Department of Economics
The Ohio State University
Final Exam Questions and Answers–Econ 8712

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Directions: *Answer all questions, carefully label all diagrams, and show all work.*

1. (40 points)

The following economy has one consumer, three firms, and three goods. The consumer has the initial endowment vector, $\omega = (0, 0, 1)$, and the utility function,

$$u(x^1, x^2, x^3) = a \log(x^1) + (1 - a) \log(x^2).$$

The parameter, a, is strictly between zero and one. Notice that the third good provides no utility, so the consumer will demand 0 units whatever the prices.

Firm 1 produces good 1 using good 3 as an input, with the following production set:

$$Y_1 = \{(y_1^1, y_1^2, y_1^3) : y_1^1 \leq -3y_1^3, \quad y_1^3 \leq 0, \quad y_1^2 = 0\}.$$

Firm 2 produces good 2 using good 3 as an input, with the following production set:

$$Y_2 = \{(y_2^1, y_2^2, y_2^3) : y_2^2 \leq -3y_2^3, \quad y_2^3 \leq 0, \quad y_2^1 = 0\}.$$

Firm 3 produces both good 1 and good 2, using good 3 as an input, with the following production set:

$$Y_3 = \{(y_3^1, y_3^2, y_3^3) : y_3^1 \leq -2y_3^3, \ \ y_3^2 \leq -2y_3^3, \quad y_3^3 \leq 0\}.$$

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (30 points) Compute the competitive equilibrium price vector and allocation for all values of the parameter, a.

Hint: Think about the possibilities for which firms are producing and which firms are shutting down, and the implications for equilibrium prices. It is simplest to normalize $p^3 = 1$.

Answer:

(a) A competitive equilibrium is a price vector, (p^1, p^2, p^3), and an allocation, $(x^1, x^2, x^3, y_1^1, y_1^2, y_1^3, y_2^1, y_2^2, y_2^3, y_3^1, y_3^2, y_3^3)$, such that

(i) (x^1, x^2, x^3) solves

$$
\begin{aligned}
& \max[a\log(x^1) + (1-a)\log(x^2)] \\
& \text{subject to} \\
p^1x^1 + p^2x^2 + p^3x^3 &\leq p^3 + \pi_1 + \pi_2 + \pi_3 \\
(x^1, x^2, x^3) &\geq 0
\end{aligned}
$$

(ii) (y_1^1, y_1^2, y_1^3) solves

$$
\begin{aligned}
& \max p^1y_1^1 + p^2y_1^2 + p^3y_1^3 \\
& \text{subject to} \\
y_1^1 &\leq -3y_1^3, \quad y_1^3 \leq 0, \quad y_1^2 = 0
\end{aligned}
$$

(iii) (y_2^1, y_2^2, y_2^3) solves

$$
\begin{aligned}
& \max p^1y_2^1 + p^2y_2^2 + p^3y_2^3 \\
& \text{subject to} \\
y_2^2 &\leq -3y_2^3, \quad y_2^3 \leq 0, \quad y_2^1 = 0
\end{aligned}
$$

(iv) (y_3^1, y_3^2, y_3^3) solves

$$
\begin{aligned}
& \max p^1y_3^1 + p^2y_3^2 + p^3y_3^3 \\
& \text{subject to} \\
y_3^1 &\leq -2y_3^3, \quad y_3^2 \leq -2y_3^3, \quad y_3^3 \leq 0
\end{aligned}
$$

(v) markets clear:

$$
\begin{aligned}
x^1 &\leq y_1^1 + y_2^1 + y_3^1 \\
x^2 &\leq y_1^2 + y_2^2 + y_3^2 \\
x^3 &\leq 1 + y_1^3 + y_2^3 + y_3^3.
\end{aligned}
$$

(b) Normalize $p^3 = 1$. Because of monotonicity, the budget constraint and market clearing conditions hold as equalities. Also, we impose $x^3 = y_1^2 = y_2^1 = 0$. Also, firms' outputs of goods 1 and 2 will be as large as possible, given the input of good 3. Because of constant returns to scale, we have $\pi_1 = \pi_2 = \pi_3 = 0$.

