1 General Equilibrium

1.1 Pure Exchange Economy

K goods, n consumers

agent i: preferences $\succsim_i$ or utility $u_i : \mathbb{R}_+^K \to \mathbb{R}$

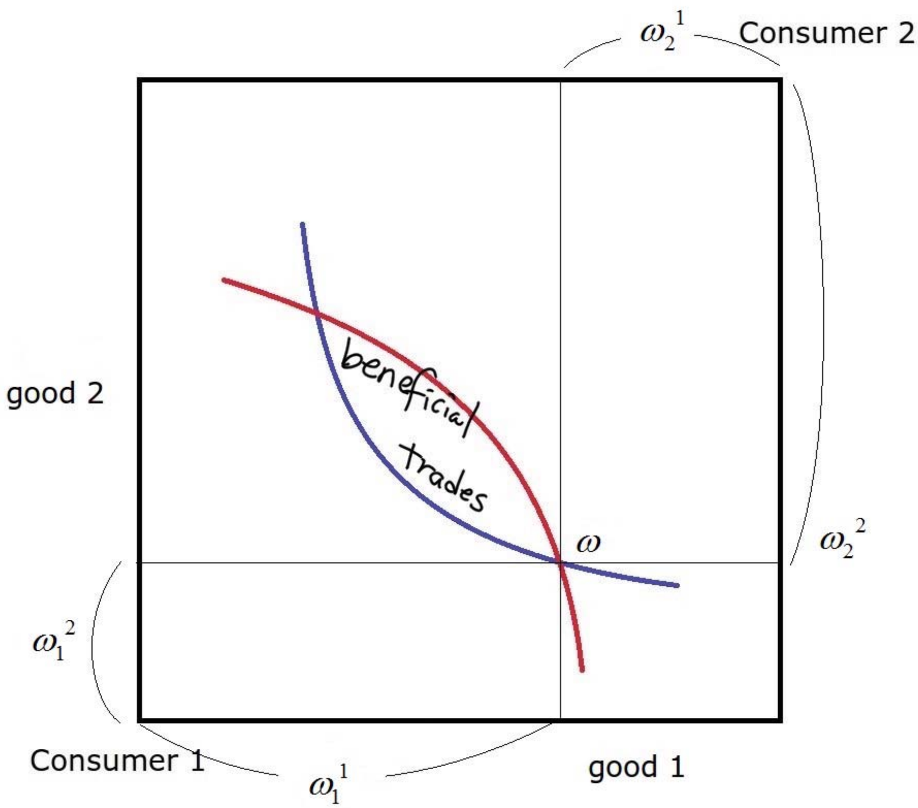
initial endowments, $\omega_i \in \mathbb{R}_+^K$

consumption bundle, $x_i = (x_i^1, \cdots, x_i^K) \in \mathbb{R}_+^K$

Definition 1 *An allocation,* $x = (x_1, \cdots, x_i, \cdots, x_n) \in \mathbb{R}_+^{nK}$ *is* <u>*feasible*</u> *if*

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \omega_i$$

An **Edgeworth Box** describes all of the feasible, non-wasteful allocations (for $n = 2$)



Prices, $p = (p^1, \cdots, p^K)$

Consumer Optimization

$$\max_{x_i} u_i(x_i) \qquad \text{s.t.} \qquad p \cdot x_i \leq p \cdot \omega_i$$

$$x_i \geq 0$$

If u_i is continuous, the UMP has a solution for all $p \in \mathbb{R}^K_{++}$

If u_i is strictly quasi-concave, for all $p \in \mathbb{R}^K_{++}$ the optimization problem gives rise to a demand *function*, $x_i(p, p \cdot \omega_i)$

Assumptions

(1) Each u_i is continuous, strictly quasi-concave (upper contour sets are convex sets) and satisfies local nonsatiation (for any bundle, there is a nearby bundle yielding higher utility)

(1′) Strict monotonicity replaces local non-satiation.

