Department of Economics
The Ohio State University
Final Exam Questions and Answers–Econ 8712

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1. (30 points)

Consider a pure exchange economy with $n \geq 3$ consumers and $K \geq 2$ commodities. For $i = 1, ..., n$, the utility function of consumer i satisfies strict concavity, differentiability, and strict monotonicity. Let x^* be an allocation that fully allocates all resources. Provide a proof of the following statement.

Statement: *If, given the allocation x^*, no pair of two consumers can find a trade such that both consumers are strictly better off, then the allocation is Pareto optimal.*

That is, show that x^* is Pareto optimal if there do not exist consumers i and h, and after-trade bundles x'_i and x'_h, such that

$$\begin{aligned} u_i(x'_i) &> u_i(x^*_i) \\ u_h(x'_h) &> u_h(x^*_h) \\ x'_i + x'_h &\leq x^*_i + x^*_h. \end{aligned}$$

Answer:

Consider a hypothetical economy with only consumers i and h. For this economy, the allocation, (x^*_i, x^*_j) is Pareto optimal, by the hypothesis of the statement. We showed in class that, with these assumptions, the consumers i and h have equal marginal rates of substitution. (Note: I forgot to specify that the allocation is interior and fully allocates all resources, but no one was hung up on this.) By cycling through hypothetical economies for all pairs of consumers, 1 and 2, 2 and 3, 3 and 4, etc., we know that all consumers have the same marginal rates of substitution at the allocation x^*. Again, with these assumptions, we showed in class that for an interior allocation that fully allocates all resources and has all marginal rates of substitution equalized, the allocation is Pareto optimal.

Note: many of you had large deductions because you proved the converse, which is pretty obvious, that if we have a Pareto optimal allocation, the condition on i and h is satisfied.

2. (35 points)

The following economy has two consumers, two firms, and three goods, where good 3 is labor/leisure. For $i = 1, 2$, consumer i receives utility from goods 1 and 2 only (and therefore inelastically supplies his/her leisure endowment as labor), according to the utility function,

$$u_i = \log(x_i^1) + \log(x_i^2).$$

Each consumer is endowed with one unit leisure, $\omega_i = (0, 0, 1)$. Consumer 1 owns firm 1 and consumer 2 owns firm 2.

Firm 1 produces good 1 using good 3 as an input. For convenience, denote the (nonnegative) quanity of labor demanded by firm 1 as L_1. Firm 1's production function (the boundary of the production set) is given by:

$$y_1^1 = (AL_1)^{1/2},$$

where A is a positive parameter.

Firm 2 has a constant returns to scale technology, where good 3 is used as an input to produce both good 1 and good 2. Letting L_2 denote the (nonnegative) quantity of labor demanded by firm 2, L_2 units of good 3 as an input produce L_2 units of good 1 **and** L_2 units of good 2.

Normalize the price of good 3 to be one, $p^3 = 1$. Calculate the value of the parameter, A, such that the competitive equilibrium price of good 1 is one-third, $p^1 = \frac{1}{3}$.

Answer:

The profit maximization problem of firm 1 can be written as choosing L_1 to maximize

$$p^1(AL_1)^{1/2} - L_1$$

Differentiating and solving, we have

$$\begin{aligned} L_1 &= \frac{A(p^1)^2}{4} \\ y_1^1 &= \frac{Ap^1}{2} \\ \pi_1 &= \frac{A(p^1)^2}{4} \end{aligned}$$

The profit of firm 2 as a function of its input L_2 is given by

$$\pi_2 = p^1 L_2 + p^2 L_2 - L_2.$$

Since firm 2 is the only firm producing good 2 and marginal utility is infinite at zero consumption, firm 2 must be producing a positive quantity in equilibrium. Since firm 2 has a constant returns to scale technology, this implies zero profits and the restriction on prices,

$$p^1 + p^2 = 1. \tag{1}$$

Consumer 1's utility maximization problem is

$$\begin{aligned} & \max \log(x_1^1) + \log(x_1^2) \\ & \text{subject to} \\ p^1 x_1^1 + p^2 x_1^2 &= 1 + \pi_1, \end{aligned}$$

yielding the demand functions

$$\begin{aligned} x_1^1 &= \frac{1+\pi_1}{2p^1} \\ x_1^2 &= \frac{1+\pi_1}{2p^2}. \end{aligned}$$

