Department of Economics
The Ohio State University
Midterm Questions and Answers–Econ 8712

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Fall 2020

Directions: *Answer all questions, be neat, and show all work. If it is appropriate, you are allowed to use propositions presented in class without proving them here.*

1. (30 points)

A decision maker (DM) is a von Neumann-Morgenstern expected utility maximizer with Bernoulli utility function over final wealth x given by the strictly concave function $u(x)$, and initial wealth W. The DM is given a choice between two lotteries, F and G. With lottery F, the DM receives final wealth $W-\alpha$ with probability one half, and the DM receives final wealth $W+\alpha$ with probability one half. With lottery G, the DM receives final wealth $W-2\alpha$ with probability one quarter, the DM receives final wealth W with probability one half, and the DM receives final wealth $W+2\alpha$ with probability one quarter. The parameter, α, is the same in both lotteries, and satisfies the condition, $0 < 2\alpha < W$.

Which lottery would the DM prefer? Give a rigorous explanation.

Answer: The DM prefers lottery F, since F second-order stochastically dominates G. The easiest way to rigorously demonstrate this is to show that G is a mean preserving spread of F. For either realization of F, we add the outcome of a mean zero distribution, H, with realization $-\alpha$ with probability one half and realization α with probability one half. Thus, with probability $\frac{1}{4}$ we have the low realization of F and the low realization of H, with final wealth $W-2\alpha$; with probability $\frac{1}{4}$ we have the low realization of F and the high realization of H, with final wealth W; with probability $\frac{1}{4}$ we have the high realization of F and the low realization of H, with final wealth W; and with probability $\frac{1}{4}$ we have the high realization of F and the high realization of H, with final wealth $W+2\alpha$. This is exactly G.

2. (35 points)

The following partial equilibrium economy has 2 consumers, 2 firms, and a government. Consumer 1 has the utility function,

$$u_1(x_1, m_1) = \log(x_1) + m_1,$$

the numeraire endowment $\omega_1^m = 1$, and owns half of each firm, $T_{1,f} = \frac{1}{2}$ for $f = 1, 2$. Consumer 2 has the utility function,

$$u_2(x_2, m_2) = 2\log(x_2) + m_2,$$

the numeraire endowment $\omega_2^m = 1$, and owns half of each firm, $T_{2,f} = \frac{1}{2}$ for $f = 1, 2$.

Firm 1 has the cost-of-production function given by

$$c_1(y_1) = (y_1)^2$$

and firm 2 has the cost-of-production function given by

$$c_2(y_2) = 2(y_2)^2.$$

The government imposes, on each firm, a tax of $\frac{5}{3}$ for each unit of output produced, and distributes half of the tax revenue to consumer 1 and half of the tax revenue to consumer 2. That is, firm 1 pays a tax of $\frac{5}{3}y_1$ units of the numeraire, firm 2 pays a tax of $\frac{5}{3}y_2$ units of the numeraire, and each consumer receives a payment from the government of $\frac{1}{2}[\frac{5}{3}y_1 + \frac{5}{3}y_2]$ units of the numeraire. Each firm's total cost is its cost of production plus its tax payment.

(i) (10 points) Define a competitive equilibrium for this economy.

(ii) (20 points) Compute the competitive equilibrium price and allocation.

(iii) (5 points) Explain whether or not the output is efficiently produced and distributed.

Answer: A competitive equilibrium is a price, p^*, and an allocation, $(x_1^*, m_1^*, x_2^*, m_2^*, y_1^*, y_2^*)$, such that

(1) x_1^*, m_1^* solves

$$\max \log(x_1^*) + m_1^*$$

subject to

$$p^* x_1^* + m_1^* = 1 + \frac{1}{2}[\frac{5}{3}y_1^* + \frac{5}{3}y_2^*] + \frac{1}{2}[p^* y_1^* - (y_1^*)^2 - \frac{5}{3}y_1^*] + \frac{1}{2}[p^* y_2^* - 2(y_2^*)^2 - \frac{5}{3}y_2^*],$$

(2) x_2^*, m_2^* solves

$$\max 2\log(x_2^*) + m_2^*$$

subject to

$$p^* x_2^* + m_2^* = 1 + \frac{1}{2}[\frac{5}{3}y_1^* + \frac{5}{3}y_2^*] + \frac{1}{2}[p^* y_1^* - (y_1^*)^2 - \frac{5}{3}y_1^*] + \frac{1}{2}[p^* y_2^* - 2(y_2^*)^2 - \frac{5}{3}y_2^*],$$

(3) y_1^* solves

$$\max p^* y_1^* - (y_1^*)^2 - \frac{5}{3}y_1^*$$

(4) y_2^* solves

$$\max p^* y_2^* - 2(y_2^*)^2 - \frac{5}{3}y_2^*$$

(5) markets clear

$$\begin{aligned} x_1^* + x_2^* &= y_1^* + y_2^* \\ m_1^* + m_2^* + (y_1^*)^2 + 2(y_2^*)^2 &= 2 \end{aligned}$$

Note: market clearing of the numeraire holds because government tax revenues equal government transfers.

