2 Production

Y_f: production possibility set for firm f

$y_f \in Y_f$: a net output vector, where negative components are inputs.

$p \cdot y_f$: firm f's profits

firm's problem

$$\max_{y_f} p \cdot y_f \ \text{ s.t. } y_f \in Y_f$$

If Y_f is a strictly convex set, then the firm's optimal supply is a function, $y_f(p)$.

Aggregate production possibility set:

$$Y \equiv \{y \in \mathbb{R}^K : y = \sum_{f=1}^{F} y_f \text{ and } y_f \in Y_f, \forall f\}$$

Proposition 9 *y maximizes aggregate profit* <u>*if and only if*</u> *each y_f maximizes firm f's profit. That is, y solves* $\max p \cdot y$ *s.t. $y \in Y$ implies there exist $y_1, ..., y_F$ with each $y_f \in Y_f$, s.t. $y = \sum_{f=1}^{F} y_f$ and $p \cdot y_f \geq p \cdot y_f'$ for all $y_1', ..., y_F'$ with each $y_f' \in Y_f$, and conversely, for $f = 1, ..., F$, if y_f solves $\max_{y_f} p \cdot y_f$ s.t. $y_f \in Y_f$ then $y = \sum_{f=1}^{F} y_f$ solves* $\max p \cdot y$ *s.t. $y \in Y$.*

Proof sketch $\Rightarrow$ Let $\pi^*(p)$ denote the aggregate profit associated with the solution to the aggregate profit maximization problem, and let $y_1, ..., y_F$ be a way to feasibly achieve $\pi^*(p)$. Suppose the conclusion is false, so there exists some $y_f' \in Y_f$, such that $p \cdot y_f' > p \cdot y_f$. Then $y_1, ..., y_f', ..., y_F$ is feasible and yields profits greater than $\pi^*(p)$, contradicting the optimality of y.

Proof sketch $\Leftarrow$ Suppose for $f = 1, ..., F$, each y_f solves $\max_{y_f} p \cdot y_f$ s.t. $y_f \in Y_f$, but the conclusion is false, so there exists feasible $y'_1, ..., y'_F$ such that $p \cdot \sum_{f=1}^{F} y'_f > p \cdot \sum_{f=1}^{F} y_f$. Then at least one component of the left sum exceeds that of the right sum, so $p \cdot y'_f > p \cdot y_f$. This contradicts the fact that y_f is profit maximizing for firm f.

What about duopoly and higher profits of forming a cartel?

Modeling labor supply

Workers have endowments of leisure. Supply of labor is $(\omega_i^l - x_i^l)$.

Must require $x_i^l \leq \omega_i^l$ for consumer i in Definition of consumption set

Accounting for Profits

$0 \leq T_{i,f} \leq 1$ consumer i's share of firm f, where $\sum\limits_{i=1}^{n} T_{i,f} = 1$ for all f

Consumer's Problem

max $u_i(x_i)$ s.t. $p \cdot x_i \leq p \cdot \omega_i + \sum\limits_{f=1}^{F} T_{i,f} p \cdot y_f(p)$ and $x_i \in X_i$

$$z(p) = \sum_{i=1}^{n} x_i(p) - \sum_{f=1}^{F} y_f(p) - \sum_{i=1}^{n} \omega_i$$

Walras' Law: If utility satisfies local nonsatiation, $p \cdot z(p) = 0$ for all p

Proof. $p \cdot z(p) = \sum_{i=1}^{n} p \cdot x_i(p) - \sum_{f=1}^{F} p \cdot y_f(p) - \sum_{i=1}^{n} p \cdot \omega_i$

and $\sum_{i=1}^{n} p \cdot x_i(p) = \sum_{i=1}^{n} p \cdot \omega_i + \sum_{i=1}^{n} \sum_{f=1}^{F} T_{i,f} p \cdot y_f(p)$ by local n.s.

$$p \cdot z(p) = \sum_{f=1}^{F} [p \cdot y_f(p) \sum_{i=1}^{n} T_{i,f}] - \sum_{f=1}^{F} p \cdot y_f(p) = 0 \quad \blacksquare$$

Definition 10 *A C.E. is a price vector* $p \in S^{K-1}$ *and an allocation,* (x, y) *satisfying*

(1) each consumer maximizes utility subject to his/her budget constraint

(2) each firm maximizes profits s.t. technology

(3) markets clear: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \omega_i + \sum_{f=1}^{F} y_f$

Existence of C.E.