Simultaneously solving the budget equation and the marginal rate of substitution condition,

$$
\frac{ax^2}{(1-a)x^1} = \frac{p^1}{p^2},
$$

we have the demand functions,

$$
x^1 = \frac{a}{p^1} \tag{1}
$$

$$
x^2 = \frac{1-a}{p^2}. \tag{2}
$$

Imposing $y_1^1 = -3y_1^3$, firm 1's profits can be written as $\pi_1 = p^1(-3y_1^3) + y_1^3$. Thus, firm 1 can make unlimited profits unless $p^1 \leq \frac{1}{3}$, it is indifferent as to its output if $p^1 = \frac{1}{3}$, and it will produce zero if $p^1 < \frac{1}{3}$.

Imposing $y_2^2 = -3y_2^3$, firm 2's profits can be written as $\pi_2 = p^2(-3y_2^3) + y_2^3$. Thus, firm 2 can make unlimited profits unless $p^2 \leq \frac{1}{3}$, it is indifferent as to its output if $p^2 = \frac{1}{3}$, and it will produce zero if $p^2 < \frac{1}{3}$.

Imposing $y_3^1 = -2y_3^3$ and $y_3^2 = -2y_3^3$, firm 3's profits can be written as $\pi_3 = p^1(-2y_3^3) + p^2(-2y_3^3) + y_3^3 = (-y_3^3)(2p^1 + 2p^2 - 1)$. Thus, firm 1 can make unlimited profits unless $p^1 + p^2 \leq \frac{1}{2}$, it is indifferent as to its output if $p^1 + p^2 = \frac{1}{2}$, and it will produce zero if $p^1 + p^2 < \frac{1}{2}$. [Remember that this is a case of joint production, where one unit of good 3 produces 2 units of good 1 **and** 2 units of good 2. For example, $y_3 = (2, 2, -1)$ satisfies all of the inequalities and is in Y_3.]

What types of CE are possible? Clearly, good 1 and good 2 must be produced since the CE allocation must be Pareto optimal. Also, firm 3 must be producing in any CE. If firm 3 does not produce, then firms 1 and 2 must produce, which would imply $p^1 = \frac{1}{3}$ and $p^2 = \frac{1}{3}$, but then firm 3 would be able to obtain unlimited profits, a contradiction.

We are left with three possible types of CE.

Case 1: firm 1 and firm 3 produce. Then we have $p^1 = \frac{1}{3}$ and $p^1 + p^2 = \frac{1}{2}$, which implies that the equilibrium price vector is $(\frac{1}{3}, \frac{1}{6}, 1)$. From (1) and (2), we have $x^1 = 3a$ and $x^2 = 6(1-a)$. Since all of good 2 is produced by firm 3, market clearing implies $y_3^2 = 6(1-a)$. Therefore, $y_3^1 = 6(1-a)$ and $y_3^3 = (-\frac{1}{2})y_3^2 = -3(1-a)$. Finally, market clearing for good 1 implies

$$\begin{aligned} 3a &= y_1^1 + 6(1-a), \text{ so we have} \\ y_1^1 &= 9a - 6 \\ y_1^3 &= -\frac{1}{3}y_1^1 = 2 - 3a. \end{aligned} \tag{3}$$

When is this a CE? Given the prices, firm 2 does not want to produce. We must check that firm 1's output is nonnegative. From equation (3), this is a CE when $a \geq \frac{2}{3}$.

Case 2: firm 2 and firm 3 produce. Then we have $p^2 = \frac{1}{3}$ and $p^1 + p^2 = \frac{1}{2}$, which implies that the equilibrium price vector is $(\frac{1}{6}, \frac{1}{3}, 1)$. From (1) and (2), we have $x^1 = 6a$ and $x^2 = 3(1-a)$. Since all of good 1 is produced by firm 3, market clearing implies $y_3^2 = 6a$. Therefore, $y_3^1 = 6a$ and $y_3^3 = (-\frac{1}{2})y_3^2 = -3a$. Finally, market clearing for good 2 implies

$$\begin{aligned} 3(1-a) &= y_2^2 + 6a, \text{ so we have} \\ y_2^2 &= 3 - 9a \\ y_2^3 &= -\frac{1}{3}y_2^2 = 3a - 1. \end{aligned} \tag{4}$$

When is this a CE? Given the prices, firm 1 does not want to produce. We must check that firm 2's output is nonnegative. From equation (4), this is a CE when $a \leq \frac{1}{3}$.

