Department of Economics
The Ohio State University
Final Exam Questions and Answers–Econ 8712

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Directions: *Answer all questions. An answer is not correct unless you show all work and provide proper explanations. When appropriate, you can use propositions proven in class without proving them here.*

1. (35 points)
Consider a pure exchange economy with 2 consumers and 2 commodities. Consumer 1 has the utility function,

$$u_1(x_1^1, x_1^2) = \log(x_1^1) + \log(x_1^2),$$

and consumer 2 has the utility function,

$$u_2(x_2^1, x_2^2) = \log(x_2^1) + \frac{2}{3}\log(x_2^2).$$

The total aggregate endowment of each good is 1.

(a) (20 points) Find the allocation, x^*, such that (i) x^* is Pareto optimal and (ii) we have $u_1(x_1^{1*}, x_1^{2*}) = u_1(\frac{1}{2}, 1)$.
(b) (15 points) If the Pareto optimal allocation, x^*, from part (a) is the initial endowment allocation and (p^*, x^*) is a competitive equilibrium, what is p^*?

Answer: (a) The allocation must satisfy the condition that the marginal rates of substitution for the two consumers are equal, and the condition that $u_1(x_1^{1*}, x_1^{2*}) = u_1(\frac{1}{2}, 1)$ holds. Using total resources to eliminate the consumption variables for consumer 2, the MRS condition is given by

$$\frac{x_1^{2*}}{x_1^{1*}} = \frac{(1 - x_1^{2*})}{\frac{2}{3}(1 - x_1^{1*})}. \tag{1}$$

The second condition is

$$\log(x_1^{1*}) + \log(x_1^{2*}) = \log(\frac{1}{2}) + \log(1)$$

which can be simplified to

$$x_1^{1*} = \frac{1}{2x_1^{2*}}. \tag{2}$$

Substituting this into (1), cross multiplying, and simplifying yields

$$4(x_1^{2*})^2 + x_1^{2*} - 3 = 0.$$

Using the quadratic equation, the positive root gives $x_1^{2*} = \frac{3}{4}$. From (2), we have $x_1^{1*} = \frac{2}{3}$. The total resources gives consumer 2's consumption, $x_2 = (\frac{1}{3}, \frac{1}{4})$.

(b) If the allocation, x^*, is the initial endowment allocation, then the price ratio, $\frac{p^{1*}}{p^{2*}}$ supporting x^* as a competitive equilibrium must be equal to the marginal rate of substitution of the consumers. Consumer 1's MRS is

$$\frac{x_1^{2*}}{x_1^{1*}} = \frac{\frac{3}{4}}{\frac{2}{3}} = \frac{9}{8}$$

so the price vector is proportional to $(\frac{9}{8}, 1)$.

2. (40 points)

The following economy has one consumer, two firms, and two goods. The consumer has the utility function,

$$u(x^1, x^2) = \log(x^1) + \log(x^2).$$

The consumer has no endowments of goods, $\omega = (0, 0)$, but owns both firms.

Firm 1 produces good 1 using good 2 as an input. Using the standard general equilibrium notation, firm 1's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_1^1 &= (-y_1^2)^{1/2}, \\ \text{where } y_1^1 &\geq 0 \text{ and } y_1^2 \leq 0. \end{aligned}$$

Firm 2 produces good 2 using good 1 as an input. Firm 2's production function (the boundary of the production set) is given by:

$$\begin{aligned} y_2^2 &= (-y_2^1)^{1/2}, \\ \text{where } y_2^2 &\geq 0 \text{ and } y_2^1 \leq 0. \end{aligned}$$

Normalize the price of good 2 to be 1, and denote the price of good 1 by p.

(a) (10 points) Define a competitive equilibrium for this economy.

(b) (20 points) Calculate the competitive equilibrium price and allocation for this economy.

(c) (10 points) Carefully explain the reason for the paradoxical outcome that the economy is endowed with no resources, yet there is a positive quantity of each good produced in the competitive equilibrium. How would you modify the specification of the commodities and the production functions to avoid this paradox?

Answer:

(a) A competitive equilibrium is a price vector, $(p, 1)$, and an allocation, $(x^1, x^2, y_1^1, y_1^2, y_2^1, y_2^2)$, such that

(i) (y_1^1, y_1^2) solves

$$\begin{aligned}
\max \pi_1 &= py_1^1 + y_1^2 \\
& \text{subject to} \\
y_1^1 &= (-y_1^2)^{1/2} \\
y_1^1 &\geq 0 \text{ and } y_1^2 \leq 0
\end{aligned}$$

(ii) (y_2^1, y_2^2) solves

$$\begin{aligned}
\max \pi_2 &= py_2^1 + y_2^2 \\
& \text{subject to} \\
y_2^2 &= (-y_2^1)^{1/2} \\
y_2^2 &\geq 0 \text{ and } y_2^1 \leq 0
\end{aligned}$$

(iii) (x^1, x^2) solves

$$\begin{aligned}
& \max \log(x^1) + \log(x^2) \\
& \text{subject to} \\
px^1 + x^2 &= \pi_1 + \pi_2
\end{aligned}$$

(iv) markets clear

$$\begin{aligned}
x^1 &= y_1^1 + y_2^1 \\
x^2 &= y_1^2 + y_2^2
\end{aligned}$$

Note: equalities are due to the fact that utility is strictly monotonic.

