1^{bg}) + \frac{1}{6}\log(x_1^{bb}) \\ & \text{subject to} \\ & p^{gg}x_1^{gg} + p^{gb}x_1^{gb} + p^{bg}x_1^{bg} + p^{bb}x_1^{bb} \\ = \; & p^{gg} + p^{gb} \\ x_1 \geq \; & 0. \end{aligned}$$

(2) x_2 solves:

$$\begin{aligned} & \max \frac{1}{3}\log(x_2^{gg}) + \frac{1}{3}\log(x_2^{gb}) + \frac{1}{6}\log(x_2^{bg}) + \frac{1}{6}\log(x_2^{bb}) \\ & \text{subject to} \\ & p^{gg}x_2^{gg} + p^{gb}x_2^{gb} + p^{bg}x_2^{bg} + p^{bb}x_2^{bb} \\ = \; & p^{gg} + p^{bg} \\ x_2 \geq \; & 0. \end{aligned}$$

(3) x_3 solves:

$$\begin{aligned} & \max \frac{1}{3}(x_3^{gg}) + \frac{1}{3}(x_3^{gb}) + \frac{1}{6}(x_3^{bg}) + \frac{1}{6}(x_3^{bb}) \\ & \text{subject to} \\ & p^{gg}x_3^{gg} + p^{gb}x_3^{gb} + p^{bg}x_3^{bg} + p^{bb}x_3^{bb} \\ = \; & 2(p^{gg} + p^{gb} + p^{bg} + p^{bb}) \\ x_3 \geq \; & 0. \end{aligned}$$

(4) Market clearing:

$$\begin{aligned} x_1^{gg} + x_2^{gg} + x_3^{gg} &= 4 \\ x_1^{gb} + x_2^{gb} + x_3^{gb} &= 3 \\ x_1^{bg} + x_2^{bg} + x_3^{bg} &= 3 \\ x_1^{bb} + x_2^{bb} + x_3^{bb} &= 2. \end{aligned}$$

(ii) Solving for consumer 1's demand functions, pick a commodity, say bb, and there will be a marginal rate of substitution condition for each of the other three commodities in relation to bb:

$$\frac{\frac{1}{3}x_1^{bb}}{\frac{1}{6}x_1^{gg}} = \frac{p^{gg}}{p^{bb}}, \quad \frac{\frac{1}{3}x_1^{bb}}{\frac{1}{6}x_1^{gb}} = \frac{p^{gb}}{p^{bb}}, \quad \frac{x_1^{bb}}{x_1^{bg}} = \frac{p^{bg}}{p^{bb}}$$

Cross-multiplying, we have:

$$\begin{aligned} p^{gg}x_1^{gg} &= p^{gb}x_1^{gb} = 2p^{bb}x_1^{bb} \\ p^{bg}x_1^{bg} &= p^{bb}x_1^{bb} \end{aligned}$$

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Substituting the above equations into the budget constraint, we have:

$$6p^{bb}x_1^{bb} = p^{gg} + p^{gb}$$

This yields the demand functions:

$$\begin{aligned} x_1^{bb} &= \frac{p^{gg}+p^{gb}}{6p^{bb}}, x_1^{bg} = \frac{p^{gg}+p^{gb}}{6p^{bg}} \\ x_1^{gg} &= \frac{p^{gg}+p^{gb}}{3p^{gg}}, x_1^{gb} = \frac{p^{gg}+p^{gb}}{3p^{gb}}. \end{aligned}$$

Consumer 2 has the same marginal rate of substitution equations and a similar budget equation. Similar analysis yields the demand functions:

$$\begin{aligned} x_2^{bb} &= \frac{p^{gg}+p^{bg}}{6p^{bb}}, x_2^{bg} = \frac{p^{gg}+p^{bg}}{6p^{bg}} \\ x_2^{gg} &= \frac{p^{gg}+p^{bg}}{3p^{gg}}, x_2^{gb} = \frac{p^{gg}+p^{bg}}{3p^{gb}}. \end{aligned}$$

