

# Dynamic Competition with Random Demand and Costless Search: A Theory of Price Posting

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## Abstract

This paper studies a dynamic model of perfectly competitive price posting under demand uncertainty. Firms must produce output in advance. After observing aggregate sales in prior periods, firms post prices for their unsold output. Consumers arrive at the market in random order, observe the posted prices, and either purchase at the lowest available price or delay their purchase decision.

Every sequential equilibrium outcome is Pareto optimal. Thus consumers endogenously sort themselves efficiently, with the highest valuations purchasing first. Transactions prices in each period rise continuously, as firms become more optimistic about demand, followed by a market correction. By the last period, prices are market clearing.

## 1 Introduction

This paper studies markets characterized by three important features. First, the good is offered for sale for an extended, but limited length of time, called the “sales season.” Consumers purchase at most once during the sales season, but can optimize in the timing of their purchases. At the end of the sales season the good loses most of its consumption value. Second, because demand is concentrated in the sales season, firms need to produce or schedule capacity in advance, so as to be able to satisfy this demand. Third, aggregate demand is highly uncertain.

Any type of good for which demand is seasonal, either because of customs in purchasing behavior

(e.g. the Christmas season or graduation time), or because its use is related to the weather pattern (e.g. skis or lawn fertilizer), and for which there is a need to smooth production over time fits this mold. Goods or services for which the variability in demand is of higher frequency, such as airline travel, hotels, vacation home rentals, and car or equipment rentals also exhibit these characteristics.

Because in these situations the strength of demand is only gradually revealed over the demand period, firms must solve two difficult problems. First, they must decide how much output to make available for the demand season. The wrong level of output may result in forgone sales, or force firms to mark down their prices excessively. In addition, firms face the difficult problem of how to price their output under partial knowledge of the demand realization. Prices may have to be adjusted upwards over time in order to respond to what appears to be strong demand, and when that guess turns out to be wrong, markdowns or sales may need to characterize the final stages of the demand season. Consumers face the equally challenging task of deciding whether to purchase at the going price, or wait to purchase later in the season. Postponing a purchase has the advantage of being able to clinch a bargain when demand turns out to be weak, but could result in the inability to purchase when the good stocks out.

While the set-up of our model is non-standard, it is nevertheless highly relevant, as a substantial fraction of commerce is traded under the conditions our paper sets forth. In the US, the Christmas shopping season alone accounts for over 18 percent of total retail sales during the year, making November and December the busiest months of the year for retailers.<sup>1,2</sup> Other significant sales seasons include the Back to School period, Graduation, Father's Day and Mother's Day, and the four apparel seasons. Furthermore, for products for which the variability in demand is of higher frequency, it should be noted that annual spending on travel in the US nearly equals the amount Americans spend during the Christmas season.<sup>3</sup>

Our assumptions of production in advance and the presence of significant demand uncertainty are well motivated. The gradual disappearance of low-priced seats for airline travel, sometimes followed by the sudden appearance of bargain tickets, is a phenomenon well-known both to airline customers and travel agents. Similarly, the occurrence of clearance sales near the end of the holiday shopping season is well documented, both in the academic press (see e.g. Warner and Barsky (1995)), and

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<sup>1</sup> "Retail Industry Indicators 2008," National Retail Federation, Washington, D.C.

<sup>2</sup> For many retail sectors, Christmas holiday sales are particularly important. For example, in 2004, 32.2% of jewelry stores' total annual sales and 24.4% of department stores' total annual sales occurred during the holiday season.

<sup>3</sup> "Economic Review of Travel in America, 2008," U.S. Travel Association, Washington, D.C.

in the popular press.<sup>4</sup> Indeed, markdowns have become so rampant in retailing that they now take up more than 35% of department stores' annual sales volume.<sup>5</sup> There is also a preponderance of evidence that consumers are strategic in the timing of their purchases,<sup>6</sup> and that firms spend significant resources to solve their difficult intertemporal pricing problem.<sup>7</sup>

Yet despite the importance of the phenomenon we study, economists have written relatively little on the topic, presumably because of the complicated problem of solving the simultaneous intertemporal arbitrage problems of firms and consumers. The few models that do exist typically have firms producing to inventory (e.g., Danthine (1977), Caplin (1985)), and treat consumer demand in a simplistic fashion. Indeed, the literature invariably postulates either a time-invariant flow demand, or else assumes that consumers are myopic, again resulting in a static demand function. One of the major innovations of our paper is to introduce intertemporal substitutability in demand, by allowing consumers to optimize in the timing of their purchase decisions.

We develop a dynamic trading model, in which aggregate demand is uncertain, and production must occur before demand is realized. There are multiple trading periods, and in each period market participants observe the sales made in prior periods, thereby updating their beliefs about the demand state. Within each period, firms start by posting price for that period, after which any remaining active customers arrive in random order to the market. Consumers have perfect information on product availability and current prices, and decide whether to purchase then or delay their purchase until a future round. The dynamic setting leads to a nontrivial decision problem for firms and consumers. In equilibrium, firms' pricing decisions and inferences about demand are based on perceived purchasing decisions of consumers, and purchasing decisions are based on currently available prices and expectations about future prices, which depend on the purchasing decisions of other consumers during the current round, as well as future pricing decisions. We demonstrate the existence of an equilibrium to our model under a regularity assumption on the distribution of

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<sup>4</sup>See e.g. "Wal-Mart Sparks War Among Big Toy Sellers," Wall Street Journal, October 9, 2008, or "Marks & Spencer fires new shot in Christmas price war," Times Online, December 8, 2008.

<sup>5</sup>"Retail Horizons 2008," Merchandising and Operating Results of Department and Specialty Stores, National Retail Federation, Washington, DC.

<sup>6</sup>"This Weekend's Shopping Deals: Another Black Friday?," Wall Street Journal, December 20, 2007; "The Price Is Right," Anne Kadet, Smart Money, December 10, 2004.

<sup>7</sup>Airlines were the first to develop sophisticated and proprietary software allowing them to adjust prices frequently in response to changes in demand. Big retailers such as Home Depot and J.C. Penney now also use such "Revenue management" or "Markdown Management" systems to manage demand uncertainty. According to a survey of the National Retail Federation earlier this year, fully 21% of retailers are already using such software, and 33% plans to have it installed within the next 18 months ("US Retailers Find New Ways To Fine-Tune Discounts", Wall Street Journal, July, 3, 2009).

demand, and characterize the properties of all such equilibria.

The model generates price dispersion within each period in which the demand state has not yet been revealed, because some firms will post a low price and be certain to sell, while others might post a higher price and only sell when demand is sufficiently high. When aggregate demand depends on a single parameter, the case on which the existing literature has focused exclusively, prices rise over time to reflect firms' increasing optimism about demand. Once demand dries up, firms hold a fire sale. Thus a period of "rational exuberance" is invariably followed by a "market correction." These equilibria have the interesting empirical implication that the markdown price is increasing in the pre-sale price, with no mark-down occurring when demand is sufficiently strong. When aggregate demand depends on several parameters, the equilibria are characterized by a succession of periods of rational exuberance followed by market corrections.

The equilibria to our model also have a surprising theoretical implication: although consumers arrive in a random order, in equilibrium they sort themselves efficiently, with high-valuation consumers purchasing early, and lower-valuation ones postponing their purchase decision (thereby exploiting their option of refusing to purchase in the future should the sale price exceed their valuation). Rather than assuming efficient rationing, we thus show that it arises endogenously in equilibrium! As a consequence, despite the existence of price pre-commitment within each period, the presence of asymmetric information about demand, and the random arrival of customers to the market, the equilibria to our model turn out to be efficient. Thus our model shows how in the presence of demand uncertainty an economy can grope its way towards competitive equilibrium.

