Department of Economics
The Ohio State University
Final Exam Answers–Econ 8712

Prof. Peck
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1. (35 points)

The following economy has two consumers, two firms, and two goods. Good 2 is leisure/labor. For $i = 1, 2$, consumer i has the initial endowment vector, $\omega_i = (0, 1)$, and the utility function,

$$u(x_i^1, x_i^2) = \log(x_i^1) + \log(x_i^2).$$

Consumer 1 owns firm 1 and consumer 2 owns firm 2.

Firm 1 produces good 1 using good 2 as an input. For convenience, denote the (nonnegative) labor input used by firm 1 as L_1. Firm 1's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_1^1 &= (\frac{2}{3}L_1)^{1/2}, \\ \text{where } L_1 &\geq 0. \end{aligned}$$

Firm 2 also produces good 1 using good 2 as an input. For convenience, denote the (nonnegative) labor input used by firm 2 as L_2. Firm 2's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_2^1 &= \frac{1}{4}L_2, \\ \text{where } L_2 &\geq 0. \end{aligned}$$

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (25 points) Compute the competitive equilibrium price vector and allocation.

Answer:

(a) A CE is a price vector (p^1, p^2) and an allocation $(x_1^1, x_1^2, x_2^1, x_2^2, y_1^1, L_1, y_2^1, L_2)$ satisfying

(i) (x_1^1, x_1^2) solves

$$\begin{aligned} & \max \log(x_1^1) + \log(x_1^2) \\ & \text{subject to} \\ p^1x_1^1 + p^2x_1^2 &\leq p^2 + \pi_1 \\ x_1 &\geq 0 \end{aligned}$$

(ii) (x_2^1, x_2^2) solves

$$\begin{aligned} & \max \log(x_2^1) + \log(x_2^2) \\ & \text{subject to} \\ p^1 x_2^1 + p^2 x_2^2 &\leq p^2 + \pi_2 \\ x_2 &\geq 0 \end{aligned}$$

(iii) (y_1^1, L_1) solves

$$\begin{aligned} \max \pi_1 &= p^1 y_1^1 - p^2 L_1 \\ & \text{subject to} \\ y_1^1 &\leq \sqrt{\frac{2}{3} L_1} \\ L_1 &\geq 0 \end{aligned}$$

(iv) (y_2^1, L_2) solves

$$\begin{aligned} \max \pi_2 &= p^1 y_2^1 - p^2 L_2 \\ & \text{subject to} \\ y_2^1 &\leq \frac{L_2}{4} \\ L_2 &\geq 0 \end{aligned}$$

(v) markets clear:

$$\begin{aligned} x_1^1 + x_2^1 &\leq y_1^1 + y_2^1 \\ x_1^2 + x_2^2 + L_1 + L_2 &\leq 2. \end{aligned}$$

(b) Normalize the price of good 2 to be 1 and denote the price of good 1 as p. Let us start with the profit maximization problems.

For firm 1, we can substitute the constraint into the objective, and derive the first order condition,

$$p \cdot \frac{1}{2} (\frac{2}{3} L_1)^{-1/2} \cdot \frac{2}{3} - 1 = 0,$$

from which we derive the supply function and profit (skipping some work you should show),

$$\begin{aligned} L_1 &= \frac{p^2}{6}, \\ y_1^1 &= \frac{p}{3}, \\ \pi_1 &= \frac{p^2}{6}. \end{aligned}$$

For firm 2, whose production function exhibits constant returns to scale, substituting the constraint into the objective we have profit as a function of L_2 given by $p\frac{L_2}{4} - L_2$, so this firm produces positive output if and only if $p = 4$. However, at that price, firm 1 would demand more than all of the economy's endowment of good 2, which is inconsistent with equilibrium. Therefore, firm 2 does does not produce in equilibrium.