- Assumption (1)

- Consumption sets are closed, convex, and bounded from below. Endowments are in the interior of consumption sets

- Y_f contains 0, is closed and convex

- irreversibility: $Y_f \cap (-Y_{f'}) = \{0\}$, $\forall f, f'$

- free disposal

Since we are allowing for constant returns, $z(p)$ might be a correspondence, requiring Kakutani's thm.

Definition 11 (x, y) *is* *<u>feasible</u>* *if* $x_i \in X_i$ *for all* i, $y_f \in Y_f$ *for all* f, *and* $\sum_{i=1}^{n} x_i \leq \sum_{f=1}^{F} y_f + \sum_{i=1}^{n} \omega_i$.

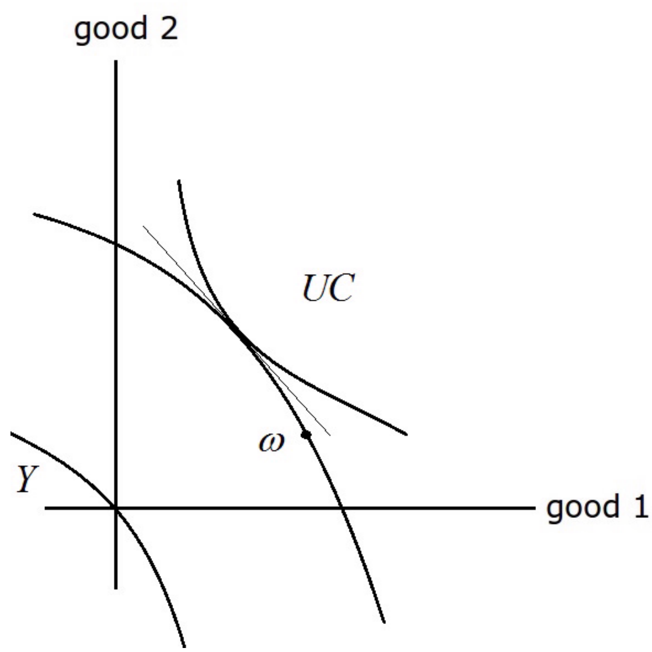
Definition 12 *A feasible allocation is* *<u>P.O.</u>* *if there is no other feasible allocation,* (x', y') *s.t.* $u_i(x'_i) \geq u_i(x_i)$ *for all* i *and* $u_h(x'_h) > u_h(x_h)$ *for some* h.

<u>FFTWE</u>: Under local nonsatiation, any C.E. allocation (x, y) is P.O.

<u>SFTWE</u>: Suppose (x^*, y^*) is P.O., $x^* \gg 0$. Preferences are convex, continuous, strictly monotonic. Y_f convex. Then for some endowments and firms' shares, (x^*, y^*) is a C.E. allocation.

<u>Sketch</u>: Separate the aggregate preferred bundles, UC, from the feasible aggregate bundles,

$$\{z : \text{for some } y_1, ..., y_F \text{ with each } y_f \in Y_f, \text{ we have } z = \sum_{i=1}^{n} \omega_i + \sum_{f=1}^{F} y_f \}$$



Note: When you are solving for a C.E.,

(i) If technologies are constant returns, for a solution to the profit maximization problem to exist in which the firm produces output, this gives an equation restricting prices (yielding zero profits). You will not be able to solve for supply functions.

(ii) With strictly convex production sets, you can solve for supply functions, but you must account for profits in the consumer opt. problems.