Consumer 3's marginal rate of substitution conditions yield:

$$\frac{p^{gg}}{p^{bb}} = 2, \frac{p^{gb}}{p^{bb}} = 2, \frac{p^{bg}}{p^{bb}} = 1.$$

Normalizing $p^{bb} = \frac{1}{6}$, we have p = (1/3, 1/3, 1/6, 1/6). Intuitively, because consumer 3 is risk neutral and we have an interior solution, prices are proportional to probabilities. Plugging these prices into the demand function for consumers 1 and 2, we get the consumption bundles

$x_1 = (2/3,\ 2/3,\ 2/3,\ 2/3)$ and $x_2 = (\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2})$. Not surprisingly, both consumers receive full and fair insurance. We have not yet used market clearing conditions, but we will do so to determine consumer 3's allocation: $x_3 = (17/6,$ $11/6,\ 11/6,\ 5/6)$.

(iii) Now suppose firms cannot tell who is low risk and who is high risk. To find the pooling contract yielding full insurance and zero expected profits, we find the point on the pooled fair odds line where $W_1 = W_2$. To find the equation for the fair odds line, notice that the probability that a consumer taken at random has an accident is $\frac{1}{2}(1/3) + \frac{1}{2}(\frac{1}{2}) = 5/12$. Then the slope of the fair odds line is - 7/5 and the equation can be written as $W_2 = (-\frac{7}{5})W_1 + \mathit{const}$. Since we know that (1,0) is on the line, the intercept, const., must equal 7/5, so the full equation is:

$$W_2 = (-\frac{7}{5})W_1 + \frac{7}{5}.$$

Setting $W_1 = W_2$ and solving, we have $W_1 = W_2 = 7/12$.

(iv) At the point (7/12, 7/12), the low risk consumer's MRS is 2 and the high risk consumer's MRS is 1. Therefore, the low risk's indifference curve is steeper than the pooled fair odds line and the high risk's indifference curve is flatter. Any contract on the pooled fair odds line below (7/12,7/12) will be accepted by the low risk consumer and not by the high risk consumer, and will therefore be profitable. For any sufficiently small and positive ε, (7/12 +ε , 7/12 - 7ε /5) works. Many other solutions are possible.

2. Consider the following version of Spence's signaling model. There are two types of workers, where type 1 workers have productivity A, and type 2 workers have productivity 2A. Type 1 workers make up a fraction q_1 of the population. There are two possible signals that workers can choose from, which are costly but do not enhance productivity.

A worker's cost of choosing y>0 units of signal Y is y for type 1 and y/2 for type 2. That is, we have $c_1(y) = y$ and $c_2(y) = y/2$. A worker's cost of choosing z>0 units of signal Z is z^2 (z squared) for type 1 and z for type 2. That is, we have $c_1(z) = z^2$ and $c_2(z) = z$.

As in the Spence model, firms are risk neutral and workers' utility is their wage minus their signal cost. The timing is that workers first choose their signal, followed by firms choosing a wage schedule.

(a) Assume that Y and Z are mutually exclusive activities, so that workers must choose either activity Y or activity Z, but not both. As a function of the nonnegative parameter, A, find the best separating equilibrium for the type 2 workers. Specify the full equilibrium, including "beliefs," as carefully as you can.

(b) Show whether or not there would be an even better separating equilibrium if workers could choose both activities. [where $c_1(y, z) = y + z^2$ and $c_2(y, z) = \frac{y}{2} + z$.]

Answer:

First, we will find the best separating equilibrium where type 2 workers choose y* units of activity Y, and then we will find the best separating equilibrium where type 2 workers choose z* units of activity Z. Type 1 workers will choose y=z=0.