The utility maximizing demands for consumer 1 satisfy the first order conditions,

$$\begin{aligned} \frac{x_1^2}{x_1^1} &= p, \\ px_1^1 + x_1^2 &= 1 + \frac{p^2}{6}, \end{aligned}$$

which (skipping some work you should show) yield the demand functions,

$$\begin{aligned} x_1^1 &= \frac{1}{2p} + \frac{p}{12}, \\ x_1^2 &= \frac{1}{2} + \frac{p^2}{12}. \end{aligned}$$

Similarly solving consumer 2's utility maximization problem yields the demand functions

$$\begin{aligned} x_2^1 &= \frac{1}{2p}, \\ x_2^2 &= \frac{1}{2}. \end{aligned}$$

Market clearing for good 2 requires

$$\begin{aligned} \frac{1}{2} + \frac{p^2}{12} + \frac{1}{2} + \frac{p^2}{6} &= 2 \\ p &= 2. \end{aligned}$$

Therefore, the allocation is

$$\begin{aligned} x_1^1 &= \frac{5}{12}, x_1^2 = \frac{5}{6}, x_2^1 = \frac{1}{4}, x_2^2 = \frac{1}{2}, \\ y_1^1 &= \frac{2}{3}, L_1 = \frac{2}{3}, y_2^1 = 0, L_2 = 0. \end{aligned}$$

2. (35 points)

The following pure-exchange economy has 2 consumers, 3 equally likely states of nature, and one physical commodity per state of nature. For $s = 1, 2, 3$ and $i = 1, 2$, denote the consumption of consumer i in state s by x_i^s. The initial endowment vectors are given by $(\omega_1^1, \omega_1^2, \omega_1^3) = (1, 2, 3)$ and $(\omega_2^1, \omega_2^2, \omega_2^3) = (A, 2, 1)$, where A is a positive parameter. For $i = 1, 2$, consumer i is an expected utility maximizer, with a "Bernoulli" utility function given by $u_i(x_i^s) = \log(x_i^s)$. The consumers trade on a complete contingent-commodity market.

(a) (5 points) Define a competitive equilibrium for this economy.
(b) (25 points) Compute the competitive equilibrium price vector and allocation.
(c) (5 points) For what values of the parameter A will the competitive equilibrium allocation Pareto dominate the initial endowment allocation?

Answer:
(a) A C.E. is a price vector (p^1, p^2, p^3) and an allocation $(x_1^1, x_1^2, x_1^3, x_2^1, x_2^2, x_2^3)$ satisfying
(i) (x_1^1, x_1^2, x_1^3) solves

$$\begin{aligned} & \max \frac{1}{3}\log(x_1^1) + \frac{1}{3}\log(x_1^2) + \frac{1}{3}\log(x_1^3) \\ & \text{subject to} \\ p^1 x_1^1 + p^2 x_1^2 + p^3 x_1^3 \leq \; & p^1 + 2p^2 + 3p^3 \\ x_1 \geq \; & 0, \end{aligned}$$

(ii) $(x_2^1, x_2^2, x_2^3))$ solves

$$\begin{aligned} & \max \frac{1}{3}\log(x_2^1) + \frac{1}{3}\log(x_2^2) + \frac{1}{3}\log(x_2^3) \\ & \text{subject to} \\ p^1 x_2^1 + p^2 x_2^2 + p^3 x_2^3 \leq \; & Ap^1 + 2p^2 + p^3 \\ x_2 \geq \; & 0, \end{aligned}$$

(iii) markets clear:

$$\begin{aligned} x_1^1 + x_2^1 &\leq 1 + A \\ x_1^2 + x_2^2 &\leq 4 \\ x_1^3 + x_2^3 &\leq 4. \end{aligned}$$

(b) Normalize $p^3 = 1$. Utility maximization is characterized by the budget equation and the MRS conditions, for $i = 1, 2$,

$$\frac{x_i^3}{x_i^1} = p^1 \quad \text{and} \quad \frac{x_i^3}{x_i^2} = p^2.$$

This allows us to solve (skipping work you should show) for the demand functions

$$\begin{aligned} x_1^1 &= \frac{p^1+2p^2+3}{3p^1}, x_1^2 = \frac{p^1+2p^2+3}{3p^2}, x_1^3 = \frac{p^1+2p^2+3}{3}, \\ x_2^1 &= \frac{Ap^1+2p^2+1}{3p^1}, x_2^2 = \frac{Ap^1+2p^2+1}{3p^2}, x_2^3 = \frac{Ap^1+2p^2+1}{3}. \end{aligned}$$