3 Uncertainty

$x_i^{j,s}$, $s \in$ states of nature $= \{1, 2, \cdots, S\}$

π_s: probability of state s

Definition 13 *A C.E. is a price vector,*

$p = \{p^{1,1}, \cdots, p^{1,S}, p^{2,1}, \cdots, p^{2,S}, \cdots, p^{K,1}, \cdots, p^{K,S}\}$
and an allocation $\{x_i^{j,s}\}|_{i=1}^{n}|_{j=1}^{K}|_{s=1}^{S} \in \mathbb{R}_+^{nKS}$ *such that*

(1) utility maximization, x_i *solves*

$$\max \sum_{s=1}^{S} \pi_s u_i(x_i^{1,s}, \cdots, x_i^{K,s}) \equiv V_i(x_i) : \text{ expected utility}$$

subject to $p \cdot x_i \leq p \cdot \omega_i$ *and* $x_i \geq 0$

(2) market clearing for all j *and* s, $\sum_{i=1}^{n} x_i^{j,s} \leq \sum_{i=1}^{n} \omega_i^{j,s}$

Note:

(1) $x_i^{j,s}$ is a state-contingent commodity. good j is delivered iff the state is s

(2) all trade takes place before any information about the state is received (no pre-existing conditions)

(3) setup presumes that the equilibrium can be enforced–questionable when commodities have an uncertainty dimension or a time dimension.

(4) Specification is not completely general, because the Bernoulli utility function does not depend on the state. Von Neumann-Morgenstern does not allow utility of consuming an umbrella to depend on the state (except through π_s)

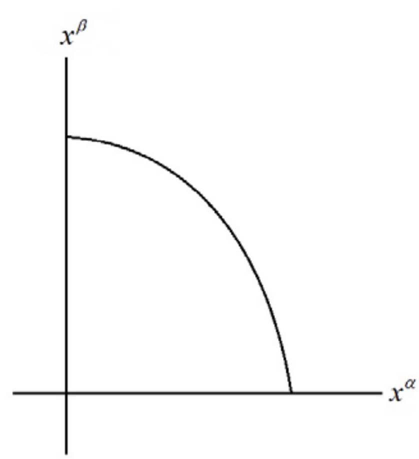
Here is an example where the Bernoulli utility function is strictly quasi-concave but the overall utility function is not.

$$V(x^\alpha, x^\beta) = \pi^\alpha (x^\alpha)^2 + \pi^\beta (x^\beta)^2$$

each $u(x) = x^2$ is strictly quasi-concave, $x^2 > y^2 \Longrightarrow [\theta x + (1-\theta)y]^2 > y^2$

V is not strictly quasi-concave

tradeoffs are being made across states - the level of utility within a state matters. Risk aversion



If each u_i is strictly increasing continuous, strictly concave, $\omega \in \mathbb{R}^{nKS}_{++}$

(1) consumers are risk-averse

(2) equilibrium exists and is Pareto optimal

(3) any P.O. allocation is a C.E. allocation for some redistribution of state-contingent endowments

Special Case: $K = 1$, no aggregate uncertainty

u_i: differentiable, strictly monotonic, strictly concave

FOC of consumer's problem: $\dfrac{\pi_s \frac{\partial u_i(x_i^s)}{\partial x_i^s}}{\pi_{s'} \frac{\partial u_i(x_i^{s'})}{\partial x_i^{s'}}} = \dfrac{p^s}{p^{s'}}$ for all s, s'

and $\sum\limits_{s=1}^{S} p^s x_i^s = \sum\limits_{s=1}^{S} p^s \omega_i^s$

Assume no aggregate uncertainty: $\sum\limits_{i=1}^{n} \omega_i^s = \sum\limits_{i=1}^{n} \omega_i^{s'}$ for all s, s'

Claim

(i) Any interior Pareto Optimal allocation satisfies

$$(*)\ x_i^s = x_i^{s'} \text{ for all } s, s' \text{ and for all } i$$

(ii) Any nonwasteful allocation satisfying $(*)$ is Pareto Optimal

Proof of (i)

Since x is P.O., each consumer has the same MRS at x.

$$\frac{\frac{\partial u_i(x_i^s)}{\partial x_i^s}}{\frac{\partial u_i(x_i^{s'})}{\partial x_i^{s'}}} = \frac{\frac{\partial u_h(x_h^s)}{\partial x_h^s}}{\frac{\partial u_h(x_h^{s'})}{\partial x_h^{s'}}}, \forall i, h, s, s'$$

