The Ohio State University
Department of Economics
Econ 808 Midterm Answers for Part 1

Profs Levin, Morelli, and Peck
Spring 2001

1. Consider an exchange economy with 2 consumers and one consumption good per state. Both consumers have the (Bernoulli) utility function over certain consumption,

$$u_i(x_i) = \log(x_i),$$

and are von Neumann-Morgenstern expected utility maximizers. Each consumer has an initial wealth of 2, which is his/her endowment in states where he/she does not have an accident. For those states in which a consumer is involved in an accident, he/she loses 1 unit of wealth, so his/her endowment is 1.

With probability $\frac{1}{2}$, there are no accidents. With probability $\frac{1}{2}$, consumer 1 is involved in an accident. (Thus, consumer 1 is involved in all accidents.) Conditional on consumer 1 having an accident, he crashes into a tree with probability $\frac{1}{2}$, and collides with consumer 2 with probability $\frac{1}{2}$. When both consumers are in an accident together, assume that neither can sue the other for damages.

(a) (14 points) Define a competitive equilibrium for this economy, where the two consumers participate on a complete state-contingent commodities market.
(b) (19 points) Calculate the competitive equilibrium price vector and allocation.

Answer:

(a) There are three states of nature: in state 1, there is no accident, and it occurs with probability $\frac{1}{2}$; in state 2, consumer 1 crashes into a tree, and it occurs with probability $\frac{1}{4}$; in state 3, both consumers are in the accident, and it occurs with probability $\frac{1}{4}$.

A *competitive equilibrium* is a price vector, (p^1, p^2, p^3), and an allocation, $(x_1^1, x_1^2, x_1^3, x_2^1, x_2^2, x_2^3)$, such that

(i) (x_1^1, x_1^2, x_1^3) solves

$$\max \frac{1}{2}\log(x_1^1) + \frac{1}{4}\log(x_1^2) + \frac{1}{4}\log(x_1^3)$$

subject to

$$p^1 x_1^1 + p^2 x_1^2 + p^3 x_1^3 \leq 2p^1 + p^2 + p^3,$$

(ii) (x_2^1, x_2^2, x_2^3) solves

$$\begin{aligned} & \max \frac{1}{2}\log(x_2^1) + \frac{1}{4}\log(x_2^2) + \frac{1}{4}\log(x_2^3) \\ & \text{subject to} \\ p^1 x_2^1 + p^2 x_2^2 + p^3 x_2^3 \leq \; & 2p^1 + 2p^2 + p^3, \end{aligned}$$

(iii) markets clear:

$$\begin{aligned} x_1^1 + x_2^1 &= 4 \\ x_1^2 + x_2^2 &= 3 \\ x_1^3 + x_2^3 &= 2. \end{aligned}$$

(b) Let us normalize $p^1 = 1$. To compute the competitive equilibrium, we first solve for consumer 1's demand function. The first order conditions are:

$$\frac{\frac{1}{2}x_1^2}{\frac{1}{4}x_1^1} = \frac{1}{p^2} \quad (1)$$

$$\frac{\frac{1}{2}x_1^3}{\frac{1}{4}x_1^1} = \frac{1}{p^3} \quad (2)$$

$$x_1^1 + p^2 x_1^2 + p^3 x_1^3 = 2 + p^2 + p^3. \quad (3)$$

Equation (1) can be rewritten as $x_1^2 = x_1^1/(2p^2)$, and equation (2) can be rewritten as $x_1^3 = x_1^1/(2p^3)$. Substituting into (3), we have

$$x_1^1 + \frac{x_1^1}{2} + \frac{x_1^1}{2} = 2 + p^2 + p^3.$$

Solving, we have

$$x_1^1 = \frac{2 + p^2 + p^3}{2},$$

which yields

$$\begin{aligned} x_1^2 &= \frac{2 + p^2 + p^3}{4p^2}, \\ x_1^3 &= \frac{2 + p^2 + p^3}{4p^3}. \end{aligned}$$

By the same procedure, we can solve for consumer 2's demand function. The first order conditions from the utility maximization problem are:

$$\begin{aligned} \frac{\frac{1}{2}x_2^2}{\frac{1}{4}x_2^1} &= \frac{1}{p^2} \\ \frac{\frac{1}{2}x_2^3}{\frac{1}{4}x_2^1} &= \frac{1}{p^3} \\ p^1 x_2^1 + p^2 x_2^2 + p^3 x_2^3 &= 2p^1 + 2p^2 + p^3, \end{aligned}$$

from which we calculate the demand function

$$\begin{aligned} x_2^1 &= \frac{2+2p^2+p^3}{2}, \\ x_2^2 &= \frac{2+2p^2+p^3}{4p^2}, \\ x_2^3 &= \frac{2+2p^2+p^3}{4p^3}. \end{aligned}$$