To solve for the equilibrium prices, market clearing for good 2 implies

$$\begin{aligned} \frac{p^1+2p^2+3}{3p^2} + \frac{Ap^1+2p^2+1}{3p^2} &= 4, \quad \text{or} \\ p^1(A+1)+4 &= 8p^2. \end{aligned} \tag{1}$$

Market clearing for good 3 implies

$$\begin{aligned} \frac{p^1+2p^2+3}{3} + \frac{Ap^1+2p^2+1}{3} &= 4, \quad \text{or} \\ p^1(A+1)+4p^2 &= 8. \end{aligned} \tag{2}$$

Subtracting (1) from (2) yields

$$\begin{aligned} 4p^2-4 &= 8-8p^2, \\ p^2 &= 1. \end{aligned}$$

Substituting $p^2=1$ into (2) yields

$$p^1 = \frac{4}{A+1}.$$

Substituting the equilibrium prices into the demand functions and simplifying, we have the C.E. allocation,

$$\begin{aligned} x_1^1 &= \frac{5A+9}{12}, x_1^2 = \frac{5A+9}{3(A+1)}, x_1^3 = \frac{5A+9}{3(A+1)}, \\ x_2^1 &= \frac{7A+3}{12}, x_2^2 = \frac{7A+3}{3(A+1)}, x_2^3 = \frac{7A+3}{3(A+1)}. \end{aligned}$$

(c) Because each consumer can afford his/her initial endowment and chooses not to demand it, we know that each consumer's utility maximizing bundle must be weakly preferred to his/her initial endowment. By concavity, x_i will be strictly preferred to ω_i (for consumer i) unless $x_i = \omega_i$ holds. Clearly this is impossible: for example, $x_1^2 = x_1^3$ but $\omega_1^2 \neq \omega_1^3$. Therefore, the C.E. allocation Pareto dominates the initial endowment allocation for all positive A.

3. (30 points)

Recall the notation and definition of a competitive equilibrium for the "Standard" Arrow Securities model with n consumers, S states, and K physical commodities per state:

A C.E. is a set of prices $\{q^s, p(s)\}|_{s=1}^S$, security holdings $\{b_i^s\}|_{s=1}^S|_{i=1}^n$, and consumption $\{x_i^j(s)\}|_{i=1}^n|_{j=1}^K|_{s=1}^S$ satisfying

(i) for $i = 1, ..., n$, $\{x_i^j(s)\}|_{j=1}^K|_{s=1}^S$ solves:

$$\max_{x_i(s), b_i^s} \sum_{s=1}^S \pi_s u_i(x_i(s))$$

subject to

$$\begin{aligned}
\sum_{s=1}^S q^s b_i^s &\leq 0 \\
\sum_{j=1}^K p^j(s) x_i^j(s) &\leq \sum_{j=1}^K p^j(s)\omega_i^j(s) + b_i^s \text{ for all } s, \\
x_i^j(s) &\geq 0,
\end{aligned}$$

(ii) market clearing:

$$\begin{aligned}
\sum_{i=1}^n b_i^s &\leq 0 \text{ for all } s \\
\sum_{i=1}^n x_i^j(s) &\leq \sum_{i=1}^n \omega_i^j(s) \text{ for all } j, s.
\end{aligned}$$

Now consider a **Modified** Arrow Securities Model, where for $s = 1, ..., S$, security s pays s units of account on the state-s spot market, and pays nothing on all of the other spot markets. Everything else is exactly as in the Standard Arrow Securities Model.

(a) (5 points) Define a competitive equibrium for the Modified Arrow Securities Model.