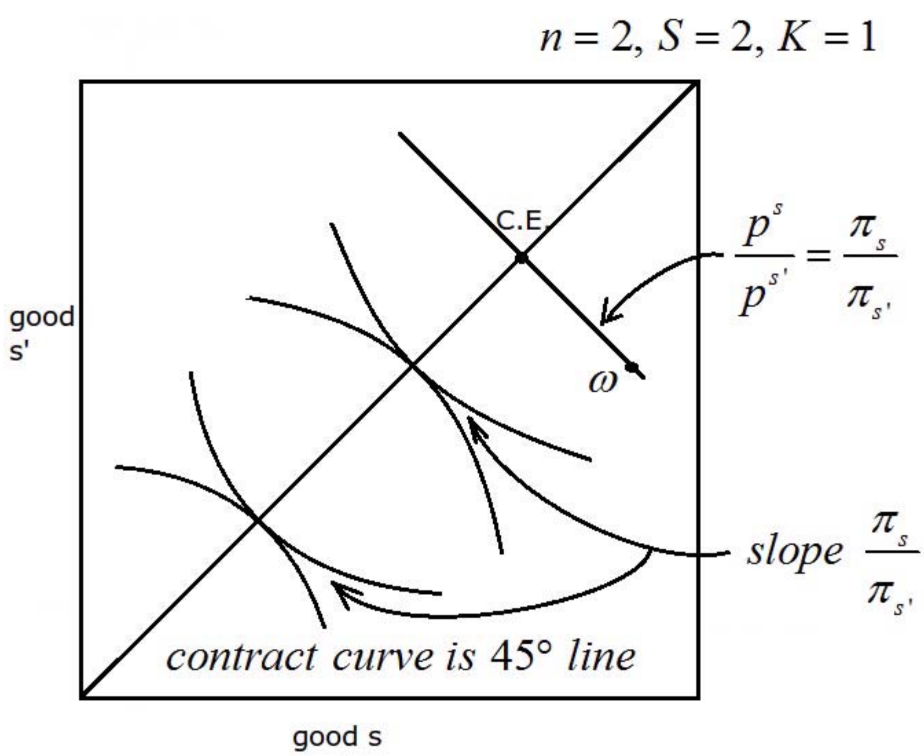
Suppose $x_i^s > x_i^{s'}$ for some i, s, s'. Since we have no aggregate uncertainty, $x_h^s < x_h^{s'}$ for some consumer h.

Since u_i and u_h are concave,

$$\frac{\partial u_i(x_i^s)}{\partial x_i^s} < \frac{\partial u_i(x_i^{s'})}{\partial x_i^{s'}}, \frac{\partial u_h(x_h^s)}{\partial x_h^s} > \frac{\partial u_h(x_h^{s'})}{\partial x_h^{s'}}$$

Contradicts Pareto optimality. Therefore, $x_i^s = x_i^{s'}, \forall i, s, s'$.

