Department of Economics
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Midterm Questions and Answers–Econ 808

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1. (30 points)

In the village of Debreuvia, there are 150 consumers and one physical commodity per state of nature. For i = 1, ... , 150, consumer i is a von Neumann-Morgenstern expected utility maximizer with (Bernoulli) utility of certain consumption given by $u_i(x_i) = \log(x_i)$.

For i = 1, ... , 100, consumer i lives in the highlands, and has an endowment of 1 is all states of nature.

For i = 101, ... , 150, consumer i lives in the flood plain. When a flood occurs, all consumers living in the flood plain receive an endowment of 0. The probability of a flood is $\frac{1}{10}$. When a flood does not occur (probability $\frac{9}{10}$), all consumers living in the flood plain receive an endowment of 1.

(a) (10 points) If the entire village of Debreuvia trades state-contingent commodities, define a competitive equilibrium for this economy.

(b) (20 points) Calculate the competitive equilibrium price vector and allocation.

Answer:

(a) There are two states of nature, corresponding to a flood (state 1) and no flood (state 2). A competitive equilibrium is a price vector, (p^1, p^2), and an allocation, $(x_i^1, x_i^2)_{i=1}^{150}$, such that

(i) for i = 1, ... , 100, (x_i^1, x_i^2) solves

$$\begin{aligned} & \max\left[\frac{1}{10}\log(x_i^1) + \frac{9}{10}\log(x_i^2)\right] \\ & \text{subject to} \\ p^1 x_i^1 + p^2 x_i^2 \quad \leq \quad & p^1 + p^2, \end{aligned}$$

(ii) for i = 101, ... , 150, (x_i^1, x_i^2) solves

$$\begin{aligned} & \max\left[\frac{1}{10}\log(x_i^1) + \frac{9}{10}\log(x_i^2)\right] \\ & \text{subject to} \\ p^1 x_i^1 + p^2 x_i^2 \quad \leq \quad & p^2, \end{aligned}$$

(iii) markets clear:

$$\begin{aligned} \sum_{i=1}^{150} x_i^1 \quad & \leq \quad 100 \\ \sum_{i=1}^{150} x_i^2 \quad & \leq \quad 150. \end{aligned}$$

(b) Normalize the price of consumption in state 2 (no flood) to be 1, $p^2 = 1$. First, we calculate the demand function for those living in the highlands (i=1, ... , 100). The first order conditions are

$$\begin{aligned} \frac{x_i^2}{9x_i^1} &= p^1 \\ p^1 x_i^1 + x_i^2 &= p^1 + 1, \end{aligned}$$

which we solve for the demand functions

$$\begin{aligned} x_i^1 &= \frac{p^1 + 1}{10p^1} \\ x_i^2 &= \frac{9(p^1 + 1)}{10}. \end{aligned}$$

Next, we calculate the demand function for those living in the flood plain (i=101, ... , 150). The first order conditions are

$$\begin{aligned} \frac{x_i^2}{9x_i^1} &= p^1 \\ p^1 x_i^1 + x_i^2 &= 1, \end{aligned}$$

which we solve for the demand functions

$$\begin{aligned} x_i^1 &= \frac{1}{10p^1} \\ x_i^2 &= \frac{9}{10}. \end{aligned}$$

Market clearing for consumption in state 2 (no flood) implies

$$100(\frac{9(p^1 + 1)}{10}) + 50(\frac{9}{10}) = 150.$$

Solving for p^1, we have $p^1 = \frac{1}{6}$. The C.E. allocation is given by

$$\begin{aligned} (x_i^1, x_i^2) &= (\frac{7}{10}, \frac{21}{20}) \quad \text{for i = 1, ... , 100,} \\ (x_i^1, x_i^2) &= (\frac{3}{5}, \frac{9}{10}) \quad \text{for i = 101, ... , 150.} \end{aligned}$$

2. (30 points)

In the following "lemons" model, there are N consumers of type 1, each endowed with 1 car and no money, and there are N consumers of type 2, each endowed with no car and a large amount of money.