Our paper also significantly innovates in its modelling of uncertainty about consumer demand, and in its analysis of the factors that govern optimal consumer purchase decisions. In our model, consumers know their valuation for the good and whether or not they are active, and hence possess private information. Since there is aggregate demand uncertainty, this information is correlated amongst consumers. As a consequence, beliefs about the demand state differ across consumers, and generally also differ from the beliefs held by firms. We describe two novel effects that sort consumers' types into those that purchase at the currently available price, and those that delay their purchase decision: an option value effect, and an information effect. In our model, as a consequence of intertemporal arbitrage by firms, the lowest available current price is an unbiased estimate of the future transaction price, i.e. prices are martingales. If a consumer holds the same beliefs about future

demand as firms, she will prefer to postpone purchasing whenever her valuation is low. Indeed, when the future transaction price exceeds her valuation, she has the option of not purchasing. Second, there is an information effect: our regularity condition ensures that high valuation customers give relatively more weight to high-demand states than other customers, so they have more pessimistic beliefs over future prices, and are more inclined to purchase early.

Our paper adds to a sparse literature on dynamic competition under demand uncertainty. The one-period version of our model is a generalization of Prescott’s (1975) static “hotels” model.<sup>8</sup> Eden (1990, 2009) and Lucas and Woodford (1993) present a quasi-dynamic version of the Prescott model in which firms change prices gradually over time, in response to observed increases in cumulative aggregate sales. However, unlike in our model, both papers assume that consumers cannot return to the market once they have chosen not to purchase.<sup>9</sup> Lazear (1986) presents a dynamic pricing model for a monopolist selling a fashion good (a “designer dress”). The firm faces a sequence of identical potential buyers over time, but is uncertain about their valuation for the good. The dress is initially offered at the static monopoly price, and price is lowered over time to reflect the seller’s increasing pessimism regarding the buyer’s valuation. Our model differs from Lazear’s in a number of important respects: we have competition on the seller side of the market, there is uncertainty about the quantity to be sold, and consumers are fully optimizing.<sup>10</sup> Pashigian (1988) introduces competition into Lazear’s model, and also obtains a declining pattern of prices. Our model bears some resemblance to the literature on durable goods (Stokey (1981), Fudenberg, Levine and Tirole (1985), Gul, Sonnenschein and Wilson (1986), Ausubel and Deneckere (1989)), where consumers are also allowed to intertemporally substitute. However, in that literature there is no production in advance, and firms end up saturating the entire market. Furthermore, the important dimension of demand uncertainty is absent there. Finally, our problem is related to the literature on peak-load pricing under stochastic demand (Crew and Kleindorfer (1986)) or priority pricing (Harris and Raviv (1981)). Unlike in our model, the demand state then is either known when the demand period arrives, or else all customers are present in the market at the same time, and private information on demand can be elicited.

In Section 2, we set up the model. Section 3 studies the one-period version of our model. We

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<sup>8</sup>The static, one-period version of this model has been studied in previous work by Prescott (1975), Bryant (1980), Eden (1990), Dana (1993, 1998, 1999), and Deneckere, Marvel, and Peck (1996).

<sup>9</sup>An equivalent assumption is that firms cannot adjust prices downward once demand dries up.

<sup>10</sup>Lazear’s model assumes that consumers are myopic.

study the dynamic model in Section 4, which contains our characterization and efficiency results. Section 5 concludes. All proofs are contained in an Appendix.

## 2 The Model

We consider a competitive market for which aggregate demand is random, and for which firms must produce output prior to the demand realization. More specifically, there is a continuum of firms, of total measure  $M$ , each of whom can produce one unit of output, at a common cost of  $c$ . On the demand side, there is a continuum of potential customers of total measure  $C < M$ . Each consumer buys at most one unit of the good. To avoid difficult technical issues, we assume that there is a finite number  $n$  of possible consumer types, characterized by the valuations  $v_1 > v_2 > \dots > v_n$ . Since  $n$  can be large, there is little economic loss from this assumption. A consumer with valuation  $v_i$  receives net utility of  $v_i - p$  when purchasing the good at a price  $p$ , and 0 otherwise. We assume that each consumer has an endowment of the numeraire (money) that is greater than their valuation. We let  $C_i$  denote the measure of potential customers of type  $i$ ; thus  $C = \sum_{i=1}^n C_i$ . To make the model interesting, we assume that  $c \leq v_1$  holds. Let  $i^*$  be the lowest valuation type whose valuation is at least  $c$ . To avoid uninteresting cases with multiple competitive equilibria, we assume that  $v_{i^*} \neq c$ .

Demand uncertainty is modelled as follows. Nature first selects the measure  $\gamma_i$  of active consumers of type  $i$  according to a continuous distribution function  $F(\gamma_1, \dots, \gamma_n)$ , whose support is the rectangle  $\prod_{i=1}^n [\underline{\gamma}_i, \bar{\gamma}_i] \subset \mathbf{R}_{++}$ . Let  $\gamma = (\gamma_1, \dots, \gamma_n)$ . Associated with  $F$  is the density  $f(\gamma) > 0$ . Having determined  $\gamma$ , for each  $i$  nature then randomly selects amongst the  $C_i$  potential customers the  $\gamma_i$  that will become active. Thus each potential customer of type  $i$  consumer is chosen to become active with probability  $\gamma_i/C_i$ , independently across consumers. This procedure guarantees that active consumers of the same type are symmetric, ex-ante.

The timing of decisions is as follows. In an initial period,  $t = 0$ , firms first decide whether or not to produce. Nature then chooses the set of active consumers. Subsequently, in each of a finite number of periods  $t = 1, \dots, T$ , with  $T \geq n + 1$ , firms who still have output available decide whether to make this output available in period  $t$ , and if so, what price to post it at. All remaining active customers (those who have not purchased in prior periods) are then put in a queue, in random order, and released sequentially into the market. When a consumer is released into the market, she observes

$a_t$ , the amount of sales that have been made so far in period  $t$ , after which she decides whether or not to purchase at the lowest remaining price. At the end of period  $t$ , all market participants who have not yet transacted observe  $\bar{a}_t$ , the aggregate sales made in period  $t$ , and play proceeds to the next period, unless no output remains to be sold, in which case the game is over.

It is important to note that in this game, consumers have private information about the state of demand, information that is not shared by the rest of the market. Specifically, a consumer of type  $i$  who is selected by nature knows that she is active. Since such a consumer is more likely to be active when  $\gamma_i$  is high, Bayesian updating will shift her beliefs toward higher realizations of  $\gamma_i$ , and through positive correlation between  $\gamma_i$  and  $\gamma_j$  for  $j \neq i$ , towards higher realizations of the demand state  $\gamma$ . Furthermore, since this correlation structure is generally different for different types, these posterior beliefs will differ across active consumers.

Our description of the market therefore defines a dynamic Bayesian Game, and we will look for sequential equilibria (extended to allow for a continuum of players, and a continuum of player strategies). We will look for equilibria in which firms use pure strategies, and in which all consumers of the same type use the same strategy. Since in equilibrium the identities of the producing firms, and the identities of firms posting prices in each period will be indeterminate, we will identify all sequential equilibria with the same aggregate behavior.

Accordingly, on the firm side, a sequential equilibrium will specify an aggregate production quantity  $q^*$ , and for each  $t = 1, \dots, T$ , and each history of sales in prior periods  $(\bar{a}_1, \dots, \bar{a}_{t-1})$  such that the market remains active in period  $t$  (i.e. such that  $\sum_{\tau=1}^{t-1} \bar{a}_\tau < q^*$ ), an aggregate quantity  $a_t^{\max}(\bar{a}_1, \dots, \bar{a}_{t-1})$  offered for sale in period  $t$ ,<sup>11</sup> and a nondecreasing function  $p_t(a_t | \bar{a}_1, \dots, \bar{a}_{t-1}) : [0, a_t^{\max}] \rightarrow \mathbf{R}_+$  indicating the price of the  $a$ -th lowest priced unit offered for sale in period  $t$ . On the consumer side, a sequential equilibrium specifies for each period  $t$  and each  $(\bar{a}_1, \dots, \bar{a}_{t-1})$  such that the market remains active in period  $t$ , and for for each consumer type  $i$ , a function  $\psi_t^i(a_t | \bar{a}_1, \dots, \bar{a}_{t-1}) : [0, a_t^{\max}] \rightarrow [0, 1]$ , indicating the probability with which she will accept the price  $p_t(a_t | \bar{a}_1, \dots, \bar{a}_{t-1})$  if this is the lowest priced unit remaining on the market when she arrives at the

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<sup>11</sup>Without loss of generality, we assume that any output that would never be sold in period  $t$  following the history  $(\bar{a}_1, \dots, \bar{a}_{t-1})$  is just not offered for sale in that period.

market, and a measure  $a_t$  of output has already been sold in period  $t$ .<sup>12,13</sup>

### 3 The Static Model

In this section, we consider the one-period version of the model, in which  $T = 1$ . In essence, this is the model of Prescott (1975), extended to allow for heterogeneous consumers, and arbitrary demand uncertainty. Dana (1999) and Eden (1990) also characterize equilibrium for variants of the model. We significantly generalize their assumption of multiplicative uncertainty and provide a formal proof of inefficiency, but our main contribution lies in the dynamic model analyzed Section 3.