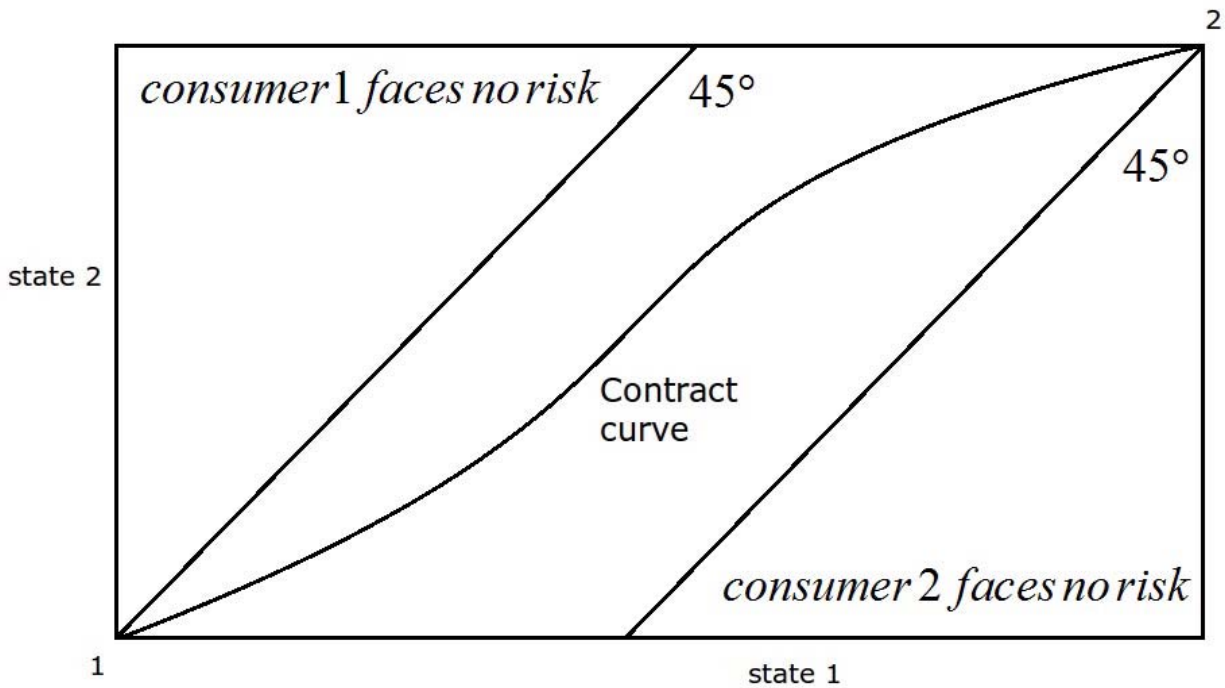
<u>Proof of (ii)</u> Any point satisfying $(*)$ has $\dfrac{\pi_s \frac{\partial u_i(x_i^s)}{\partial x_i^s}}{\pi_{s'} \frac{\partial u_i(x_i^{s'})}{\partial x_i^{s'}}} =$ $\frac{\pi_s}{\pi_{s'}}$ for all i, s, s' and is therefore Pareto optimal. ■



Note: <u>unique C.E.</u> $(p^1, \cdots, p^s, \cdots, p^S)$

$= (\pi^1, \cdots, \pi^s, \cdots, \pi^S)$, only relies on $K = 1$

Aggregate Uncertainty ($K = 1$)



The argument for the no-aggregate-uncertainty case also says that, with aggregate uncertainty, Pareto optimality implies that for any s, s' with $\sum_{i=1}^{n} \omega_i^s > \sum_{i=1}^{n} \omega_i^{s'}$, all consumers receive higher Bernoulli utility in state s than state s'. This illustrates how incomplete our markets really are.

Incompleteness of Markets

(1) A country experiencing an earthquake is disproportionally hit financially

(2) State of nature must specify whose hard disk crashes, who has an accident, who is having a tough time concentrating,...

(3) moral hazard problems

(4) asymmetric information–adverse selection can destroy insurance

(5) it is costly to set up markets - is there a more economical way? Arrow Securities

3.1 Arrow Securities Market

Stage 1: before the state is observed, consumers buy and sell securities, where a security, s, pays 1 unit of account on the state$-s$ spot market.

Stage 2: after the state of nature is observed, securities are redeemed and commodities are traded on a spot market.

b_i^s: consumer i's holdings of security s. $b_i^s > 0$ implies i receives b_i^s "dollars" in state s.

q^s: the price of security s

$x_i^j(s)$: consumption of commodity j received from the state-s spot market

$p^j(s)$: price of commodity j on the state$-s$ spot market

Utility Max Problem

$$\max_{x_i(s),b_i^s} V_i(\cdot) = \sum_{s=1}^{S} \pi_s u_i(x_i(s)) \text{ s.t. } \sum_{s=1}^{S} q^s b_i^s \leq 0 \text{ and}$$
$$\sum_{j=1}^{K} p^j(s) x_i^j(s) \leq \sum_{j=1}^{K} p^j(s)\omega_i^j(s) + b_i^s \text{ for all } s,\ x_i^j(s) \geq 0$$