(2) $\omega_i \gg 0$

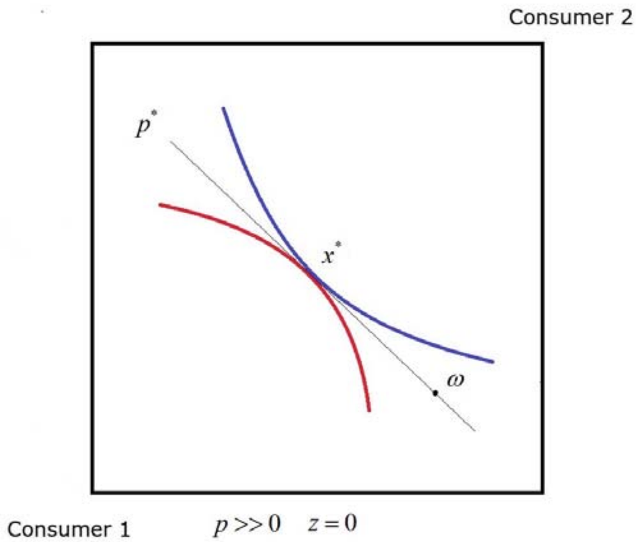
(3) free disposal (not needed if each u_i is monotone, rules out negative prices)

Definition 2 *A competitive equilibrium is a pair* (p^*, x^*) *such that*

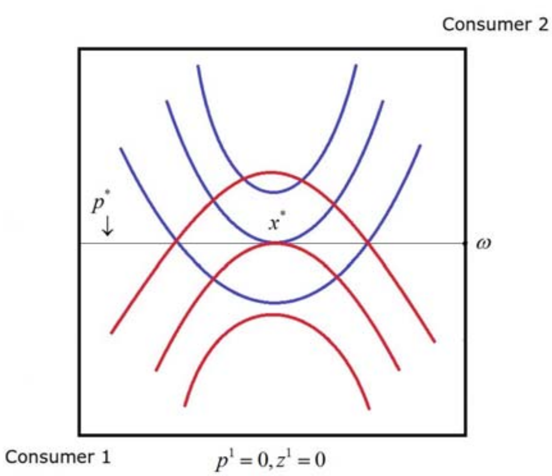
1. given p^*, each x_i^* solves the consumer optimization problem

2. markets clear: $\sum_{i=1}^{n} x_i^* \leq \sum_{i=1}^{n} \omega_i \qquad (K \text{ equations})$

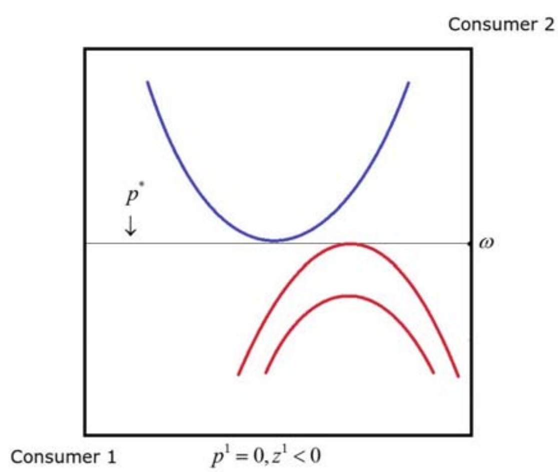
Alternatively, we can combine the two conditions using the notation $x_i(p, p \cdot \omega_i)$ for the utility maximizing demand function. A CE is a pair, (p^*, x^*), such that $\sum_{i=1}^{n} x_i(p^*, p^* \cdot \omega_i) \leq \sum_{i=1}^{n} \omega_i$ holds, where x^* is determined by $x_i^* = x_i(p^*, p^* \cdot \omega_i)$.



$p >> 0 \quad z = 0$



$p^1 = 0, z^1 = 0$



$p^1 = 0, z^1 < 0$

Fact: $x_i(p, p{\cdot}\omega_i)$ is homogeneous of degree zero in prices. (It requires assumption (1) to ensure $x_i(p, p \cdot \omega_i)$ is a function.)

We can, without loss of generality, normalize prices: $p^1 = 1$ (as long as the good is not a free good) or $p \in S^{K-1}$.

Definition 3 *The price simplex, S^{K-1}, is*

$$\{p \in R^K_+ \mid \sum_{j=1}^{K} p^j = 1\}.$$

<u>Walras' Law</u> Assuming local nonsatiation, for any p in S^{K-1}, we have $p \cdot z(p) = 0$, where $z(p)$ is excess demand,

$$z(p) = \sum_{i=1}^{n} [x_i(p, p \cdot \omega_i) - \omega_i].$$

<u>Proof of Walras' Law</u>

$$p \cdot z(p) = p \cdot [\sum_{i=1}^{n} x_i(p, p \cdot \omega_i) - \sum_{i=1}^{n} \omega_i] =$$
$$\sum_{i=1}^{n} [p \cdot x_i(p, p \cdot \omega_i) - p \cdot \omega_i]$$

Since each u_i satisfies local nonsatiation, the budget constraint is satisfied with equality.

$$\therefore \qquad \forall i, \qquad p \cdot x_i(p, p \cdot \omega_i) - p \cdot \omega_i \qquad \blacksquare$$

Note: If utility is not continuous or if some good has a zero price, then excess demand might not be well defined. If utility is not strictly quasi-concave, then excess demand might be a correspondence, in which can all selections from the correspondence satisfy $p \cdot z(p) = 0$.