Our justification for analyzing the static model is two-fold. First, the static model provides a benchmark against which we can evaluate the performance of the full-fledged dynamic model. Second, for general  $T$  the solution we derive will be in effect in the final period, whenever the market is still active, and uncertainty about the state of demand remains.

To simplify notation, we will drop the subscript  $T = 1$  in this section. Since in equilibrium all produced output will be offered for sale, and since firms must be able to recoup their ex-ante production cost of  $c$ , the price schedule is a non-decreasing function  $p(\cdot) : [0, q^*] \rightarrow \mathbb{R}_+$  satisfying  $p(0) \geq c$ . Given a price schedule  $p(\cdot)$ , let  $\pi(a)$  denote the probability that the  $a$ 'th lowest priced unit sells. Then the equilibrium is characterized as follows:

**Proposition 1** *There exists a unique sequential equilibrium price function, in which the aggregate output  $q^*$ , the price function  $p(\cdot)$ , and the probability of trade  $\pi(\cdot)$  satisfy:*

$$p(a)\pi(a) - c = 0, \text{ for all } a \leq q^* \tag{1}$$

$$p(0) = c, \text{ and } p(q^*) = v_1, \tag{2}$$

$$\pi(a) = \int_{\{\gamma: \gamma_1 \geq h(a, \gamma_2, \dots, \gamma_i^*)\}} f(\gamma) d\gamma \tag{3}$$

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<sup>12</sup>As is common in the literature (see e.g. Gul, Sonnenschein and Wilson (1986)), we do not specify players' strategies following histories that involve simultaneous deviations. Such histories can never arise following single deviations from a candidate equilibrium profile, and hence do not affect the incentive to deviate.

A single deviation by a firm cannot shrink the range of prices offered in equilibrium, but could lead the consumer to observe a price below  $p_t(0|\bar{a}_1, \dots, \bar{a}_{t-1})$ , or above  $p_t(a_t^{\max}|\bar{a}_1, \dots, \bar{a}_{t-1})$ . For the latter case we may set  $\psi_t^i = 0$  for all  $i$ , and for the former case, we would have  $\psi_t^i = 1$  for all  $i$  for which  $\psi_t^i(0|\bar{a}_1, \dots, \bar{a}_{t-1}) > 0$ , so such a price would be sure to be accepted. Firms would therefore prefer to set  $p_t(0|\bar{a}_1, \dots, \bar{a}_{t-1})$  instead.

<sup>13</sup>Firms only observe the aggregate sales history, so their pricing strategies can only depend on this history. Consumers also observe the prices that were available when they were released from the queue in prior periods, so their strategies could conceivably depend on more than the history  $(\bar{a}_1, \dots, \bar{a}_{t-1})$  and  $a_t$ . However, in deciding whether or not to purchase, the consumer only needs to consider the lowest available current price and the distribution of future prices, and those are completely determined by  $(\bar{a}_1, \dots, \bar{a}_{t-1})$  and  $a_t$ .

where the function  $h(a, \gamma_2, \dots, \gamma_{i^*})$  is increasing in  $a$ , strictly decreasing in  $(\gamma_2, \dots, \gamma_{i^*})$ , and satisfies  $h(a, \underline{\gamma}_2, \dots, \underline{\gamma}_{i^*}) = \underline{\gamma}_1$  for all  $a \leq \sum_{i=1}^{i^*} \underline{\gamma}_i$ .

According to Proposition 1 there is price dispersion, because firms “bet on the demand state”. Since  $\pi(a)$  is decreasing in  $a$ , firms that charge higher prices will be less likely to sell. Furthermore, since competition drives expected profits down to zero, the expected revenue of unit  $a$  is just sufficient to recoup the ex-ante cost of production  $c$ . Units that are sure to sell, i.e.  $a \leq \sum_{i=1}^{i^*} \underline{\gamma}_i$ , are priced at marginal cost. Over the remainder of the range  $(\sum_{i=1}^{i^*} \underline{\gamma}_i, q^*]$  the price function is strictly increasing.

We now address the question of efficiency. Of course, we require feasible allocations to respect the constraint that output must be chosen before demand is realized.

**Definition 1** *A feasible allocation is an aggregate quantity,  $q^*$ , and consumption probabilities for each active consumer type in each aggregate state,  $\phi_i(\gamma)$ , such that:*

$$\sum_{i=1}^n \phi_i(\gamma) \leq q, \text{ for all } \gamma$$

*A feasible allocation is efficient if there is no other feasible allocation yielding higher surplus, where surplus is given by*

$$\int \left[ \sum_{i=1}^n v_i \gamma_i \phi_i(\gamma) \right] f(\gamma) d\gamma - cq^*.$$

We then have:

**Proposition 2** *The equilibrium of the static model is not efficient if  $n > 1$ .<sup>14</sup>*

There are three sources of inefficiency. For some realizations of  $\alpha$ , there will be unsold goods and low valuation consumers who do not purchase. This is because the market price cannot adjust to dispose of the excess supply. The second source of inefficiency is that there will be demand states in which, due to the random arrival of consumers, some lower-valuation consumers can receive the good while some higher-valuation consumers do not. The third source of inefficiency is that the aggregate quantity of output produced is generally not the quantity that a planner would produce, if the planner could also distribute output as desired.

The proof of Proposition 2 relies on the first source of inefficiency described above, based on the fact that prices cannot adjust to allow low valuation consumers to purchase when demand turns

<sup>14</sup>As noted by Prescott (1975), the equilibrium of the static model is efficient when  $n = 1$ .

out to be low. As we show in the next section, the dynamic model allows firms to adjust prices so that unsold output can be allocated to later periods. More surprisingly, under some conditions, consumers will choose to sort themselves efficiently over time.

## 4 The Dynamic Model

Equilibrium in the dynamic model with  $T \geq n + 1$  is considerably more complicated than in the static model. The presence of multiple sales periods gives consumers intertemporal substitution possibilities and firms intertemporal arbitrage opportunities that are absent in the static model. When faced with the lowest remaining price  $p_t(a_t|\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{t-1})$  in some period  $t < T$ , a consumer whose valuation exceeds  $p_t(a_t|\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{t-1})$  may decide to postpone purchasing, in the hopes of clinching a lower future price. The benefits of such a strategy depend both on the likelihood of future stockouts, and the prices that would prevail if a stockout does not occur. As explained in the model section, a consumer's assessment of the likelihood of these events is based upon private information about the state of demand, given the available public information  $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{t-1}, a_t)$ . Firms no longer need to put up for sale all the available output in the first period, instead reserving some of it for sale in future periods. Furthermore, firms can dynamically adjust the pricing of unsold output upon the basis of publicly observed information.

Indeed, solving for equilibrium becomes a very arduous task. Prices in future periods will depend upon the residual demand and supply remaining at the end of the current period. Both of these in turn depend upon consumers' purchase behavior in the current period, and this behavior is governed by expectations of future prices. In principle, it is possible to simplify the computation of equilibrium by using backward induction. However, backward induction is made difficult by the high dimensionality of the state variable, the number of consumers of each type remaining in every possible demand state.

Our approach to solving for equilibrium is therefore based upon conjecturing the nature of equilibrium, and then provide conditions under which the proposed candidate equilibrium is indeed the outcome of a sequential equilibrium of the game. In some cases (see Section 4.1, below), we are able to show that this equilibrium outcome is (essentially) the unique possible one.