To find the competitive equilibrium price, we compute the demand functions and supply functions. For consumer 1, utility maximization requires

$$\begin{aligned} \frac{1}{x_1^*} &= p^* \quad \text{or the demand function} \\ x_1^* &= \frac{1}{p^*}. \end{aligned}$$

Similarly, for consumer 2 we compute the demand function,

$$x_2^* = \frac{2}{p^*}.$$

For firm 1, profit maximization requires

$$p^* - 2(y_1^*) - \frac{5}{3} = 0,$$

yielding the supply function,

$$y_1^* = \frac{p^* - \frac{5}{3}}{2}.$$

Similarly, for firm 2 we compute the supply function,

$$y_2^* = \frac{p^* - \frac{5}{3}}{4}.$$

Market clearing for the good is given by

$$\begin{aligned} \frac{1}{p^*} + \frac{2}{p^*} &= \frac{p^* - \frac{5}{3}}{2} + \frac{p^* - \frac{5}{3}}{4} \\ \frac{3}{p^*} &= 3\frac{p^* - \frac{5}{3}}{4} \\ 3(p^*)^2 - 5p^* - 12 &= 0. \end{aligned}$$

Since only one root is positive, the solution is $p^* = 3$. From the demand and supply functions, we have $x_1^* = \frac{1}{3}, x_2^* = \frac{2}{3}, y_1^* = \frac{2}{3}, y_2^* = \frac{1}{3}$. From the budget equations, we have $m_1^* = \frac{7}{6}$ and $m_2^* = \frac{1}{6}$.

The output is efficiently produced and allocated, because marginal utilities are equated across consumers and marginal costs are equated across firms. (This is not the quantity that maximizes Marshallian surplus, however, due to the tax that creates a deadweight loss.)

3. (35 points)

The following pure exchange economy has two goods and two consumers. Consumer 1 has the utility function,

$$u_1(x_1^1, x_1^2) = \frac{x_1^1 x_1^2}{x_1^1 + x_1^2}$$

and the initial endowment vector, $\omega_1 = (1, 0)$. Consumer 2 has the utility function,

$$u_2(x_2^1, x_2^2) = \log(x_2^1) + \log(x_2^2)$$

and the initial endowment vector, $\omega_2 = (0, 2)$. Normalize the price of good 2 to be one and denote the price of good 1 by p.

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (25 points) Calculate the competitive equilibrium price of good 1 relative to good 2, p^*. (Note: Just the price is enough. You do not need to specify the allocation. Substantial credit will be given if you find the correct equation but have trouble solving.)

Answer: A competitive equilibrium is a (normalized) price vector, $(p^*, 1)$, and an allocation, $(x_1^{1*}, x_1^{2*}, x_2^{1*}, x_2^{2*})$, such that

(1) (x_1^{1*}, x_1^{2*}) solves

$$\begin{aligned} & \max \frac{x_1^1 x_1^2}{x_1^1 + x_1^2} \\ & \text{subject to} \\ p^* x_1^1 + x_1^2 &\leq p^* \\ x_1 &\geq 0 \end{aligned}$$

(2) (x_2^{1*}, x_2^{2*}) solves

$$\begin{aligned} & \max[\log(x_2^1) + \log(x_2^2)] \\ & \text{subject to} \\ p^* x_2^1 + x_2^2 &\leq 2 \\ x_2 &\geq 0 \end{aligned}$$

(3) markets clear

$$\begin{aligned} x_1^1 + x_2^1 &\leq 1 \\ x_1^2 + x_2^2 &\leq 2. \end{aligned}$$

First we compute the demand functions and then use market clearing to determine p^*. Because utility is strictly monotonic, budget constraints and market clearing will hold as equalities.

Consumer 1's utililty maximization is characterized by the budget equation and the marginal rate of substitution condition,

$$\frac{(x_1^2)^2}{(x_1^1)^2} = p.$$

Substituting the MRS equation into the budget equation yields

$$px_1^1 + \sqrt{p}x_1^1 = p$$

Solving for x_1^1 and substituting back into the MRS condition yield the demand functions

$$\begin{aligned} x_1^1 &= \frac{p}{p+\sqrt{p}} \\ x_1^2 &= \frac{p^{3/2}}{p+\sqrt{p}}. \end{aligned}$$

Consumer 2's utililty maximization is characterized by the budget equation and the marginal rate of substitution condition,

$$\frac{x_2^2}{x_2^1} = p.$$

Substituting the MRS equation into the budget equation yields

$$2px_2^1 = p$$

Solving for x_2^1 and substituting back into the MRS condition yield the demand functions

$$\begin{aligned} x_2^1 &= \frac{1}{p} \\ x_2^2 &= 1. \end{aligned}$$

Market clearing of good 1 gives us the equation for the equilibrium price.

$$\frac{p}{p+\sqrt{p}} + \frac{1}{p} = 1.$$

Multiplying both sides by $p(p+\sqrt{p})$ yields

$$\begin{aligned} p^2 + p + \sqrt{p} &= p(p+\sqrt{p}) \text{ or} \\ p + \sqrt{p} &= p\sqrt{p} \end{aligned}$$

Therefore, by combining the square root terms, we have

$$\begin{aligned} p &= \sqrt{p}(p-1) \\ \frac{p}{p-1} &= \sqrt{p} \\ \frac{p^2}{(p-1)^2} &= p \\ \frac{p}{(p-1)^2} &= 1 \end{aligned}$$

Finally, we use the quadratic formula to choose the root in which p is greater than one,

$$p^* = \frac{3 + \sqrt{5}}{2}.$$