Fact: (Free Goods) Assuming (1) and (3), in a competitive equilibrium, either $z^j(p^*) = 0$ or [if $z^j(p^*) < 0$] $p^{j*} = 0$.

Proof

By free disposal, each $p^{j*} \geq 0$. In any C.E. $z^j(p^*) \leq 0$. By Walras' Law,

$$0 = p^* \cdot z(p^*) = \sum_{j=1}^{K} p^{j*} z^j(p^*)$$

Suppose it is possible to have $z^j(p^*) < 0$ and $p^{j*} > 0$ for some commodity j. Then the j^{th} term in the above sum is negative. All other terms are non-positive, a contradiction.

Therefore, either $z^j(p^*) = 0$ or $p^{j*} = 0$ for all j. ■

Equilibrium Conditions: Counting Equations and Unknowns

Solve each consumer's UMP for demand functions, and (if utility is strictly monotonic), we have:

K market clearing equations (where supply equals demand)

1 equation is redundant, by Walras' law. If $D = S$ in $K-1$ markets, the Kth term in $\sum_{j=1}^{K} p^{j*} z^j(p^*)$ is also zero.

K components of the price vector.

1 of the prices can be normalized, so there are $K-1$ unknowns to solve for.

1.2 Existence of C.E.

Price normalization: replace $\widehat{p}$ with p given by

$$p^j = \frac{\widehat{p}^j}{\sum\limits_{j'=1}^{K} \widehat{p}^{j'}}$$

This puts prices in the simplex, S^{K-1}, where $S^{K-1} = \{p \in \mathbb{R}_+^K \mid \sum\limits_{j=1}^{K} p^j = 1\}$.

The key Mathematical result:

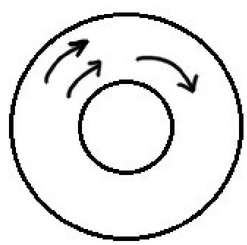
1.3 Brouwer's Fixed Point Theorem

If S is a convex, compact set and f is a continuous function from S to itself, $f : S \longrightarrow S$, then f has a fixed point. That is, there exists $x \in S$ such that $x = f(x)$.

Note:

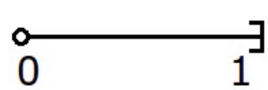
1. the simplex, S^{K-1}, is a convex, compact set. (Reason why we normalized prices so that they were bounded.)

2. If S is not convex:



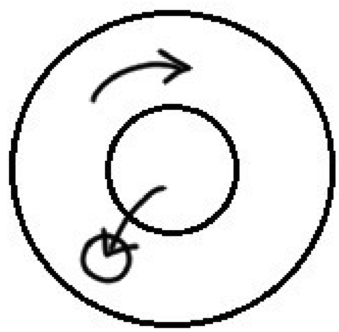
rotate by 90° - no fixed point. ex. $S = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $f(x) = 1 - x$

3. If S is not closed: $f(x) = \frac{x}{2}$ and $x \in (0, 1] = S$



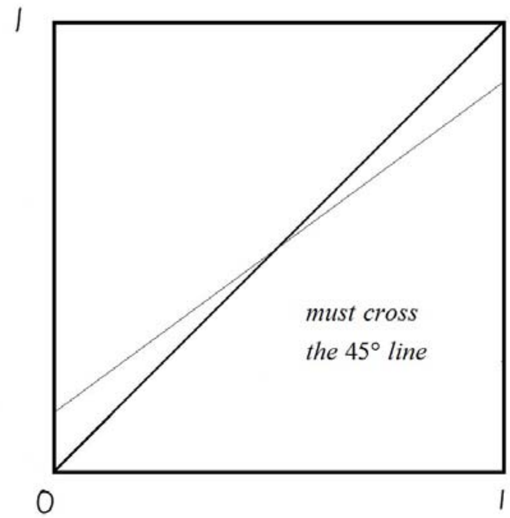
4. If S is not bounded: $f(x) = x + 1$ and $S = \mathbb{R}$

5. f is not continuous



S : unit circle. outside of donut rotates by 90°, donut hole is mapped to outside the hole.