Market Clearing

$$\sum_{i=1}^{n} b_i^s \leq 0 \text{ for all } s$$

$$\sum_{i=1}^{n} x_i^j(s) \leq \sum_{i=1}^{n} \omega_i^j(s) \text{ for all } j \text{ and } s$$

Definition 14 *A C.E. is a set of prices* $\{q, p(s)\}$*, security holdings* $\{b_i^s\}|_{s=1}^{S}|_{i=1}^{n}$*, and consumption* $\{x_i^j(s)\}|_{i=1}^{n}|_{j=1}^{K}|_{s=1}^{S}$ *satisfying market clearing and utility maximization.*

Notice:

(1) $\pi_s > 0$ and monotonicity implies $q^s > 0$

(2) We can normalize $q^1 = 1$ without affecting the set of affordable bundles for any consumer

$$AFF(q, p) = AFF(\lambda q, p)$$

(3) We can normalize $p^1(s) = 1$ without loss of generality, for each s,

$$AFF(q^1, \cdots, q^s, \cdots, q^S, p(1), \cdots, p(s), \cdots p(S)) =$$
$$AFF(q^1, \cdots, \lambda q^s, \cdots, q^S, p(1), \cdots, \frac{p(s)}{\lambda}, \cdots, p(S))$$

Theorem 15 *(Arrow) The contingent commodities model and the Arrow Securities model are equivalent.*

$\{x_i^{j,s}\}$ *is a C.E. allocation (of the cont. comm. model)* $\Rightarrow$ $\{x_i^j(s), b_i^s\}$ *is a C.E. allocation (of the Arrow sec. model) for some* $\{b_i^s\}$, *where* $x_i^j(s) = x_i^{j,s}$, $\forall i, j, s$

AND $\{x_i^j(s), b_i^s\}$ *is a C.E. allocation (of the Arrow sec. model)* $\Rightarrow$ $x_i^{j,s}$ *is a C.E. allocation (of the cont. comm. model), where* $x_i^{j,s} = x_i^j(s)$

Proof sketch

Construct the Arrow Security Prices from the A-D prices and vice versa.

Relative price of consumption in the Arrow securities model determined as follows:

- reduce consumption of $x^1(s)$ by one unit
- receive $p^1(s)$ additional units of account on the spot market
- allows you to demand $p^1(s)$ fewer state-s securities
- allows you to have $p^1(s) \cdot q^s$ more income on the securities market

- allows you to buy $\frac{p^1(s)q^s}{q^{s'}}$ more state-s' securities

- allows you to buy $\frac{p^1(s)q^s}{p^2(s')q^{s'}}$ more units of $x^2(s')$

- price of $x^1(s)$ [in terms of $x^2(s')$] is $\frac{p^1(s)q^s}{p^2(s')q^{s'}}$

example: $u_i(x_i^s) = \log(x_i^s)$, $\pi_\alpha = 1/4$, $\pi_\beta = 3/4$, $\omega_1 = (1,2)$, $\omega_2 = (2,1)$

normalize $p(\alpha) = p(\beta) = 1$

consumer 1: $\max\limits_{x_1, b_1} \frac{1}{4}\log(x_1^\alpha) + \frac{3}{4}\log(x_1^\beta)$

$$\text{s.t. } q^\alpha b_1^\alpha + q^\beta b_1^\beta = 0,\ x_1^\alpha = 1 + b_1^\alpha,\ x_1^\beta = 2 + b_1^\beta$$

substitute spot market budget constraints into sec. market, to get

$$q^\alpha(x_1^\alpha - 1) + q^\beta(x_1^\beta - 2) = 0 \cdots\cdots (eq.1)$$

FOC (multiplies λ_1):

$$\frac{1}{4x_1^\alpha} - \lambda_1 q^\alpha = 0 \cdots\cdots (eq.2)$$

$$\frac{3}{4x_1^\beta} - \lambda_1 q^\beta = 0 \cdots\cdots (eq.3)$$

solving, $\lambda_1 = \frac{1}{4q^\alpha x_1^\alpha} = \frac{3}{4q^\beta x_1^\beta}$,

$$\text{so } x_1^\beta = \frac{3q^\alpha x_1^\alpha}{q^\beta} \cdots\cdots (eq.4)$$