Case 3: only firm 3 produces. Then we have $p^1 < \frac{1}{3}$, $p^2 < \frac{1}{3}$, and $p^1 + p^2 = \frac{1}{2}$. Market clearing of good 3 implies $y_3^3 = -1$, which implies $y_3^1 = y_3^2 = 2$. Market clearing of goods 1 and 2 imply $x^1 = x^2 = 2$. We find the CE prices by substituting the CE consumption into (1) and (2), yielding the CE price vector, $(\frac{a}{2}, \frac{1-a}{2}, 1)$. When is this a CE? We must check that firms 1 and 2 do not want to produce. This requires $p^1 = \frac{a}{2} < \frac{1}{3}$ and $p^2 = \frac{1-a}{2} < \frac{1}{3}$. Thus, this is a CE when $\frac{1}{3} < a < \frac{2}{3}$.

2. (30 points)

The following pure-exchange economy has 2 consumers, 2 states of nature, and one physical commodity per state of nature. For $i = 1, 2$, consumer i is a von Neumann-Morgenstern expected utility maximizer with the Bernoulli utility function $u_i(x_i^s) = \log(x_i^s)$. The probability of state 1 is $\frac{1}{4}$ and the probability of state 2 is $\frac{3}{4}$. Consumer 1 has the initial endowment vector, $\omega_1 = (1, 0)$, and consumer 2 has the initial endowment vector, $\omega_2 = (0, 2)$.

(a) (10 points) Before the state of nature is observed, consumers trade state-contingent commodities. Define a competitive equilibrium for this economy.

(b) (15 points) Compute the competitive equilibrium price vector and allocation.

(c) (5 points) Is the allocation from part (b) Pareto optimal? Very briefly explain.

Answer:

(a) A competitive equilibrium is a price vector, (p^1, p^2) and an allocation, $(x_1^1, x_1^2, x_2^1, x_2^2)$, such that

(i) (x_1^1, x_1^2) solves

$$\begin{aligned} & \max \frac{1}{4}\log(x_1^1) + \frac{3}{4}\log(x_1^2) \\ & \text{subject to} \\ p^1 x_1^1 + p^2 x_1^2 &\leq p^1 \\ x_1 &\geq 0, \end{aligned}$$

(ii) (x_2^1, x_2^2) solves

$$\begin{aligned} & \max \frac{1}{4}\log(x_2^1) + \frac{3}{4}\log(x_2^2) \\ & \text{subject to} \\ p^1 x_2^1 + p^2 x_2^2 &\leq 2p^2 \\ x_2 &\geq 0, \end{aligned}$$

(iii) markets clear:

$$\begin{aligned} x_1^1 + x_2^1 &\leq 1 \\ x_1^2 + x_2^2 &\leq 2. \end{aligned}$$

(b) Due to monotonicity, the budget constraints and market clearing conditions will hold as equalities. Normalize $p^2 = 1$. Simultaneously solving the budget equation for consumer 1 and the marginal rate of substitution condition,

$$\frac{\frac{1}{4}x_1^2}{\frac{3}{4}x_1^1} = p^1,$$

we have

$$\begin{aligned} x_1^1 &= \frac{1}{4} \\ x_1^2 &= \frac{3p^1}{4}. \end{aligned}$$

Using the budget equation and MRS condition for consumer 2, we can solve for consumer 2's demand functions,

$$\begin{aligned} x_2^1 &= \frac{1}{2p^1} \\ x_2^2 &= \frac{3}{2}. \end{aligned}$$

Market clearing for good 2 requires

$$\begin{aligned} \frac{3p^1}{4} + \frac{3}{2} &= 2 \\ p^1 &= \frac{2}{3}. \end{aligned}$$

Substituting the price into the demand functions, we have the CE allocation, $(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{2})$.

(c) The allocation from part (b) must be Pareto optimal, since the condition for the FFTWE is met (monotonicity implies local non-satiation). Alternatively, you can show that all resources are allocated and marginal rates of substitution are equal across consumers.