Proof of Brouwer's Thm is very difficult, and the 2-good one dimensional case in Varian is misleadingly simple.



1.4 Existence of C.E.

Under assumptions $(1') - (2)$, there exists a C.E. (Argument assumes $z(p)$ exists and is continuous, so we will have to patch the argument to deal with prices that can be zero.)

<u>Proof</u>

Let $p \in S^{K-1}$ and consider the map, $p \rightarrow \tilde{p}$,

$$\tilde{p}^j = \frac{p^j + \max(0, z^j(p))}{1 + \sum_{j'=1}^{K} \max(0, z^{j'}(p))},$$

note: it will not work to use the simpler function, $p^j + z^j(p)$.

Since we have $\tilde{p}^j \geq 0$ and $\sum_{j=1}^{K} \tilde{p}^j = 1$, then $\tilde{p} \in S^{K-1}$. The function $p \rightarrow \tilde{p}$ is continuous since composition of continuous functions are continuous.

Apply Brouwer's theorem: there exists p^* s.t.

$$p^{j*} = \frac{p^{j*} + \mathsf{max}(0, z^j(p^*))}{1 + \sum_{j'=1}^{K} \mathsf{max}(0, z^{j'}(p^*))} \text{ for } j = 1, \cdots, K.$$

Cross multiply: for all j,

$$p^{j*} \sum_{j'=1}^{K} \mathsf{max}(0, z^{j'}(p^*)) = \mathsf{max}(0, z^j(p^*))$$

multiply by $z^j(p^*)$

$$z^j(p^*) p^{j*} \sum_{j'=1}^{K} \mathsf{max}(0, z^{j'}(p^*)) = z^j(p^*) \mathsf{max}(0, z^j(p^*))$$

sum over j

$$[\sum_{j=1}^{K} p^{j*} z^j(p^*)][\sum_{j'=1}^{K} \mathsf{max}(0, z^{j'}(p^*))] = [\sum_{j=1}^{K} z^j(p^*) \mathsf{max}(0, z^j(p^*))]$$

from Walras' Law, $LHS = 0$.

Each term of RHS sum is either 0 or $[z^j(p^*)]^2$, but for the sum to be 0, each term is therefore 0, $\mathsf{max}(0, z^j(p^*)) = 0$ for $\forall j$

$\therefore \quad z^j(p^*) \leq 0$ for all j, which implies p^* is a C.E.

$\blacksquare$

This proof (from Varian) assumes that $z(p)$ is a continuous function, but this assumption generally does not hold at the boundary of the simplex.

- demand can be infinite

- we can't forget about zero prices, because Brouwers' theorem requires S to be closed

1.4.1 Existence "Patch"

Let $x_i^b(p)$ be the solution to the *bounded* utility max problem:

$$\max_{x_i} u_i(x_i) \qquad \text{s.t.} \qquad p \cdot x_i \leq p \cdot \omega_i$$

$$x_i \geq 0, \;\; x_i^j \leq \sum_{h=1}^{n} \omega_h^j + 1 \text{ for } j = 1, \cdots, K$$

<u>Note</u>: $x_i^b(p)$ is a continuous function, even at some zero prices.

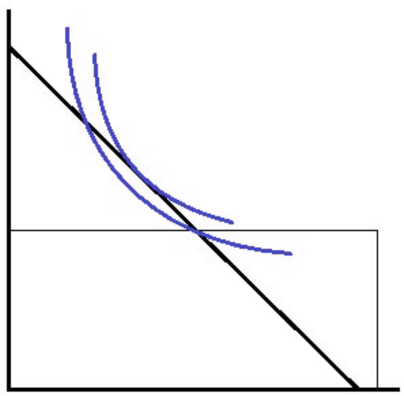
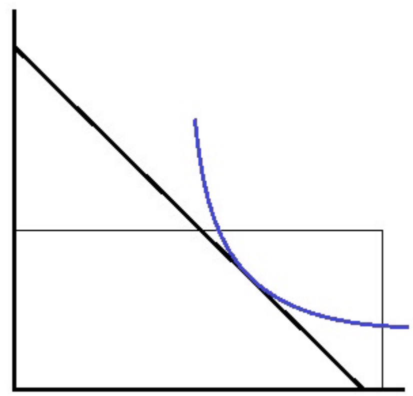
Claim: If $(x_i^j)^b(p) < \sum_{h=1}^{n} \omega_h^j + 1$ for $\forall j$ (so artificial bound is not binding), then the bounded demand equals the demand without the artificial constraint,

$$x_i^b(p) = x_i(p)$$

Proof of Claim

By strict quasi-concavity, a local optimum in the (unbounded) utility max problem is a global optimum in that problem.