The candidate equilibrium has the following intuitive structure: in equilibrium consumers of

type  $i$ , those with the valuation  $v_i$ , purchase in period  $i$  if output is available at any price less than or equal to  $v_i$ . As a consequence, purchase behavior in period  $i$  will reveal  $\gamma_i$ , so that the demand state is gradually revealed over time.

Let  $D(p, \gamma)$  denote the ordinary demand function when the realized state of demand is  $\gamma$ . Thus  $D(v_1, \gamma) = \gamma_1$ ,  $D(v_2, \gamma) = \gamma_1 + \gamma_2$ , and so on. Similarly,  $P(q, \gamma)$  denotes the inverse demand function in demand state  $\gamma$ , and satisfies  $P(\gamma_1, \gamma) = v_1$ ,  $P(\gamma_1 + \gamma_2, \gamma) = v_2$ , and so on. We may now define:

**Definition 2** A ‘successive revelation candidate’ for the dynamic model with  $T \geq n + 1$  periods is characterized as follows:

(i)  $\psi_t^i(a_t | \bar{a}_1, \dots, \bar{a}_{t-1}) = 1$  if  $t \geq i$ ,  $p_t(a_t | \bar{a}_1, \dots, \bar{a}_{t-1}) \leq v_i$ , and  $a_t \leq a_t^{\max}$ ; in all other cases  $\psi_t^i(a_t | \bar{a}_1, \dots, \bar{a}_{t-1}) = 0$ .

(ii) For all  $t = 1, \dots, n$ ,

$$p_t(a_t; \bar{a}_1, \dots, \bar{a}_{t-1}) = E(P(q^*, \gamma) | \gamma_t \geq a_t, \gamma_1 = \bar{a}_1, \dots, \gamma_{t-1} = \bar{a}_{t-1}) = \frac{\int_{a_t}^{\bar{\gamma}_t} \cdots \int_{\gamma_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{t-1}, \gamma_t, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{t-1}, \gamma_t, \dots, \gamma_n) d\gamma_n \cdots d\gamma_t}{\int_{a_t}^{\bar{\gamma}_t} \cdots \int_{\gamma_n}^{\bar{\gamma}_n} f(\bar{a}_1, \dots, \bar{a}_{t-1}, \gamma_t, \dots, \gamma_n) d\gamma_n \cdots d\gamma_t}. \quad (4)$$

(iii)

$$c = \int_{\gamma_1}^{\bar{\gamma}_1} \cdots \int_{\gamma_n}^{\bar{\gamma}_n} P(q^*, \gamma_1, \dots, \gamma_n) f(\gamma_1, \dots, \gamma_n) d\gamma_n \cdots d\gamma_1$$

(iv)  $a_t^{\max}(\bar{a}_1, \dots, \bar{a}_{t-1}) = \min\{\bar{\gamma}_t, q^* - \sum_{i=1}^{t-1} \bar{a}_i\}$ .

If the successive revelation candidate is consistent with sequential equilibrium, it is a successive revelation equilibrium.

Condition (i) says that in period 1 the only consumers who purchase are those with the highest valuation,  $v_1$ . The strong law of large numbers then implies that aggregate sales in period 1,  $\bar{a}_1$ , reveals  $\gamma_1$ . If any output remains to be sold in period 2, aggregate sales  $\bar{a}_2$  reveal  $\gamma_2$ , and so on. Thus, when period  $(n + 1)$  arrives, the demand state  $(\gamma_1, \dots, \gamma_n)$  has been fully revealed, and either no further customers remain, or all available output has been sold. In either case, the game is over. Note, however, that in a perfectly competitive market, the price in period  $(n + 1)$  would then just be the market clearing price on the residual demand curve. Thus, if no demand remains, we have  $p_{n+1} = 0$ , and if no output remains  $p_{n+1}$  equals the highest valuation on the residual demand curve. Importantly, because consumers endogenously sort themselves efficiently over time, this price equals

the market clearing price  $P(q^*, \gamma)$ . Thus we could say that  $p_{n+1}(\gamma) = P(q^*, \gamma)$ . Condition (ii) says that in every period  $t < n$ , and for every public history  $(\bar{a}_1, \dots, \bar{a}_{t-1}, a_t)$  the price  $p_t$  is the expected value of the final price  $p_{n+1}$ , conditional upon the information revealed by the history. Condition (iii) says that the price of the first unit to be sold in period 1,  $p_1(0)$ , must equal the marginal cost of production. Note that this condition uniquely pins down the equilibrium aggregate output  $q^*$ . Finally, condition (iv) says that the amount of output made available in period  $t$  is sufficient to serve all customers of type  $t$ , provided this is feasible.

Successive revelation equilibria display a remarkable efficiency property. In spite of the fact that consumers of different valuations arrive randomly to the market in each period, consumers *endogenously* sort themselves efficiently, with the highest valuation consumers purchasing in period 1, the second highest valuation consumers in period 2 and so on, until period  $n$  when the lowest valuation consumers purchase. Also, unlike the static model, the dynamic model has the equilibrium price adjusting to information revealed in previous periods, with price falling below  $c$  if demand turns out to be sufficiently small. In fact, we can show that the allocation is efficient. That is, a planner choosing output before demand is realized, and distributing that output to the consumers with the highest valuations, cannot increase total expected surplus.

**Proposition 3** *In a successive revelation equilibrium the resulting allocation is efficient.*

We now turn to conditions under which successive revelation equilibria exist. We start by studying the special degenerate case in which there is a single parameter determining the demand state. One reason for doing so is that this is the case on which the existing literature has exclusively focused. A second reason is that successive revelation equilibria then take on the simplest possible form, allowing a deeper investigation of the set of sequential equilibria.

## 4.1 Single Parameter Demand

### 4.1.1 Existence

Consider the limiting case of our model (in the sense of convergence in distribution, also known as weak convergence) in which there is a single parameter  $\alpha$  determining  $\gamma$ . With some abuse of notation, let us denote the measure of type  $i$  consumers when the state of nature equals  $\alpha$  by the function  $\gamma_i(\alpha)$ . We assume that  $\alpha$  has support on the non-degenerate interval  $[\underline{\alpha}, \bar{\alpha}]$ , on which it

has a density  $f(\alpha) > 0$ . Higher  $\alpha$  values are associated with higher demand, i.e.  $\gamma_i(\alpha)$  is strictly increasing in  $\alpha$  for each  $i$ . Thus we have  $\underline{\gamma}_i = \gamma_i(\underline{\alpha})$  and  $\bar{\gamma}_i = \gamma_i(\bar{\alpha})$ . We furthermore assume that the function  $\gamma(\cdot)$  is continuous, so that its graph traces out a 1-dimensional curve in  $\prod_{i=1}^n [\underline{\gamma}_i, \bar{\gamma}_i]$ .

Using the shorthand notation  $P(q, \alpha) = P(q, \gamma(\alpha))$ , condition (ii) in Definition 2 becomes:

$$p_1(a_1) = E(P(q^*, \alpha) | \gamma_1(\alpha) \geq a_1), \text{ for } a_1 \in [0, \min\{\gamma_1(\bar{\alpha}), q^*\}], \quad (5)$$

$$p_t(a_t; \bar{a}_1, \dots, \bar{a}_{t-1}) = P(q^*, \gamma_1^{-1}(\bar{a}_1)), \text{ for all } t > 1, \quad (6)$$

and condition (iii) becomes

$$c = \int_{\underline{\alpha}}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha) d\alpha \quad (7)$$

To understand equation (6) note that the purchase behavior of consumers reveals  $\gamma_1 = \bar{a}_1$  at the end of the first purchase period. Because  $\gamma_1$  is a strictly increasing function of  $\alpha$ , this also reveals the state of nature,  $\alpha = \gamma_1^{-1}(\bar{a}_1)$ . Thus in the case of single parameter demand the successive revelation equilibrium is perfectly revealing in period 1. As a consequence, by the beginning of period 2, it is public information how many consumers of type  $i > 1$  remain, and each price  $p_t$  must clear the market on the residual demand curve. Because the only consumers who previously bought are type 1 consumers, this market clearing price must equal the market clearing price on the original demand curve when the state of nature is  $\alpha$ , i.e.  $P(q^*, \alpha)$ .