3. (30 points)

Consider the following standard Arrow securities market, with n consumers, S states of nature, and K physical commodities per state. Before the state of nature is revealed, consumers trade a complete set of Arrow securities, after which the state is revealed, securities are redeemed, and we have a spot market. Prices are written as (q, p), where $q = (q^1, ..., q^S)$ are the securities prices and $p = (p(1), ..., p(S))$ are the prices of each good on each spot market. For $i =$

$1, ..., n$, consumer i has the utility maximization problem

$$\max_{x_i, b_i} \sum_{s=1}^{S} \pi_s u_i(x_i(s))$$

subject to

$$\begin{aligned}
\sum_{s=1}^{S} q^s b_i^s &\leq 0 \\
p(s) \cdot x_i(s) &\leq p(s) \cdot \omega_i(s) + b_i^s \quad \text{for } s = 1, ..., S \\
x_i(s) &\geq 0 \quad \text{for } s = 1, ..., S.
\end{aligned}$$

The Bernoulli utility function u_i is strictly concave, strictly monotonic, and continuous. Denote the demand functions for securities and spot market consumption solving the utility maximization problem as $b_i^s[q, p]$ and $x_i(s)[q, p]$.

For each of the following statements, prove the statement if it is true, and find a counterexample if it is false.

(a) (15 points) For all strictly positive prices (q, p), we have

$$\sum_{i=1}^{n} \sum_{s=1}^{S} p(s) \cdot x_i(s)[q, p] = \sum_{i=1}^{n} \sum_{s=1}^{S} p(s) \cdot \omega_i(s).$$

(b) (15 points) For any competitive equilibrium prices, (q^*, p^*), we have

$$\sum_{i=1}^{n} \sum_{s=1}^{S} p^*(s) \cdot x_i(s)[q^*, p^*] = \sum_{i=1}^{n} \sum_{s=1}^{S} p^*(s) \cdot \omega_i(s).$$

Answer:

(a) This statement is false. For the simplest counterexample, suppose $n = 1$, $K = 1$, and $S = 2$. Suppose the probability of each state is one half, and suppose the endowment in each state is 1. Let $u(x(s)) = \log(x(s))$ and let prices be given by $q = (q^1, 1)$ and $p = (1, 1)$. Then the UMP is

$$\max_{x,b} \frac{1}{2} \log(x(1)) + \frac{1}{2} \log(x(2))$$

subject to

$$\begin{aligned}
q^1 b^1 + b^2 &= 0 \\
x(s) &= 1 + b^s \quad \text{for } s = 1, 2.
\end{aligned}$$

To solve the UMP,solve the spot market constraints for the security holdings, and substitute into the securities market constraint, yielding the equivalent problem

$$\begin{aligned}
& \max_x \frac{1}{2}\log(x(1)) + \frac{1}{2}\log(x(2)) \\
& \text{subject to} \\
q^1(x(1)-1) + (x(2)-1) \quad = \quad & 0.
\end{aligned}$$

Solving the Lagrangean problem yields the demand functions

$$\begin{aligned}
x(1)[p,q] &= \frac{1+q^1}{2} \\
x(2)[p,q] &= \frac{1+q^1}{2q^1}.
\end{aligned}$$

In order for the statement to be true, we must have

$$\frac{1+q^1}{2} + \frac{1+q^1}{2q^1} = 2,$$

but this is only true if $q^1 = 1$. (There is nothing special about this counterexample. Almost any example will serve as a counterexample.)

(b) This statement is true. We want to show that

$$\sum_{i=1}^{n}\sum_{s=1}^{S} p^*(s) \cdot x_i(s)[q^*, p^*] = \sum_{i=1}^{n}\sum_{s=1}^{S} p^*(s) \cdot \omega_i(s)$$

holds at equilibrium prices. Reversing the order of summation and subtracting the right hand side expression from both sides, we have the equivalent condition

$$\sum_{s=1}^{S}\sum_{i=1}^{n} p^*(s) \cdot [x_i(s)[q^*, p^*] - \omega_i(s)] = 0.$$

Since $p^*(s)$ does not depend on i, we can factor it out of the inner sum, yielding the equivalent condition,

$$\sum_{s=1}^{S} p^*(s) \cdot \left(\sum_{i=1}^{n} [x_i(s)[q^*, p^*] - \omega_i(s)] \right) = 0. \tag{5}$$

Due to strict monotonicity of the utility functions, the market clearing condition for each spot market requires that, at equilibrium prices, the expressions in parentheses in (5) equal zero for each s. This verifies that the statement is true.