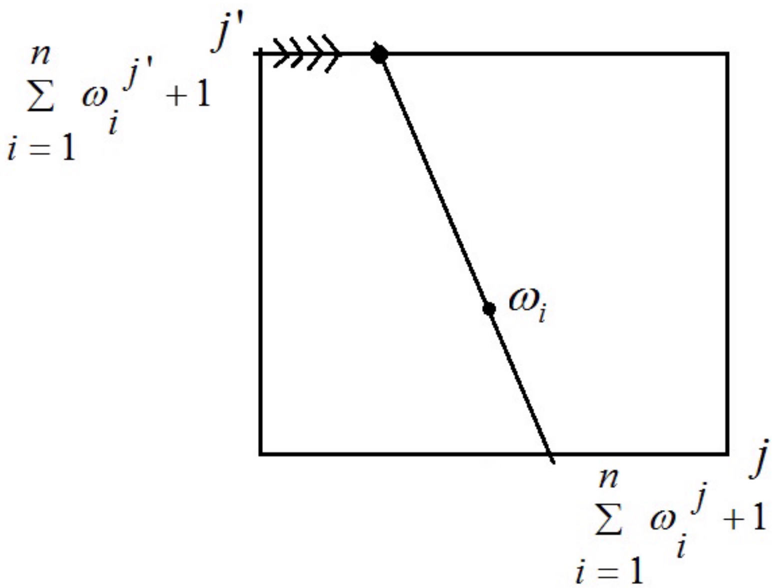
$x_i^b(p)$ is a local optimum in the unbounded problem, because the extra constraint is slack in the bounded problem, so the set of local bundles satisfying the constraints is the same in both problems. It therefore is a global optimum in the unconstrained problem.

Constraint binds   Constraint does not bind

Does $z^b(p)$ satisfy Walras' Law?

Strict monotonicity implies $p \cdot z^b(p) = 0$.



replace z^b for z in the mapping $\widetilde{p}$.

Since $z^b(p)$ is continuous for all prices in the simplex, we can use it instead of $z(p)$ in the mapping to conclude there is a fixed point. Since $z^b(p)$ satisfies Walras' Law, we can go through the previous algebra to conclude that the fixed point satisfies $z^{j,b}(p^*) \leq 0$ for all j.

Since aggregate bounded excess demand is non-positive, this implies the strict inequality, $\sum_h x_h^{j,b}(p^*) < \sum_h \omega_h^j + 1$. Therefore, we have $(x_i^j)^b(p^*) < \sum_{h=1}^{n} \omega_h^j + 1$ for $\forall j$. From our claim, $x_i^b(p^*) = x_i(p^*)$ holds, so $z(p^*) = z^b(p^*)$.

Therefore, p^* is a C.E. price.

1.5 Pareto Optimality (efficiency)

Definition 4 *A feasible allocation, x, is weakly P.O. if there is no feasible allocation, x' such that all agents strictly prefer x' to x: $u_h(x'_h) > u_h(x_h)$ for $\forall h$.*

Definition 5 *A feasible allocation, x, is strongly P.O. if there is no feasible allocation, x' such that $u_h(x'_h) \geq u_h(x_h)$ for all h and $u_i(x'_i) > u_i(x_i)$ for some i.*

$$x\text{: strongly P.O.} \Longrightarrow x\text{: weakly P.O.}$$

If you can't make one person strictly better off, you can't make everyone strictly better off.

Theorem 6 *If assumption* $(1')$ *holds (strict monotonicity), then* x *is weakly P.O.* $\Longrightarrow$ x *is strongly P.O.*

<u>Proof</u>

Suppose x is not strongly P.O., but is weakly P.O.

$$\exists\ x' \text{ s.t. } u_h(x'_h) \geq u_h(x_h) \text{ for all } h \text{ and}$$
$$u_i(x'_i) > u_i(x_i) \text{ for some } i.$$

x'', defined below, is a feasible allocation:

$$x_i'' = \theta x_i'$$

$$x_h'' = x_h' + \frac{1-\theta}{n-1} x_i' \text{ for } h \neq i$$

By continuity, i is still strictly better off for θ close enough to 1.

By monotonicity, all other consumers are strictly better off.

x is not weakly P.O., a contradiction.