Each type 1 consumer has the utility function,

$$u_1 = M + \sum_{i=1}^{N} x_i,$$

where M represents numeraire (money) consumption and x_i is the quality of car i if that car is consumed by this consumer. That is, the utility per unit of quality consumed is 1. Each type 2 consumer has the utility function,

$$u_2 = M + \sum_{i=1}^{N} \frac{3}{2} x_i,$$

so the utility per unit of quality consumed is $\frac{3}{2}$.

Owners know the true quality of their car, but everyone else treats the quality as random. Car qualities are uniformly distributed over the interval, $[1, 5]$. That is, $x_i \sim U[1, 5]$.

Note: This is exactly Akerlof's model, except that the range of qualities is [1,5] rather than [0,2].

(a) What will be the average quality of cars supplied to the used car market, as a function of the price, p?

(b) Calculate the market supply function, which depends on p and N.

(c) Calculate the equilibrium price and quantity supplied.

Answer:

(a) If the price is p, all cars with quality at or below p will be supplied. Therefore, for prices between 1 and 5, the qualities of cars on the market are uniformly distributed over $[1, p]$. The average quality of cars supplied is given by

$$\begin{aligned} \mu(p) &= \frac{p+1}{2} \quad \text{if } 1 \leq p \leq 5 \\ \mu(p) &= 3 \quad \text{if } 5 < p. \end{aligned}$$

For $p < 1$, no cars will be supplied, so the average quality is indeterminate.

(b) To calculate the quantity of cars supplied, we multiply the number of cars with the probability of a car being supplied. Since all cars with quality between 1 and p are supplied, the probability with which a car is supplied is given by

$$pr.(\text{car is supplied}) = \frac{p-1}{5-1} = \frac{p-1}{4}.$$

Therefore, the supply function is given by

$$\begin{aligned} S(p,N) &= 0 \quad \text{for} \;\; p < 1 \\ S(p,N) &= \frac{p-1}{4}N \quad \text{for } 1 \le p \le 5 \\ S(p,N) &= N \quad \text{for } 5 < p. \end{aligned}$$

(c) For prices between 1 and 5, the expected quality is $\frac{p+1}{2}$, so the incremental utility of a car is $\frac{3}{2}\frac{p+1}{2}$, which equals $(\frac{3}{4}p + \frac{3}{4})$. This is a situation in which the market partially unravels, but not completely. If $(\frac{3}{4}p + \frac{3}{4}) > p$, then type 2 consumers will want to spend all of their income on cars, leading to excess demand (since their money endowment is large). If $(\frac{3}{4}p + \frac{3}{4}) < p$, then type 2 consumers will not want to buy cars, because the quality is too low. The market will clear when $(\frac{3}{4}p + \frac{3}{4}) = p$ holds, or $p = 3$. At this price, the quantity supplied is $\frac{N}{2}$.

3. (40 points)

Consider the following principal-agent problem, with two possible effort choices, e_H and e_L. The project is either a failure, in which case profits (not including wage payments) are zero, $\pi = 0$, or the project is a success, in which case profits (not including wage payments) are $\widehat{\pi}$, $\pi = \widehat{\pi}$. If high effort is chosen, the project succeeds with probability $\frac{2}{3}$ and fails with probability $\frac{1}{3}$. If low effort is chosen, the project succeeds with probability $\frac{1}{2}$ and fails with probability $\frac{1}{2}$.

The principal cannot observe the agent's effort, so a contract must specify the wage paid to the agent when the project succeeds, w_1, and the wage paid to the agent when the project fails, w_0. The principal is risk neutral, seeking to maximize the expected value of $\pi - w$.

The agent seeks to maximize expected utility, with a (Bernoulli) utility function over certain consumption given by

$$u(w,e) = \log(w) - g(e).$$

Assume that the agent has a reservation utility of zero, $\overline{u} = 0$, and that $g(e_L) = 0$ and $g(e_H) = \log(2)$.

(a) Calculate the optimal contract, (w_0, w_1), in which the agent chooses low effort.

(b) Specify the optimization problem which can be solved for the optimal contract, (w_0, w_1), in which the agent chooses high effort. Label the relevant constraints (IR, IC, etc.).

(c) Calculate the optimal contract, (w_0, w_1), in which the agent chooses high effort.