(b) (25 points) Prove that if prices $\{q^{s*}, p^*(s)\}|_{s=1}^S$, security holdings $\{b_i^{s*}\}|_{s=1}^S|_{i=1}^n$, and consumption $\{x_i^{j*}(s)\}|_{i=1}^n|_{j=1}^K|_{s=1}^S$ are a C.E. of the Modified Arrow Securities Model, then the allocation $\{b_i^{s*}\}|_{s=1}^S|_{i=1}^n$ and $\{x_i^{j*}(s)\}|_{i=1}^n|_{j=1}^K|_{s=1}^S$ is a C.E. allocation of the Standard Arrow Securities Model. (In other words, prove there are prices $\{q^{s**}, p^{**}(s)\}|_{s=1}^S$ such that $\{q^{s**}, p^{**}(s)\}|_{s=1}^S$, $\{b_i^{s*}\}|_{s=1}^S|_{i=1}^n$, and $\{x_i^{j*}(s)\}|_{i=1}^n|_{j=1}^K|_{s=1}^S$ is a C.E. of the Standard Arrow Securities Model.) As in any proof, explain your reasoning clearly.

Answer:

(a) A C.E. for the Modified Arrow Securities Model is a set of prices $\{q^s, p(s)\}|_{s=1}^S$, security holdings $\{b_i^s\}|_{s=1}^S|_{i=1}^n$, and consumption $\{x_i^j(s)\}|_{i=1}^n|_{j=1}^K|_{s=1}^S$ satisfying

(i) for $i = 1, ..., n$, $\{x_i^j(s)\}|_{j=1}^K|_{s=1}^S$ solves:

$$\max_{x_i(s), b_i^s} \sum_{s=1}^S \pi_s u_i(x_i(s))$$
$$\text{subject to}$$
$$\begin{aligned} \sum_{s=1}^S q^s b_i^s &\leq 0 \\ \sum_{j=1}^K p^j(s) x_i^j(s) &\leq \sum_{j=1}^K p^j(s)\omega_i^j(s) + s b_i^s \text{ for all } s, \\ x_i^j(s) &\geq 0, \end{aligned}$$

(ii) market clearing:

$$\begin{aligned} \sum_{i=1}^n b_i^s &\leq 0 \ \text{ for all } s \\ \sum_{i=1}^n x_i^j(s) &\leq \sum_{i=1}^n \omega_i^j(s) \ \text{ for all } j, s. \end{aligned}$$

The only difference from the Standard Model are the terms sb_i^s in the spot market budget constraints.

(b) To prove this statement, we will use the C.E. prices of the Modified Arrow Securities Model to construct the C.E. prices for the Standard Arrow Securities Model. For $s = 1, ..., S$, let

$$\begin{aligned} q^{s**} &= q^{s*} \\ p^{**}(s) &= \frac{p^*(s)}{s}. \end{aligned} \tag{3}$$

At these prices, the utility maximization problem for consumer i in the

Standard Model is

$$\max_{x_i(s),b_i^s} \sum_{s=1}^{S} \pi_s u_i(x_i(s))$$

subject to

$$\begin{aligned}
\sum_{s=1}^{S} q^{s*} b_i^s &\leq 0 \\
\sum_{j=1}^{K} \frac{p^{j*}(s)}{s} x_i^j(s) &\leq \sum_{j=1}^{K} \frac{p^{j*}(s)}{s} \omega_i^j(s) + b_i^s \text{ for all } s, \\
x_i^j(s) &\geq 0.
\end{aligned}$$

If we multiply both sides of the spot market budget constraint by s, we see that the utility maximization problem for consumer i in the Standard Model is exactly the same as in the Modified Model. Therefore, since $(b_i^*, x_i^*(s))$ solves the UMP for the Modified Model, it must solve the UMP for the Standard Model. Since markets clear in the Modified Model, markets must clear in the Standard Model, so prices (3) and the allocation $\{b_i^{s*}\}|_{s=1}^{S}|_{i=1}^{n}$ and $\{x_i^{j*}(s)\}|_{i=1}^{n}|_{j=1}^{K}|_{s=1}^{S}$ form a C.E. to the Standard Model.