From (5), we see that  $p_1(a_1)$  is the expectation of the market clearing price based on the realized  $\alpha$ , given the quantity  $q^*$  and given that the measure of type  $n$  consumers is at least  $a_1$ . Computing the expectation explicitly yields

$$p_1(a_1) = \begin{cases} c, & \text{for } a_1 \in [0, \gamma_1(\underline{\alpha})] \text{ and} \\ \frac{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha) d\alpha}{1 - F(\gamma_1^{-1}(a_1))}, & \text{for } \gamma_1(\underline{\alpha}) \leq a_1 < \min\{\gamma_1(\bar{\alpha}), q^*\}. \end{cases} \quad (8)$$

Denoting the highest market clearing price in period 2 as  $\bar{p}$ , it follows from (8) and (6) that  $\bar{p} = P(q^*, \bar{\alpha})$  holds. Note that it is necessarily the case that  $\bar{p} \leq v_1$ . There are two types of equilibria, based on the  $q^*$  solving (7).

In Case 1 we have  $\gamma_1(\bar{\alpha}) \geq q^*$ . This is the easy case, where there can potentially be enough type 1 consumers to purchase all of the output in period 1. The highest possible price in period 1 is  $v_1$ ,

and we have

$$\bar{p} \equiv P(q^*, \bar{\alpha}) = v_1.$$

If there is strict inequality, i.e.  $\gamma_1(\bar{\alpha}) > q^*$ , then there is a positive probability that the price reaches  $v_1$  and some type 1 consumers are rationed. The demand state is then only revealed to be above a threshold level, but revelation is moot as all output is then sold in period 1. Case 2 is characterized by the inequality  $\gamma_1(\bar{\alpha}) < q^*$ . In this case, all type 1 consumers are able to purchase in period 1, and there will always be output remaining to be sold in period 1. The highest price at which transactions occur is<sup>15</sup>

$$\bar{p} \equiv P(q^*, \bar{\alpha}) < v_1.$$

The reason we know  $\bar{p} < v_1$  occurs is that output is held for consumers with valuation  $v_2$  or lower, so the highest price in period 2, which is  $\bar{p}$  as implied by equation (6), can be at most  $v_2$ .<sup>16</sup>

**Proposition 4** *Suppose that in the single-parameter model*

$$\frac{\gamma_i(\alpha)}{\gamma_1(\alpha)} \text{ is weakly decreasing in } \alpha \text{ for all } i \neq 1. \quad (9)$$

*Then there exists a successive revelation equilibrium to the dynamic game.*

Condition (9) seems reasonable, as it holds whenever demand shifts towards higher valuations in higher demand states, i.e. when demand becomes more inelastic as  $\alpha$  increases. To understand Proposition 4, note that a firm that reserves output for the second period will sell at the price  $P(q^*, \alpha)$ , and hence by (7) make zero expected profits. Meanwhile, the martingale condition (5) ensures that a firm that posts a price  $p_1(a_1)$  in period 1 will also earn zero expected profits.

Existence thus revolves around the optimality of purchase behavior, as embodied in Definition 2(i). There are two effects that induce type 1 consumers to purchase in period 1, and induce other consumers to wait. First, there is an *option-value effect*. Competition by firms ensures that, for any  $a_1$ , the current price,  $p_1(a_1)$ , equals the expected price in period 2, conditional on the currently available public information, so that a consumer who only knows the market information (i.e. does

<sup>15</sup>Substituting  $a_1 = \gamma_1(\bar{\alpha})$  into equation (8) gives  $\frac{0}{0}$ , but l'Hopital's rule yields  $p_1(\gamma_1(\bar{\alpha})) = P(q^*, \bar{\alpha})$ , provided  $P(q^*, \cdot)$  is continuous at  $\alpha = \bar{\alpha}$ . If this is not the case, then the correct formula equals  $\bar{p} = \lim_{\alpha \uparrow \bar{\alpha}} P(q^*, \alpha)$ .

<sup>16</sup>In case 2, there always exists a left neighborhood of  $\bar{\alpha}$  over which  $P(q^*, \alpha) = \bar{p}$ . For such  $\alpha$ , prices rise to  $\bar{p}$  in period 1, after which additional transactions occur at that price until no type 1 customer remain. Thus  $p_1(\cdot)$  contains a flat section at  $\bar{p}$ .

not condition on her type when she is able to purchase at the current price) is indifferent between paying the current price and always purchasing in period 2. However, a consumer whose valuation is above the current price but below  $\bar{p}$  can receive even higher utility by waiting until period 2 and purchasing only when her valuation exceeds  $p_2(\alpha)$ . Thus, consumers for whom  $\bar{p}$  exceeds their valuation strictly prefer to wait until period 2, no matter what price they face in period 1, due to the option value of not purchasing in period 2. Second, there is an *information effect*. Condition (9) ensures that type 1 consumers give relatively more weight to high-demand states than other consumers, so they have more pessimistic beliefs over future prices than other consumers. This makes them strictly prefer to purchase in period 1, while other types strictly prefer to wait.

To understand the information effect, consider a consumer of type  $i$  that is released from the queue and observes the lowest remaining price  $p_1(a_1)$ . She has two pieces of information on which to update the prior  $f(\alpha)$ : that she is selected to be active and is of type  $i$ , and that  $a_1$  units have been sold prior to her arrival. On the basis of this information, she computes the posterior density on the state of nature to be

$$f(\alpha | \text{active type } i, \text{ observe } p_1(a_1)) = \frac{f(\alpha)\gamma_i(\alpha)\frac{1}{\gamma_1(\alpha)}}{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} f(\tilde{\alpha})\gamma_i(\tilde{\alpha})\frac{1}{\gamma_1(\tilde{\alpha})}d\tilde{\alpha}}, \text{ for } \alpha \geq \gamma_1^{-1}(a_1). \quad (10)$$

The numerator in (10) reflects the requirement that the state is  $\alpha$ , that a consumer of type  $i$  was selected to be active, and that there are  $a_1$  consumers of type 1 ahead of this type  $i$  consumer in the queue. For a consumer of type 1, the latter two effects exactly cancel out each other, so she holds the same beliefs about the state of nature as the market does.<sup>17</sup> Thus by the martingale condition purchasing in period 1 is optimal for consumers of type 1. Meanwhile, since  $\frac{\gamma_i(\alpha)}{\gamma_1(\alpha)}$  is decreasing in  $\alpha$ , the expected period 2 price for consumers of type  $i \neq 1$  is strictly below  $p_1$ . As a consequence, all consumers of type  $i > 1$  will strictly prefer not to purchase at  $p_1(a_1)$ .

We can summarize the pricing behavior in the successive revelation equilibrium as follows:

**Proposition 5** *In the equilibrium constructed in Proposition 4, the pattern of prices consists of (1) a constant price equal to marginal cost as the units guaranteed to sell are sold, (2) gradual markups*

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<sup>17</sup>When the state equals  $\alpha$ , each consumer in the pool of measure  $C_i$  is equally likely to be selected to become active, so type  $i$  is selected to be active with probability  $\frac{\gamma_i(\alpha)}{C_i}$ . Consider the event that there are  $a_1$  consumers of type 1 ahead of an active consumer of type  $i$  in the queue when the state is  $\alpha$ . The probability density of this event is just the probability density of the event that when a consumer of type  $i$  gets randomly thrown into a pool of customers of type 1 of total measure  $\gamma_1(\alpha)$ , she ends up in position  $a_1$ , i.e. is given by the factor  $\frac{1}{\gamma_1(\alpha)}$ .

as lower demand states are gradually ruled out, followed by (3) a nonnegative markdown to the market clearing price for the realized state. The markdown price is weakly increasing in the highest pre-markdown price.