$\therefore$ x is strongly P.O. $\blacksquare$

1.6 First Fundamental Theorem of Welfare Economics

FFTWE - Assume that all consumers satisfy local nonsatiation. Let (p^*, x^*) be a C.E. Then, x^* is strongly P.O.

<u>Proof</u>

Suppose x^* is not strongly P.O. There exists a dominating allocation x such that

$$u_i(x_i) \geq u_i(x_i^*) \text{ for all } i$$

$$u_h(x_h) > u_h(x_h^*) \text{ for some } h.$$

From local nonsatiation, $p^* \cdot x_i \geq p^* \cdot \omega_i$ for all i. (Otherwise, you can afford an open neighborhood around x_i, so there is some x_i' such that $p^* \cdot x_i' \leq p^* \cdot \omega_i$ and $u_i(x_i') > u_i(x_i) \geq u_i(x_i^*)$ contradicts the fact that x_i^* is demanded.)

$$p^* \cdot x_h > p^* \cdot \omega_h$$

Otherwise, x_h is in the budget set and strictly preferred to x_h^*.

Sum these inequalities over $i = 1, \cdots, n$,

$$p^* \cdot \sum_{i=1}^{n} (x_h - \omega_h) > 0$$

Therefore, for some commodity j, we have

$$(p^j) \sum_{i=1}^{n} (x_i^j - \omega_i^j) > 0.$$

$\sum_{i=1}^{n} x_i^j > \sum_{i=1}^{n} \omega_i^j$, so x is not feasible. It contradicts x is a dominating allocation (feasible)

$$\therefore \quad x^* \text{ is strongly P.O.} \quad \blacksquare$$

Remark 7 *The argument does not need quasi-concavity or continuity. Without these assumptions, we cannot guarantee existence, but if a CE exists, it is PO.*

Remark 8 *Implicitly assumes that* n *and* K *are finite, or at least that the value of aggregate resources is finite. If* n *and* K *are infinite, we might not be able to conclude that aggregate consumption exceeds endowment for some* j*. (The overlapping generations model has a CE that is not PO.)*

SFTWE - Under assumptions $(1')$, every P.O. allocation $x^* \gg 0$ is a C.E. for the economy with endowments $\omega_i = x_i^*$. (no need for free disposal if monotonicity holds)

<u>Proof</u>

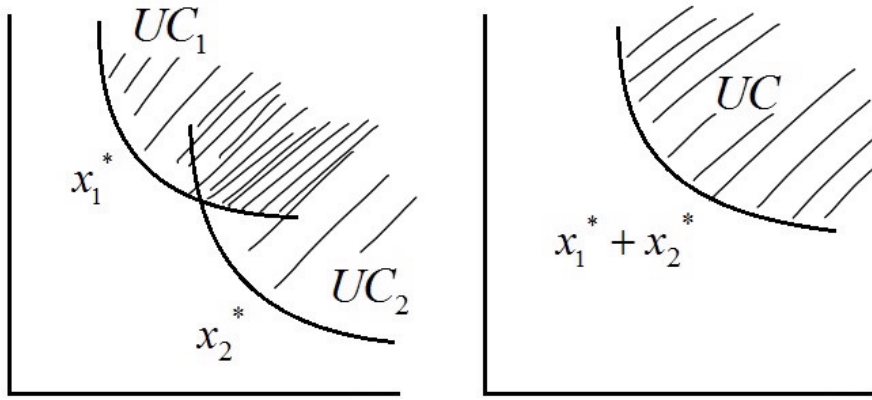
Define $UC_i = \{x_i \in \mathbb{R}_+^K : u_i(x_i) > u_i(x_i^*)\}$ and upper contour set, $UC = \sum_{i=1}^{n} UC_i = \{z \in \mathbb{R}_+^K : z = \sum_{i=1}^{n} x_i,$ $x_i \in UC_i\}$

z is a commodity bundle, not an excess demand.