Consumer 2's utility maximization is similar, except that her firm does not produce profits, yielding demand functions

$$\begin{aligned} x_2^1 &= \frac{1}{2p^1} \\ x_2^2 &= \frac{1}{2p^2}. \end{aligned}$$

To solve for the competitive equilibrium, we will use market clearing equations for goods 1 and 2, along with the restriction on prices given in (1), to solve for the three unknowns p^1, p^2, and L_2. The market clearing equations are

$$\begin{aligned} \frac{1+\frac{A(p^1)^2}{4}}{2p^1} + \frac{1}{2p^1} &= \frac{Ap^1}{2} + L_2 \\ \frac{1+\frac{A(p^1)^2}{4}}{2p^2} + \frac{1}{2p^2} &= L_2. \end{aligned}$$

After some algebra eliminating L_2 and using (1) to eliminate p^2, we have one equation in the remaining variable, p^1, given by

$$\frac{1}{p^1} = \frac{3Ap^1}{8} + \frac{4}{3-2p^1}.$$

This equation is hard to solve for p^1, but since we are told $p^1 = \frac{1}{3}$, we can easily solve for $A = \frac{72}{7}$.

3. (35 points)

The following pure-exchange economy has 2 consumers, 2 *equally likely* states of nature, and two physical commodities per state of nature. For physical commodities $j = 1, 2$, states $s = 1, 2$, and consumers $i = 1, 2$, denote the consumption of consumer i of physical commodity j in state s by $x_i^{j,s}$. The initial endowment vectors are given by $\omega_1 = (1, 1, 1, 1)$ and $\omega_2 = (1, 1, 1, 1)$. For $i = 1, 2$, consumer i is an expected utility maximizer, but with a "Bernoulli" utility function, given in brackets below, that depends on the state. Specifically, overall utility functions, V_1 and V_2, are given by

$$\begin{aligned} V_1 &= \frac{1}{2}[2\log(x_1^{1,1}) + \log(x_1^{2,1})] + \frac{1}{2}[\log(x_1^{1,2}) + \log(x_1^{2,2})], \\ V_2 &= \frac{1}{2}[\log(x_2^{1,1}) + \log(x_2^{2,1})] + \frac{1}{2}[2\log(x_2^{1,2}) + \log(x_2^{2,2})]. \end{aligned}$$

(a) (10 points) Define a competitive equilibrium for this economy, with complete contingent commodity markets.

(b) (20 points) Calculate the competitive equilibrium price vector and allocation.

(c) (5 points) Is the competitive equilibrium allocation Pareto optimal? Explain your reasoning.

Answer:

(a) I am omitting the asterisks and imposing equalities due to monotonicity. A competitive equilibrium is a price vector, $(p^{1,1}, p^{2,1}, p^{1,2}, p^{2,2})$ and an allocation, $(x_1^{1,1}, x_1^{2,1}, x_1^{1,2}, x_1^{2,2}, x_2^{1,1}, x_2^{2,1}, x_2^{1,2}, x_2^{2,2})$, such that

(i) $(x_1^{1,1}, x_1^{2,1}, x_1^{1,2}, x_1^{2,2})$ solves

$$\begin{aligned} & \max \frac{1}{2}[2\log(x_1^{1,1}) + \log(x_1^{2,1})] + \frac{1}{2}[\log(x_1^{1,2}) + \log(x_1^{2,2})] \\ & \text{subject to} \\ p^{1,1}x_1^{1,1} + p^{2,1}x_1^{2,1} + p^{1,2}x_1^{1,2} + p^{2,2}x_1^{2,2} &= p^{1,1} + p^{2,1} + p^{1,2} + p^{2,2} \\ x_1 &\geq 0 \end{aligned}$$