(d) What is the cutoff value for $\widehat{\pi}$ above which the principal is better off implementing high effort?

Answer:

(a) Since low effort is all that is required, incentive compatibility is automatically satisfied, and we need to maximize expected profits subject to the individual rationality constraint. One could work out the first order conditions, but we showed in class that the solution always fully insures the worker against wage fluctuations, since the principal is risk neutral and the agent is risk averse. Thus we have $w_0 = w_1 = w$, which is determined to give the agent a utility of zero.

$$\log(w) - g(e_L) = 0$$

implies $\log(w) = 0$. Taking the exponential of both sides, we have $w = 1$.

(b) To implement high effort, we must satisfy individual rationality and incentive compatibility. That is, the agent must weakly prefer high effort to low effort. The optimization problem is the following.

$$\begin{aligned} & \max\left[\frac{2}{3}\widehat{\pi} - \frac{1}{3}w_0 - \frac{2}{3}w_1\right] \\ & \text{subject to} \\ \frac{1}{3}\log(w_0) + \frac{2}{3}\log(w_1) \quad \geq \quad & \log(2) \quad \text{[IR]} \\ \frac{1}{3}\log(w_0) + \frac{2}{3}\log(w_1) - \log(2) \quad \geq \quad & \frac{1}{2}\log(w_0) + \frac{1}{2}\log(w_1) \quad \text{[IC]} \end{aligned}$$

Notice that if the agent considers low effort, he lowers the probability of success but does not change his wage profile, because the principal cannot observe that he is shirking.

(c) [IR] must bind, or else the principal can lower both wages and have the agent still accept the contract. [IC] must bind as well, because otherwise the principal can provide the agent with more insurance (higher w_0 and lower w_1) and still induce high effort. This would allow the principal to pay a lower expected wage. Since [IR] and [IC] both hold with equality, the binding constraints determine the optimal contract by themselves! In other words, solve [IR] and [IC] for w_0 and w_1. From [IR], we have

$$\begin{aligned} \log(w_0) + 2\log(w_1) &= 3\log(2)\text{ , or} \\ w_0(w_1)^2 &= 2^3 = 8. \qquad (1) \end{aligned}$$

Since [IR] implies that the left side of [IC] is zero, [IC] can be simplified to

$$\begin{aligned} \frac{1}{2}\log(w_0) + \frac{1}{2}\log(w_1) &= 0\text{, or} \\ w_0 w_1 &= 1. \qquad (2) \end{aligned}$$

Simultaneously solving (1) and (2), we have our answer:

$$w_0 = \frac{1}{8}, \quad w_1 = 8.$$

You did not have to do this, but let us also take the first order conditions with respect to w_0 and w_1, if only to determine the multipliers γ and μ. The first order condition with respect to w_0 is

$$-\frac{1}{3} + \gamma\frac{1}{3w_0} + \mu[\frac{1}{3w_0} - \frac{1}{2w_0}] = 0.$$

Simplifying, we have

$$w_0 = \gamma - \frac{\mu}{2}. \tag{3}$$

Equation (3) shows that the wage must be lowered when the project fails, to help provide the incentive for high effort. The first order condition with respect to w_1 is

$$-\frac{2}{3} + \gamma\frac{2}{3w_1} + \mu[\frac{2}{3w_1} - \frac{1}{2w_1}] = 0.$$

Simplifying, we have

$$w_1 = \gamma + \frac{\mu}{4}. \tag{4}$$

Equation (4) shows that the wage must be increased when the project succeeds, to help provide the incentive for high effort. Simultaneously solving equations 1-4, we have

$$\begin{aligned} w_0 &= \frac{1}{8}, \quad w_1 = 8, \\ \gamma &= \frac{43}{8}, \quad \mu = \frac{21}{2}. \end{aligned}$$

(d) With low effort, the principal's payoff is

$$\frac{\widehat{\pi}}{2} - 1$$

With high effort, the principal's payoff is

$$\frac{2\widehat{\pi}}{3} - (\frac{1}{3} \cdot \frac{1}{8}) - (\frac{2}{3} \cdot 8)$$

Equating these two expressions and solving for $\widehat{\pi}$, we have $\widehat{\pi} = \frac{105}{4}$.