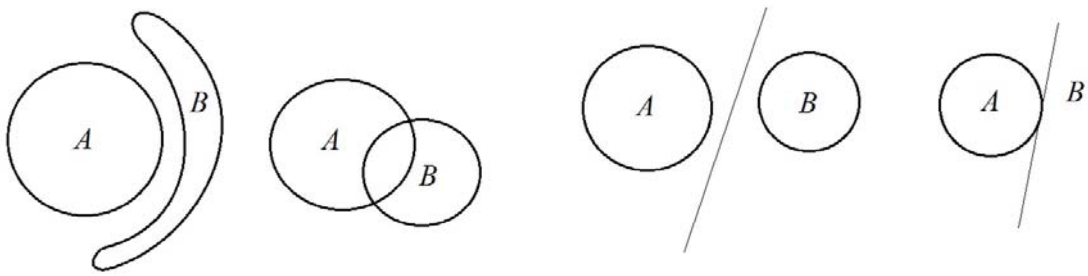
Since each UC_i is a convex set, UC is also convex. To see this, suppose $z, z' \in UC$. Then there exist allocations (not necessarily feasible) x and x' such that $x_i \in UC_i$, $x'_i \in UC_i$, $z = \sum_{i=1}^{n} x_i$ and $z' = \sum_{i=1}^{n} x'_i$ hold.

Now consider $z'' = \lambda z + (1-\lambda)z'$. We need to show that $z'' \in UC$. Let $x''_i = \lambda x_i + (1-\lambda)x'_i$ for all i. We know

$$
\begin{aligned}
\sum_{i=1}^{n} x''_i &= \lambda \sum_{i=1}^{n} x_i + (1-\lambda) \sum_{i=1}^{n} x'_i = \lambda z + (1-\lambda) z' = z'', \\
u_i(x''_i) &> u_i(x^*_i) \quad \text{(by strict quasi-concavity).}
\end{aligned}
$$



Since x^* is P.O., $\sum_{i=1}^{n} x_i^* \notin UC$.



1.6.1 Separating Hyperplane Theorem

If $A \subset \mathbb{R}^K$ and $B \subset \mathbb{R}^K$ are disjoint, convex sets (non-empty also), there exists a linear functional $p \neq 0$ such that $p \cdot x \geq p \cdot y$ for all $x \in A$ and $y \in B$.

Let $A \equiv UC$, $B = \{\sum\limits_{i=1}^{n} x_i^*\}$: single point.

From the Sep. Hyp. Th., $p \cdot z \geq p \cdot \sum\limits_{i=1}^{n} x_i^*$ for $\forall z \in UC$

[Equation 1] $\qquad p \cdot (z - \sum\limits_{i=1}^{n} x_i^*) \geq 0$ for $\forall z \in UC$

(Step 1) $p \geq 0$, $e^j \equiv (0, \cdots, 0, \underbrace{1}_{j^{th}\text{ component}}, 0, \cdots, 0)$ in $\mathbb{R}^K$

<u>Proof</u>

$\sum\limits_{i=1}^{n} x_i^* + e^j \in UC$ by monotonicity from (Eq 1), $p \cdot e^j \geq 0$ for $j = 1, \cdots, K$ which implies $p \geq 0$ $\qquad \blacksquare$

(Step 2) If $u_h(y_h) > u_h(x_h^*)$, then $p \cdot y_h \geq p \cdot x_h^*$ (holds for all h)

<u>Proof</u>

Construct a bundle in UC by taking a little away from h and giving it to other agents

$$y'_h = (1-\theta)y_h$$

$$y'_i = x_i^* + \tfrac{\theta}{n-1}y_h \text{ for } h \neq i$$

Monotonicity implies $u_i(y'_i) > u_i(x_i^*)$ for $h \neq i$. Continuity implies $u_h(y'_h) > u_h(x_h^*)$ for θ close to 0.

$\therefore \sum\limits_{i=1}^{n} y'_i \in UC$, so from (Eq 1), $p \cdot \sum\limits_{i=1}^{n} y'_i \geq p \cdot \sum\limits_{i=1}^{n} x_i^*$

Hence, $p \cdot [y_h(1-\theta) + y_h\theta + \sum\limits_{h \neq i} x_i^*] \geq p \cdot [x_h^* + \sum\limits_{i \neq h} x_i^*]$

$$\therefore \qquad p \cdot y_h \geq p \cdot x_h^* \qquad \blacksquare$$

(Step 3) For all h, $u_h(y_h) > u_h(x_h^*)$ implies $p \cdot y_h > p \cdot x_h^*$.

<u>Proof</u>

From Step 2 and $u_h(y_h) > u_h(x_h^*)$, we have $p \cdot y_h \geq p \cdot x_h^*$. Suppose the conclusion of Step 3 is false. Then it must be that $p \cdot y_h = p \cdot x_h^*$.

By continuity, $u_h(\theta y_h) > u_h(x_h^*)$ for some θ close to 1 (different θ from step 2).