Since there are three goods and two independent prices to calculate, we must use two of the market clearing conditions. Market 1 clearing implies

$$\frac{2+p^2+p^3}{2} + \frac{2+2p^2+p^3}{2} = 4 \tag{4}$$

and market 2 clearing implies

$$\frac{2+p^2+p^3}{4p^2} + \frac{2+2p^2+p^3}{4p^2} = 3. \tag{5}$$

Equation (4) can be simplified to

$$3p^2 + 2p^3 = 4,$$

and equation (5) can be simplified to

$$4 = 9p^2 - 2p^3.$$

Solving, we have

$$p^2 = \frac{2}{3}, \quad \text{and} \quad p^3 = 1. \tag{6}$$

Plugging into the demand functions, the final allocation is:

$$x_1 = (\frac{11}{6}, \frac{11}{8}, \frac{11}{12}) \quad \text{and} \quad x_2 = (\frac{13}{6}, \frac{13}{8}, \frac{13}{12}). \tag{7}$$

2. In the following variant of Spence's signaling model, everything is as in the original game (with the workers receiving education, followed by firms offering wage schedules), except that education increases productivity. Letting $x_i(y)$ denote the productivity of type i with education y, we have

$$\begin{aligned} c_1(y) &= y \\ x_1(y) &= 1+\sqrt{y} \\ c_2(y) &= \frac{y}{2} \\ x_2(y) &= 2+\sqrt{y}. \end{aligned}$$

The fraction of type 1 workers in the population is q_1.

(a) (11 points) Assuming full information, where firms can directly identify a worker's type, what will be the equilibrium education chosen and wage received by each type?

(b) (11 points) Assuming as Spence does that a worker's type is her private information, is the full-information first-best allocation from part (a) possible as an equilibrium outcome? Explain.

(c) (11 points) Again assuming that a worker's type is her private information, calculate the education chosen and wage received in the best separating equilibrium.

Answer:

(a) With full information, competition by firms ensures that each worker is paid her productivity. Thus, a worker of type 1 chooses y_1 to solve

$$\max 1 + \sqrt{y} - y.$$

Setting the derivative equal to 0 and solving, we have

$$y_1 = \frac{1}{4}, \quad w(\frac{1}{4}) = \frac{3}{2}, \quad \text{utility} = \frac{5}{4}. \tag{8}$$

A worker of type 2 chooses y_2 to solve

$$\max 2 + \sqrt{y} - \frac{y}{2}.$$

Setting the derivative equal to 0 and solving, we have

$$y_2 = 1, \quad w(1) = 3, \quad \text{utility} = \frac{5}{2} \tag{9}$$

(b) With private information, we must see if either type would prefer to choose the education meant for the other type, and receive the corresponding wage. If a type 1 worker chooses education $y = 1$, and receives a wage of 3, her utility would be $3 - y$ or 2. The utility in part (a) is only $\frac{5}{4}$, so the first best is not incentive compatible. This deviation breaks the first-best as a possible equilibrium.

(c) In all separating equilibria, the type 1 workers are treated as in the first-best, since there is no reason to reduce their utility to prevent the more productive from claiming to be less productive. Thus, we have

$$y_1 = \frac{1}{4}, \quad w(\frac{1}{4}) = \frac{3}{2}, \quad \text{utility} = \frac{5}{4}. \tag{10}$$

The high productivity type must overinvest in education until type 1 consumers do not want to imitate. The best separating equilibrium makes the type 1 consumers indifferent between choosing $y_1 = \frac{1}{4}$ and choosing education y^*. Thus, y^* solves

$$2 + \sqrt{y^*} - y^* = \frac{5}{4}.$$

Solving, we have a quadratic, with roots $\frac{1}{4}$ and $\frac{9}{4}$. Clearly, the correct root is greater than one, so we have

$$y_2 = y^* = \frac{9}{4}, \quad w(\frac{9}{4}) = \frac{7}{2}, \quad \text{utility} = \frac{7}{2} - \frac{9}{8} = \frac{19}{8}. \tag{11}$$