**Example 1:** The following example illustrates these features of the equilibrium. Suppose that there are three types of consumers, with respective valuations  $v_1 = 100$ ,  $v_2 = 80$ , and  $v_3 = 60$ . Let there be multiplicative uncertainty, with an equal proportion of consumers of each valuation in every state, so that  $\gamma_1(\alpha) = \gamma_2(\alpha) = \gamma_3(\alpha) = \alpha$ . The state  $\alpha$  is distributed uniformly on  $[\frac{1}{20}, \frac{21}{20}]$ , and the marginal cost is 55. Then we have:

$$P(q^*, \alpha) = \begin{cases} 0 & \text{if } \alpha < \frac{q^*}{3} \\ 60 & \text{if } \frac{q^*}{3} \leq \alpha < \frac{q^*}{2} \\ 80 & \text{if } \frac{q^*}{2} \leq \alpha < q^* \\ 100 & \text{if } q^* \leq \alpha \end{cases}, \quad (11)$$

From (11) we can solve for equilibrium output,  $q^* = 1$ . Prices in the first period are given by:

$$p_1(a_1) = \begin{cases} 55 & \text{if } a_1 < \frac{1}{20} \\ \frac{1100}{21-20a_1} & \text{if } \frac{1}{20} \leq a_1 < \frac{1}{3} \\ \frac{1500-1200a_1}{21-20a_1} & \text{if } \frac{1}{3} \leq a_1 < \frac{1}{2} \\ \frac{1700-1600a_1}{21-20a_1} & \text{if } \frac{1}{2} \leq a_1 \leq 1 \end{cases}. \quad (12)$$

Note that  $p_1(a_1)$  is continuous and strictly increasing for  $a_1 \in (\frac{1}{20}, 1]$ . Only type 1 customers purchase in period 1, so in equilibrium the price will gradually rise until the last such customer arrives, i.e.  $a_1 = \alpha$ . Since the measure of type 1's in the market is always at least  $\frac{1}{20}$ , firms can offer the quantity  $\frac{1}{20}$  at marginal cost, and be guaranteed not to lose any money. Output beyond this level is priced above marginal cost, to compensate firms for the possibility that too few type 1 customers will show up, and any unsold output will have to be disposed of at fire prices in period 2. We also have:

$$p_t(\alpha) = \begin{cases} 0 & \text{if } \alpha < \frac{1}{3} \\ 60 & \text{if } \frac{1}{3} \leq \alpha < \frac{1}{2} \\ 80 & \text{if } \frac{1}{2} \leq \alpha < 1 \\ 100 & \text{if } 1 \leq \alpha \leq \frac{21}{20} \end{cases}, \text{ for all } t \geq 2 \quad (13)$$

Note that our single-parameter model has the implication that price dispersion declines over the sales season (it vanishes starting in period 2). Moreover, if demand is sufficiently low ( $\alpha < \frac{1}{3}$ , in the example) competition to sell in the second period leads to a price that falls below marginal cost.

### 4.1.2 Uniqueness

In this subsection, we investigate whether, under the regularity assumption of Proposition 4, there exist other sequential equilibria to our model, and if so, what properties they exhibit. In any sequential equilibrium, let  $q_t^*(\alpha)$  and  $\gamma_t^i(\alpha)$  respectively denote the measure of output and the measure of consumers of type  $i$  remaining at the beginning of period  $t$  when the demand state equals  $\alpha$ . Then we have:

**Proposition 6** *Consider the dynamic model with a single demand parameter and  $T \geq 2$ . Then in any sequential equilibrium there exists a period  $S \leq T-1$  and a vector  $(\hat{a}_1^{\max}, \dots, \hat{a}_S^{\max})$  with  $\hat{a}_S^{\max} > 0$  such that<sup>18</sup>*

- (i) *if  $t \leq S$  is the first period such that  $\bar{a}_t < \hat{a}_t^{\max}$  holds, then the demand state  $\alpha$  is revealed in period  $t$  and all transactions after period  $t$  occur at the market clearing price  $P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha))$ ;*
- (ii) *otherwise, if  $t = S$  and  $\bar{a}_S = \hat{a}_S^{\max}$ , then the market clearing price  $P(q_{S+1}^*(\alpha), \gamma_{S+1}(\alpha))$  is revealed in period  $S$  to be independent of  $\alpha$ , any output remaining after period  $S$  is allocated efficiently, and all transactions after period  $S$  occur at the market clearing price.*

Every sequential equilibrium must therefore share two main properties of the successive revelation equilibrium: over time the demand state is either revealed or it becomes common knowledge that the market clearing price on the residual demand curve is independent of the demand state. In either of these two cases, the remaining output is sold at the market clearing price on residual demand.<sup>19</sup> However, equilibria can differ in the purchase behavior of consumers. Our next result pins down this behavior, under an additional regularity assumption.

**Proposition 7** *Suppose that*

$$\frac{\gamma_i(\alpha)}{\gamma_j(\alpha)} \text{ is strictly increasing in } \alpha, \text{ for all } j > i \tag{14}$$

<sup>18</sup>The number  $\hat{a}_t^{\max}$  is defined recursively by  $\hat{a}_t^{\max} = a_t^{\max}(\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})$ .

<sup>19</sup>If  $\bar{a}_S = \hat{a}_S^{\max}$ , then in period  $S$  the state is only revealed to lie above a threshold value, but in order to achieve efficient allocation of the remaining output (as proposition 6 asserts), the state must be revealed at some future period before  $T$ .

Then in any period  $t \in \{1, \dots, S\}$ , following the equilibrium history  $(\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})$ , and for any  $a_t \in [0, \hat{a}_t^{\max})$  such that  $p_t(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) < P(q_{S+1}^*(\bar{\alpha}), \gamma_{S+1}(\bar{\alpha}))$  holds we have  $\psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) = 0$  for all  $i > 1$ , and  $\psi_t^1(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) > 0$ . Consequently, in any sequential equilibrium the available output is allocated efficiently to consumers, and the produced output level is efficient.

The condition that  $\gamma_i(\alpha)/\gamma_j(\alpha)$  is strictly increasing in  $\alpha$  strengthens the regularity assumption of Proposition 4. Essentially, it says that demand becomes more concave as  $\alpha$  increases. Proposition 7 asserts that until  $a_t$  is sufficiently high that the market clearing price is known to equal  $P(q^*, \bar{\alpha})$ , the only consumer type that purchases with positive probability is the highest valuation type. Meanwhile, there may exist an interval of  $a_S$  such that  $p_S(a_S; (a_1^{\max}, \dots, a_{S-1}^{\max})) = P(q^*, \bar{\alpha})$ , yet  $a_S < a_S^{\max}$ . The state has then not yet been fully revealed, but it is already common knowledge that all further transactions will take place at the ex-post market clearing price  $P(q^*, \bar{\alpha})$ .

The successive revelation equilibrium is not the only one consistent with Proposition 7. First, there exist equilibria in which the state of nature is learned gradually over time. For example, let  $p_1(a_1)$  be identical to (8), up to an arbitrary cutoff price. At some price below  $\bar{p}$ , consumers simply refuse to purchase. The market learns  $\alpha$  in period 1 if demand ceases below the cutoff price, but only learns that demand is above a certain level otherwise. In period 2, if the previous cutoff price is not reached, transactions resume as if no interruption has occurred, and there may be a second cutoff price at which consumers refuse to purchase. Thus, demand can be learned gradually over 2, 3, or more periods, and once  $\alpha$  is learned, the price in remaining periods clears the remaining market. It is also possible to construct a continuum of equilibria that differ from the successive revelation equilibrium in the acceptance probability of type 1 consumers in the first period. For example, we can let  $\psi_1(a_1) = \varsigma \in (0, 1)$  for all  $a_1 \in [0, \min\{\varsigma\gamma_1(\bar{\alpha}), q^*\}]$ . Thus, a fraction  $(1 - \varsigma)$  of type 1's now trade in period 2 at the price  $P(q^*, \alpha)$ . Finally, because there is no discounting in our model, consumers and firms are indifferent to delay, creating an additional source of indeterminacy. For example, it would be an equilibrium for firms to make no output available in the first (or any finite number of periods less than  $T - 2$ ), and then play the successive revelation equilibrium of (8). Also, once  $\alpha$  is learned, the timing of future purchases is indeterminate. These equilibria share the efficiency properties and other qualitative features of the successive revelation equilibrium.

## 4.2 The General Multi-Parameter model

In this section, we assume that the support of the distribution  $F(\gamma)$  is a compact set of positive measure contained in the rectangle  $\prod_{i=1}^n [\underline{\gamma}_i, \bar{\gamma}_i]$ , and present sufficient conditions under which successive revelation is a sequential equilibrium of the dynamic game.