Solving $(eq.1)$ and $(eq.4)$, we have

$$q^\alpha x_1^\alpha - q^\alpha + 3q^\alpha x_1^\alpha - 2q^\beta = 0$$

$$\therefore x_1^\alpha = \frac{q^\alpha + 2q^\beta}{4q^\alpha} \text{ and } x_1^\beta = \frac{3(q^\alpha + 2q^\beta)}{4q^\beta}$$

Going through the same steps for consumer 2 yields

$$x_2^\alpha = \frac{2q^\alpha + q^\beta}{4q^\alpha} \text{ and } x_2^\beta = \frac{3(2q^\alpha + q^\beta)}{4q^\beta}$$

Let us normalize, $q^\beta = 1$

Market clearing on the $\beta-$spot market:

$$\frac{3(q^\alpha + 2)}{4} + \frac{3(q^\alpha + 2)}{4} = 3$$

$\therefore q^\alpha = 1/3$

$x_1 = (\frac{7}{4}, \frac{7}{4})$, $x_2 = (\frac{5}{4}, \frac{5}{4})$, $b_1 = (\frac{3}{4}, -\frac{1}{4})$, $b_2 = (-\frac{3}{4}, \frac{1}{4})$

note: no aggregate uncertainty

3.2 Production and Uncertainty in the Contingent Commodities Model

now Y_f is a set in $\mathbb{R}^{KS}$

- usually, output in state s only depends on inputs in state s.

- often, input of commodity (j, s) must be the same as input of (j, s'). For example, a worker is hired to work, no matter what the state is.

- Setup allows for "technology shocks," since output depends on s, even for the same inputs.

- usually, there is joint production, since output is not only produced in one state.

- The firm's profits, $p \cdot y_f$, do not depend on the realized state. It buys and sells state-contingent contracts <u>before</u> the state is known.

- Models in which profits depend on the state and firms maximize expected profits are based on profits being received on a spot market, not complete contingent commodity trading.

- If firms are risk neutral, how can a C.E. be Pareto Optimal, then? Ultimately, consumption risk must fall on consumers, since firms do not consume.

introducing more complicated securities

(an example with 5 states)

firm's gross return, $R(s) = \begin{pmatrix} 0 \\ 1 \\ 5 \\ 5 \\ 10 \end{pmatrix}$ vs an Arrow security

such as $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

One unit of security R is the equivalent of a portfolio of 1 state-2 Arrow security, 5 state-3 Arrow securities, etc.

Security R could be written to pay off based only on the returns received by this firm. The example partitions the zillions of states into 5 relevant "events." More realistic than specifying all aspects of the state of nature.

debt level: 3

$$D(s) = \mathsf{min}[R(s), 3],\ D(s) = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 3 \end{pmatrix}$$

equity: $E(s) = R(s) - D(s)$, $E(s) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 7 \end{pmatrix}$

- no-arbitrage condition: $p^E + p^D = p^R$
- spot market budget constraint is $p(s) \cdot x_i(s) \leq p(s) \cdot \omega_i(s) + b^R R(s) + b^D D(s) + b^E E(s)$
- real vs nominal securities

In general, if there are $r = 1, ..., R$ nominal securities traded and one unit of security r on the spot market in state s pays $a^{r,s}$ units of account, the spot market budget constraint in state s is given by

$$p(s) \cdot x_i(s) \leq p(s) \cdot \omega_i(s) + \sum_{r=1}^{R} b^r a^{r,s}$$

We can express the securities market by the $S \times R$ matrix with typical component $a^{r,s}$. Then one can prove that the competitive equilibrium is equivalent to complete markets if and only if the matrix has full rank, S.

Even if there are fewer securities than states so the market is incomplete, if a limited number of securities are designed to cover the most important risky events faced by the economy, maybe the CE is close to what would prevail under complete markets.