Again using Step 2, $p \cdot \theta y_h \geq p \cdot x_h^* = p \cdot y_h$.

$$\therefore (1-\theta) p \cdot y_h \leq 0 \qquad \Longrightarrow \qquad p \cdot y_h \leq 0.$$

But, $x_h^* \gg 0$ implies $p \cdot x_h^* > 0$

$$\therefore \qquad p \cdot y_h > 0, \text{ a contradiction} \qquad \blacksquare$$

Step 3 says that anything preferred to x_h^* is not affordable. Therefore, x_h^* is utility maximizing (x_h^* is affordable since it is the initial endowment). Since x^* is feasible, utility maximization and market clearing are satisfied, so (p, x^*) is a C.E. $\blacksquare$

1.6.2 Maximize a weighted sum of utilities

(P) max $\sum_{i=1}^{n} a_i u_i(x_i)$ s.t. $\sum_{i=1}^{n} x_i^j \leq \sum_{i=1}^{n} \omega_i^j$ for $j = 1, \cdots, K$

Assume that utility is differentiable, strictly concave, and strictly monotonic.

Lagrangeans for (P): λ^j

Necessary and sufficient FOC:

FOC (P): [w.r.to x_i^j] : $a_i \frac{\partial u_i}{\partial x_i^j} = \lambda^j$ for $\forall j$ and $\sum_i x_i^j = \sum_i \omega_i^j$

Claim: Any $x^* \gg 0$ that solves (P) is Pareto optimal.

Proof

First, note that a solution to (P) must be feasible. If some feasible x' dominates x^*, then it satisfies the constraint and yields a higher value to the objective. $\blacksquare$

Claim: Any P.O. allocation $x^* \gg 0$ solves (P) for some weights, a.

Proof

By the SFTWE, x^* is a C.E. allocation for the economy with endowments, $\omega_i = x_i^*$, for some price vector p^*.

$\therefore x_i^*$ solves (for all i)

(C: consumer's problem)

max $u_i(x_i)$ s.t. $\sum\limits_j p^{*j} x_i^j \leq \sum\limits_j p^{*j} x_i^{*j}$ and $x_i^j \geq 0$ for $\forall j$.

FOC (C): $\frac{\partial u_i(x_i^*)}{\partial x_i^j} = \widehat{\lambda}_i p^{*j}$ for some $\widehat{\lambda}_i$.

We want to find a_i and $\{\lambda^j\}_{j=1}^K$ that solve the necessary and *sufficient* FOC (P), with $x_i = x_i^*$ for $\forall i$.

Let $a_i = \frac{1}{\widehat{\lambda}_i}$ and $\lambda^j = p^{*j}$.

FOC (C), which we know is true, implies FOC (P), which we needed to show.■

Claim: A feasible allocation, $x^* \gg 0$ is P.O. iff $\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} \omega_i$ and

$$\frac{\frac{\partial u_i(x_i^*)}{\partial x_i^j}}{\frac{\partial u_i(x_i^*)}{\partial x_i^{j'}}}$$

is the same for all i.

Proof$\Longrightarrow$

If x^* is P.O., it solves FOC (P) for some weights. From FOC (P), it follows that $\frac{\frac{\partial u_i(x_i^*)}{\partial x_i^j}}{\frac{\partial u_i(x_i^*)}{\partial x_i^{j'}}} = \frac{\lambda^j}{\lambda^{j'}}$ for $\forall i$, and monotonicity implies $\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} \omega_i$.

Proof$\Leftarrow$

Suppose $MRS_i = MRS_h$ for $\forall i, h$. Then, define $\frac{\frac{\partial u_i(x_i^*)}{\partial x_i^j}}{\frac{\partial u_i(x_i^*)}{\partial x_i^1}} \equiv \widetilde{\lambda}^j$ for $\forall i, j$.

Then, $p^* = \widetilde{\lambda}$ is a CE price, because FOC (C) are satisfied for $\widehat{\lambda}_i = \frac{\partial u_i(x_i^*)}{\partial x_i^1}$, because $\frac{\partial u_i(x_i^*)}{\partial x_i^j} = \widehat{\lambda}_i p^{*j}$ becomes

$$\frac{\partial u_i(x_i^*)}{\partial x_i^j} = \frac{\partial u_i(x_i^*)}{\partial x_i^1} \widetilde{\lambda}.$$

Thus, $(x^*, \widetilde{\lambda})$ is a C.E. for the economy with $\omega_i = x_i^*$.

$\therefore$ By the FFTWE, x^* is P.O. $\blacksquare$