In order for successive revelation to be compatible with equilibrium, it must be that the price functions  $p_t(a_t; \bar{a}_1, \dots, \bar{a}_{t-1})$  from Definition 2(ii) are nondecreasing in  $a_t$ . In the single parameter demand case, this property is guaranteed to hold. Indeed, by equation (8) we have for all  $a_1 \in (\gamma_1(\underline{\alpha}), \min\{\gamma_1(\bar{\alpha}), q^*\})$ :

$$\frac{dp_1}{da_1} = f(\gamma_1^{-1}(a_1)) \frac{d\gamma_1^{-1}(a_1)}{da_1} \frac{\left( \int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} [P(q^*, \alpha) - P(q^*, \gamma_1^{-1}(a_1))] f(\alpha) d\alpha \right)}{(1 - F(\gamma_1^{-1}(a_1)))^2}$$

which is strictly positive whenever  $P(q^*, \gamma_1^{-1}(a_1)) < \bar{p}$ . In general, however, it cannot be guaranteed that all  $p_t(a_t; \bar{a}_1, \dots, \bar{a}_{t-1})$  are nondecreasing in  $a_t$ , unless some additional regularity assumptions are imposed on the distribution of  $\gamma$ . We have:

**Proposition 8** *Suppose that*

$$\frac{\partial^2 \ln f}{\partial \gamma_i \partial \gamma_j} > 0, \text{ for all } i \text{ and } j. \quad (15)$$

*Then  $p_t(a_t; \bar{a}_1, \dots, \bar{a}_{t-1})$  is increasing in  $a_t$ , for all  $(\bar{a}_1, \dots, \bar{a}_{t-1})$  and all  $t$ .*

In economics, log supermodularity of the density function is a familiar assumption from auction theory, where it is called affiliation.

As in the one-parameter demand case, the martingale property of prices ensures that all active firms make zero expected profits, and that no inactive firm can profitably deviate by entering the market. Furthermore, as in the one-parameter case, the beliefs of a consumer of type  $t$  in period  $t$  coincide with the public beliefs, making purchasing in period  $t$  an optimal. To guarantee that it is optimal for any type type  $t' \leq t$  to postpone purchasing until period  $t$ , we make the following assumption:

**Assumption:** For each  $i = 1, \dots, n-1$ , each history  $(\bar{a}_1, \dots, \bar{a}_{i-1})$  satisfying  $\sum_{t=1}^{i-1} \bar{a}_t < q^*$ , and each

$j > i$  we have:

$$\begin{aligned} & \frac{\int_{\underline{\gamma}_n}^{\bar{\gamma}_n} \cdots \int_{a_i}^{\bar{\gamma}_i} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_i \cdots d\gamma_n}{\int_{\underline{\gamma}_n}^{\bar{\gamma}_n} \cdots \int_{a_i}^{\bar{\gamma}_i} f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_i \cdots d\gamma_n} \\ & \geq \frac{\int_{\underline{\gamma}_n}^{\bar{\gamma}_n} \cdots \int_{a_i}^{\bar{\gamma}_i} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) \frac{\gamma_i}{\gamma_i} f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_i \cdots d\gamma_n}{\int_{\underline{\gamma}_n}^{\bar{\gamma}_n} \cdots \int_{a_i}^{\bar{\gamma}_i} \frac{\gamma_i}{\gamma_i} f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_i \cdots d\gamma_n} \end{aligned} \quad (16)$$

It should be emphasized that the regularity condition (16) is there to guarantee that the information effect is always positive. Correspondingly weaker conditions guarantee the existence of a successive revelation equilibrium in cases where the option value effect is positive.

**Proposition 9** *Suppose Assumptions (15) and (16) hold. Then there exists a successive revelation equilibrium to the dynamic game.*

One notable feature of the sequential revelation equilibrium with general multidimensional uncertainty is that along every equilibrium path of the game, there is price dispersion in every period in which the market is active. The game ends no later than in period  $n$ , and ends with either all output sold, or no consumers remaining. At the end of every period  $t \leq n$ , the market finds out one dimension of market uncertainty, the measure of consumer of type  $t$ . Thus transactions are characterized by increasing firm optimism and increasing prices within a period, until sales to type  $t$  cease. This phase might be called “rational exuberance.” Once the market finds out the value of  $\gamma_t \leq \bar{\gamma}_t$ , prices are revised downward, so there is a market correction at the beginning of period  $t+1$ .

To evaluate the strength of condition (16), let us consider the case  $n = 2$ . As in the single parameter model, there will be two cases, depending upon whether  $P(q^*, \bar{\gamma}_1 + \underline{\gamma}_2) = v_1$  or  $P(q^*, \bar{\gamma}_1 + \underline{\gamma}_2) = v_2$ .

First, let us consider the case where  $P(q^*, \bar{\gamma}_1 + \underline{\gamma}_2) = v_2$ , so that the highest period 1 price equals  $v_2$ . Let  $\underline{\gamma}_2(\gamma_1) = \inf\{\gamma_2 : f(\gamma_1, \gamma_2) > 0\}$  and  $\bar{\gamma}_2(\gamma_1) = \sup\{\gamma_2 : f(\gamma_1, \gamma_2) > 0\}$ , and define  $\gamma_1^H = \min\{a_1 : p_1(a_1) = v_2\}$ . Finally define

$$\lambda(\gamma_1) = \frac{\int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2}{\int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} f(\gamma_1, \gamma_2) d\gamma_2}.$$

Then we have

**Proposition 10** *Suppose that  $n = 2$  and  $\lambda(\gamma_1)$  is strictly decreasing in  $\gamma_1$ . Then condition (16) holds for all  $a_1$ , provided there is sufficiently little dispersion in  $\gamma_2$ .*

Since  $\lambda(\gamma_1)$  is the expected value of the ratio  $\gamma_2/\gamma_1$ , the requirement that  $\lambda(\gamma_1)$  be decreasing can be seen as a generalization of the requirement for the single parameter case that  $\gamma_2/\gamma_1$  be decreasing in  $\alpha$ . To see why the condition is needed, note that condition (16) may be rewritten as:

$$\frac{\int_{a_1}^{\bar{\gamma}_1} \mu(\gamma_1) \int_{\underline{\gamma}_2}^{\bar{\gamma}_2} P(q^*, \gamma_1, \gamma_2) f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}{\int_{a_1}^{\bar{\gamma}_1} \lambda(\gamma_1) \int_{\underline{\gamma}_2}^{\bar{\gamma}_2} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1} \leq \frac{\int_{a_1}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2}^{\bar{\gamma}_2} P(q^*, \gamma_1, \gamma_2) f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}{\int_{a_1}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2}^{\bar{\gamma}_2} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1} \quad (17)$$

where

$$\mu(\gamma_1) \equiv \frac{\int_{\underline{\gamma}_2}^{\bar{\gamma}_2} P(q^*, \gamma_1, \gamma_2) \frac{\gamma_2}{\gamma_1} f(\gamma) d\gamma_2}{\int_{\underline{\gamma}_2}^{\bar{\gamma}_2} P(q^*, \gamma_1, \gamma_2) f(\gamma) d\gamma_2}$$

In the proof of Proposition 10, we show that  $\mu(\gamma_1) \geq \lambda(\gamma_1)$ . If  $\lambda(\cdot)$  were a strictly increasing function, then we would have

$$\frac{\int_{a_1}^{\bar{\gamma}_1} \lambda(\gamma_1) \int_{\underline{\gamma}_2}^{\bar{\gamma}_2} P(q^*, \gamma_1, \gamma_2) f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}{\int_{a_1}^{\bar{\gamma}_1} \lambda(\gamma_1) \int_{\underline{\gamma}_2}^{\bar{\gamma}_2} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1} > \frac{\int_{a_1}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2}^{\bar{\gamma}_2} P(q^*, \gamma_1, \gamma_2) f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}{\int_{a_1}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2}^{\bar{\gamma}_2} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}$$

and so (16) would be violated. Therefore we will need to require that  $\lambda(\gamma_1)$  be a decreasing function.