Y-equilibrium: We have to find the value of y* where type 1 workers are indifferent between signaling and not. Firms will pay a wage of A if y < y* and a wage of 2A if y≥ y*, supported by the beliefs that workers with y < y* are of type 1 with probability 1, and that workers with y≥ y* are of type 2 with probability 1. Thus, for type 1's to be indifferent, we have: A = 2A - y*. Solving, we have y* = A, and type 2 workers receive utility of 2A - y*/2 = 3A/2.

Z-equilibrium: We have to find the value of z* where type 1 workers are indifferent between signaling and not. Firms will pay a wage of A if z < z* and a wage of 2A if z $\geq$z*, supported by the beliefs that workers with z < z* are of type 1 with probability 1, and that workers with z$\geq$ z* are of type 2 with probability 1. [For this to be an equilibrium, we must have z* > 1, or else the type 2's will have a higher signaling cost than the type 1's.] Thus, for type 1's to be indifferent, we have: A = 2A - $(z^*)^2$. Solving, we have z* = $\sqrt{A}$, and type 2 workers receive utility of 2A - z* = 2A $-\sqrt{A}$.

For A < 4, the Y-equilibrium yields type 2 workers higher utility and is therefore the best separating equilibrium, and for A > 4, the Z-equilibrium yields type 2 workers higher utility and is therefore the best separating equilibrium.

If workers could choose both activities, then we could look for an equilibrium in which type 1 workers choose y=z=0 and receive a wage of A, and in which type 2 workers choose (y**,z**) and receive a wage of 2A. Firms believe that if y $\geq$y** and z$\geq$ z**, the worker is of type 2 with probability 1, and that otherwise the worker is of type 1 with probability 1. Without loss of generality, we can restrict attention to (y**,z**) where type 1 workers are indifferent between signaling and not, so we have A = 2A - y** - $(z^{**})^2$, which simplifies to y** = A $-$ $(z^{**})^2$.

Type 2 workers then receive utility of 2A - y**/2 - z** , which equals: 2A - (A - $(z^{**})^2$)/2 - z**.

This function is convex in z**, because its derivative is z** - 1 and its second derivative is 1. This is a parabola with a minimum at z** = 1. The maximum is at one of the endpoints, either $(z^{**} = 0, y^{**} = A)$ or $(z^{**} = \sqrt{A}, y^{**} = 0)$. Thus, there cannot be a better separating equilibrium where workers can choose both activities.

3. In the Ackerlof lemons model with asymmetric information, suppose that the 3/2 in the utility function of type 2 consumers is replaced with 5/2. Solve for the equilibrium price, and determine whether the equilibrium is efficient.

Answer:

Now the utility advantage of transferring a car from a type 1 owner to a type 2 buyer is stronger than the adverse selection disadvantage. If the price is p, the average quality of cars on the market will be:

μ(p) = 1 if p > 2,

p/2 if p$\leq$ 2.

Therefore, if p > 2, (expected) marginal utility is 5/2, and if p$\leq$ 2, marginal utility is 5p/4. Notice that the marginal utility is greater than the price. For all prices less than 5/2, type 2 consumers will spend all their income on cars. The demand curve has a horizontal segment at 5/2, until all income is spent; then, the equation becomes D(p)=Y_2/p. The supply curve is given by S(p) = pN/2, reflecting the fact that N cars are sold at price 2, 0 cars sold at price 0, etc. The equilibrium falls into one of three cases.

Case 1: N > Y_2/2.

Consumers spend all their income, but cannot afford all the cars. $p = (2Y_2/N)^{1/2}$. There is some adverse selection, since the highest quality cars are kept off the market. Nevertheless, type 2 consumers are buying all the quality they can afford, and the equilibrium is efficient. (You can't give the high-quality owners more utililty without hurting the low-quality owners.)

Case 2: $2Y_2/5 < N < Y_2/2$.

Consumers spend all their income, and all cars are purchased. $p = Y_2/N$. Since all cars change hands, this is clearly efficient.

Case 3: $N < 2Y_2/5$.

All cars are purchased. The price is bid up to the willingness to pay of the type 2 consumers,

p = 5/2. Since all cars change hands, this is clearly efficient.