(b) To help avoid math mistakes, I will use the notation $L_2^1 = -y_2^1$ and $L_1^2 = -y_1^2$. Then by substituting the production function into firm 1's profit function, the optimal input solves

$$\max p(L_1^2)^{1/2} - L_1^2.$$

Differentiating and solving, we have

$$\begin{aligned}
L_1^2 &= \frac{p^2}{4}, \text{ and therefore,} \\
y_1^1 &= \frac{p}{2}, \pi_1 = \frac{p^2}{4}.
\end{aligned}$$

Similarly, firm 2's optimal input solves

$$\max (L_2^1)^{1/2} - L_2^1.$$

Differentiating and solving, we have

$$L_2^1 = \frac{1}{4p^2}, y_2^2 = \frac{1}{2p}, \pi_2 = \frac{1}{4p}.$$

The consumer's demand functions are computed from the MRS condition,

$$\frac{x^2}{x^1} = p,$$

and the budget equation,

$$px^1 + x^2 = \frac{p^2}{4} + \frac{1}{4p}.$$

Eliminating one consumption variable from the MRS equation, subsituting into the budget equation, and solving yields the demand functions

$$\begin{aligned} x^1 &= \frac{p}{8} + \frac{1}{8p^2} \\ x^2 &= \frac{p^2}{8} + \frac{1}{8p}. \end{aligned}$$

Use market clearing to solve for p. Using good 1, we have

$$\begin{aligned} \frac{p}{8} + \frac{1}{8p^2} + \frac{1}{4p^2} &= \frac{p}{2} \\ \frac{3}{8p^2} &= \frac{3p}{8} \\ p &= 1 \end{aligned}$$

The allocation is therefore given by $(x^1, x^2, y_1^1, y_1^2, y_2^1, y_2^2) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2})$.

(c) The paradox has nothing to do with the convention that inputs are negative outputs. It is related to the fact that the irreversibility assumption used to prove existence does not hold here, but that is not getting to the heart of the issue. Each firm is using the other firm's output as its own input, and this circularity is not consistent with production as a process that takes time. If inputs must be acquired *before* the output is produced, and not simultaneously as this model specifies, then there would be no inputs available. To correct the model to require time to produce, one could specify two time periods, 0 and 1. A commodity specifies and physical commodity (1 or 2) and a time dimension (0 or 1). Firm 1 uses good (2,0) as an input to produce good (1,1) and firm 2 uses good (1,0) as an input to produce good (2,1). We can model the consumer as receiving utility from consumption in period 1 of goods (1,1) and (2,1). This specification respects the fact that it takes time to produce, and there can be no production since there are no resources available in period 0.

3. (25 points)

The following economy has 2 consumers, 3 states of nature, and one physical commodity per state. For $s = 1, 2, 3$, denote the probability of state s by π_s. For $i = 1, 2$, consumer i is an expected utility maximizer, with a Bernoulli utility function given by $u_i(x_i)$, which is strictly monotonic, strictly concave, and differentiable. Consumer 1's initial endowment vector is $(\omega_1^1, \omega_1^2, \omega_1^3) = (1, 5, 4)$ and consumer 2's initial endowment vector is $(\omega_2^1, \omega_2^2, \omega_2^3) = (5, 4, 2)$.

The market structure consists of a complete set of contingent commodity markets, so all trading occurs before the state is observed. Denote the competitive equilibrium by (p^*, x^*). For the following questions, suppose that you are given the facts that $p^* = (1, 1, 1)$ and $x_1^{1*} = 3$.

(a) (15 points) What is the full equilibrium allocation, x^*? Show your work and explain your reasoning clearly.

(b) (10 points) Carefully explain why $\pi_1 = \pi_3$ must hold.

Answer:

(a) We are given $x_1^{1*} = 3$. Market clearing must hold with equality, so we have $x_2^{1*} = 3$.

Because the total aggregate endowment in both state 1 and state 3 is 6, we showed in class that $x_1^{3*} = x_1^{1*}$ and $x_2^{3*} = x_2^{1*}$. You could prove this by supposing it is not true, where one consumer consumes more in state 1 than in state 3, and the other consumer consumers more in state 3 than in state 1, and using concavity to contradict the requirement that marginal rates of substitution between these two states are equal. Therefore, we have $x_1^{3*} = 3$ and $x_2^{3*} = 3$.

Now we use the price vector, $(1, 1, 1)$, and consumer 1's budget constraint, which is

$$x_1^{1*} + x_1^{2*} + x_1^{3*} = 1 + 5 + 4.$$

Since we know $x_1^{3*} = x_1^{1*} = 3$, the budget constraint gives $x_1^{2*} = 4$. Doing the same calculation for consumer 2, or using market clearing, yields $x_2^{2*} = 5$. This, $x_1^* = (3, 4, 3)$ and $x_2^* = (3, 5, 3)$.

(b) Consumer 1's MRS between states 1 and 3 is equal to the price ratio,

$$\frac{\pi_1 u_1'(3)}{\pi_3 u_1'(3)} = 1$$

and therefore we have $\pi_1 = \pi_3$.