(ii) $(x_2^{1,1}, x_2^{2,1}, x_2^{1,2}, x_2^{2,2})$ solves

$$\begin{aligned} & \max \frac{1}{2}[\log(x_2^{1,1}) + \log(x_2^{2,1})] + \frac{1}{2}[2\log(x_2^{1,2}) + \log(x_2^{2,2})] \\ & \text{subject to} \\ p^{1,1}x_2^{1,1} + p^{2,1}x_2^{2,1} + p^{1,2}x_2^{1,2} + p^{2,2}x_2^{2,2} &= p^{1,1} + p^{2,1} + p^{1,2} + p^{2,2} \\ x_2 &\geq 0 \end{aligned}$$

(iii) market clearing:

$$\begin{aligned} x_1^{1,1} + x_2^{1,1} &= 2 \\ x_1^{2,1} + x_2^{2,1} &= 2 \\ x_1^{1,2} + x_2^{1,2} &= 2 \\ x_1^{2,2} + x_2^{2,2} &= 2. \end{aligned}$$

(b) Normalize $p^{2,2} = 1$. For consumer 1, the marginal rate of substitution conditions are

$$\begin{aligned}
\frac{2x_1^{2,2}}{x_1^{1,1}} &= p^{1,1} \\
\frac{x_1^{2,2}}{x_1^{2,1}} &= p^{2,1} \\
\frac{x_1^{2,2}}{x_1^{1,2}} &= p^{1,2}.
\end{aligned}$$

Using the MRS conditions to eliminate all demands except $x_1^{2,2}$, and then substituting back, we have the demand functions,

$$\begin{aligned}
x_1^{1,1} &= \frac{2(p^{1,1}+p^{2,1}+p^{1,2}+1)}{5} \\
x_1^{2,1} &= \frac{p^{1,1}+p^{2,1}+p^{1,2}+1}{5p^{2,1}} \\
x_1^{1,2} &= \frac{p^{1,1}+p^{2,1}+p^{1,2}+1}{5p^{1,2}} \\
x_1^{2,2} &= \frac{p^{1,1}+p^{2,1}+p^{1,2}+1}{5}.
\end{aligned}$$

Going through the same steps for consumer 2, we have the demand functions,

$$\begin{aligned}
x_2^{1,1} &= \frac{p^{1,1}+p^{2,1}+p^{1,2}+1}{5} \\
x_2^{2,1} &= \frac{p^{1,1}+p^{2,1}+p^{1,2}+1}{5p^{2,1}} \\
x_2^{1,2} &= \frac{2(p^{1,1}+p^{2,1}+p^{1,2}+1)}{5p^{1,2}} \\
x_2^{2,2} &= \frac{p^{1,1}+p^{2,1}+p^{1,2}+1}{5}.
\end{aligned}$$

To solve for the prices, let us simplify the algebra by defining $P = (p^{1,1} + p^{2,1} + p^{1,2} + 1)$. Good $(2,2)$ clearing becomes

$$\begin{aligned}
\frac{2P}{5} &= 2, \\
P &= 5.
\end{aligned}$$

Good $(1,1)$ clearing is given by

$$\begin{aligned}
\frac{3P}{5p^{1,1}} &= 2, \\
p^{1,1} &= \frac{3}{2}.
\end{aligned}$$

Good $(2,1)$ clearing is given by

$$\begin{aligned} \frac{2P}{5p^{2,1}} &= 2, \\ p^{2,1} &= 1. \end{aligned}$$

This implies that the price vector is $(p^{1,1}, p^{2,1}, p^{1,2}, p^{2,2}) = (\frac{3}{2}, 1, \frac{3}{2}, 1)$. The allocation is given by

$$\begin{aligned} x_1 &= (\frac{4}{3}, 1, \frac{2}{3}, 1) \\ x_2 &= (\frac{2}{3}, 1, \frac{4}{3}, 1). \end{aligned}$$

(c) The competitive equilibrium allocation is Pareto optimal, because the utility functions are strictly monotonic, and therefore satisfy local non-satiation. We can apply the first fundamental theorem of welfare economics.