For the multidimensional case, because there is dispersion in  $\gamma_2$ , we generally have  $\mu(\gamma_1) > \lambda(\gamma_1)$ . However, as the dispersion in  $\gamma_2$  collapses,  $\mu$  converges to  $\lambda$ . Therefore, if  $\lambda$  is strictly decreasing, inequality (17) will hold whenever there is sufficiently little dispersion in  $\gamma_2$ .<sup>20</sup>

In practice, when  $\lambda(\cdot)$  is strictly decreasing and condition (49) holds, condition (16) can hold even when the dispersion in  $\gamma_2$  is quite substantial, in which case a successive revelation equilibrium exists, as the following example demonstrates.

**Example 2:** There are two types of consumers, with  $v_1 = 2$  and  $v_2 = 1$ . Then we have

$$P(q^*, \gamma_1, \gamma_2) = \begin{cases} 0 & \text{if } \gamma_1 + \gamma_2 < q^* \\ 1 & \text{if } \gamma_1 \leq q^* \leq \gamma_1 + \gamma_2 \\ 2 & \text{if } q^* < \gamma_1 \end{cases} .$$

<sup>20</sup>As the proof of Proposition 10 indicates, some care needs to be taken to ensure that the required inequality holds for  $a_1$  in a left neighborhood of  $\gamma_1^H$ . This is because condition (17) holds with equality for  $a_1 \geq \gamma_1^H$ , so the limiting argument need to be more subtle.

We assume that the joint distribution of  $(\gamma_1, \gamma_2)$  is uniform over the triangle defined by the vertices  $(\frac{1}{2}, 0)$ ,  $(1, 0)$ , and  $(1, 1)$ , and that  $c = \frac{1}{6}$ . From Definition 2(iii), we have  $q^* = \frac{3}{2}$ . Prices in the first period are given by:

$$p_1(a_1) = \begin{cases} \frac{1}{6(1-a_1)^2} & \text{if } a_1 < \frac{1}{2} \\ \frac{6a_1 - 6(a_1)^2 - 1}{3(1-a_1)^2} & \text{if } \frac{1}{2} \leq a_1 < \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \leq a_1 < \frac{3}{2} \end{cases} .$$

Since  $\bar{a}_1 = \gamma_1$ , prices in the second period can be written as a function of  $a_2$  and  $\gamma_1$ . If  $\gamma_1 < \frac{1}{2}$  holds, supply is known to exceed demand, and we have  $p_2(a_2; \gamma_1) = 0$  for all  $a_2 < \frac{3}{2} - \gamma_1$ . If  $\frac{1}{2} \leq \gamma_1 < \frac{2}{3}$  holds, we have

$$p_2(a_2; \gamma_1) = \begin{cases} \frac{2\gamma_1 - 1}{1 - \gamma_1} & \text{if } a_2 < \frac{1}{2} + \frac{1}{2}\gamma_1 \\ \frac{2\gamma_1 - 1}{2(1 - a_2)} & \text{if } \frac{1}{2} + \frac{1}{2}\gamma_1 \leq a_2 < \frac{3}{2} - \gamma_1 \end{cases} .$$

Finally, if  $\gamma_1 \geq \frac{2}{3}$  holds, we have  $p_2(a_2; \gamma_1) = 1$  for all  $a_2 < \frac{3}{2} - \gamma_1$ . Notice that  $P(q^*, \gamma_1, \gamma_2)$  never exceeds the valuation of any consumer, so there is no option-value effect. We omit the computations that establish condition (16).

The equilibrium is shown in Figure 1, which plots the price as a function of the cumulative sales, either in period 1 or in period 2 after one of five depicted realizations of  $\gamma_1$ . If  $\gamma_1 = \frac{1}{3}$ , the price in period 1 rises continuously to 0.375, at which point type-1 demand ceases and the price collapses to 0 in period 2. If  $\gamma_1 = \frac{55}{100}$ , the price in period 1 rises continuously to 0.798, at which point type-1 demand ceases and the price collapses to 0.222 in period 2; after type-2 sales exceed its lower support of  $\frac{31}{40}$ , the period 2 price rises continuously to 1 (unless type-2 demand ceases before all output is sold). If  $\gamma_1 = \frac{60}{100}$ , the price in period 1 rises continuously to 0.917, at which point type-1 demand ceases and the price collapses to 0.5 in period 2; after type-2 sales exceed its lower support of  $\frac{4}{5}$ , the period 2 price rises continuously to 1. If  $\gamma_1 = \frac{65}{100}$ , the price in period 1 rises continuously to 0.993, at which point type-1 demand ceases and the price collapses to 0.857 in period 2; after type-2 sales exceed its lower support of  $\frac{33}{40}$ , the period 2 price rises continuously to 1. If  $\gamma_1 \geq \frac{2}{3}$  holds, the price in period 1 rises continuously to 1, and stays there until type-1 demand ceases; the price remains at 1 in period 2 until all output is sold.

Let us now turn to the case where  $P(q^*, \bar{\gamma}_1 + \underline{\gamma}_2) = v_1$ , so that the highest period 1 price equals  $v_1$ . We may then redefine  $\gamma_1^H = \min\{a_1 : p_1(a_1) = v_1\}$ , and Proposition 10 continues to hold, as

stated. Because the option value effect is always positive in this case, the dispersion in  $\gamma_2$  can be even more substantial, as the next example demonstrates.

**Example 3:** There are two types of consumers, with  $v_1 = \frac{3}{2}$  and  $v_2 = 1$ . We assume that the joint distribution of  $(\gamma_1, \gamma_2)$  is uniform over the unit square (independent types), and that  $c = \frac{3}{4}$ . From Definition 2(iii), we can compute  $q^* = 0.82288$ . Prices in the first period are given by:

$$p_1(a_1) = \frac{3 - (6 - 2\sqrt{7})a_1 - 2(a_1)^2}{4(1 - a_1)} \quad \text{if } a_1 < 0.82288 \quad .$$

Since  $\bar{a}_1 = \gamma_1$ , prices in the second period can be written as a function of  $a_2$  and  $\gamma_1$ . For  $\gamma_1 < q^*$ , there is output remaining for period 2 and we have

$$p_2(a_2; \gamma_1) = \frac{3 - \sqrt{7} + 2\gamma_1}{2(1 - a_2)} \quad \text{if } a_2 < 0.82288 - \gamma_1 \quad .$$

Notice that  $P(q^*, \gamma_1, \gamma_2)$  sometimes exceeds  $v_2$ , so there is an option-value effect for type-2 consumers. It turns out that condition (16) does not hold for all  $a_1$ , in which case  $p_1(a_1)$  is less than the expected market clearing price for a type-2 consumer. However, the option value effect dominates, and type-2 consumers prefer to wait, so a successive revelation equilibrium exists.

The equilibrium is shown in Figure 2, which plots the price as a function of the cumulative sales, either in period 1 or in period 2 after one of two depicted realizations of  $\gamma_1$ . If  $\gamma_1 = \frac{1}{5}$ , the price in period 1 rises continuously to 0.8682, at which point type-1 demand ceases and the price collapses to 0.3771 in period 2; then the period 2 price rises continuously to 1 (unless type-2 demand ceases before all output is sold). If  $\gamma_1 = \frac{7}{10}$ , the price in period 1 rises continuously to 1.2700, at which point type-1 demand ceases and the price collapses to 0.8771 in period 2; then the period 2 price rises continuously to 1 (unless type-2 demand ceases before all output is sold).

## 5 Conclusion

In this paper, we developed a dynamic trading model of a “sales season” characterized by uncertainty in demand and production in advance. Information about aggregate demand is dispersed across consumers, resulting in informational asymmetries amongst consumers, and between consumers and firms. Firms can adjust their prices from one period to the next, to reflect information about

aggregate demand extracted from observing the aggregate quantity sold in each period. Consumers who have not yet purchased visit the market once each period, and in light of their private information and information extracted from observing the past history of sales and the lowest remaining posted price, decide whether to purchase or wait to try and purchase in later periods. In the equilibria of our model, demand is gradually learned over time, in a recurring pattern of increasing transaction prices within a period, reflecting increased optimism by firms regarding the demand state, followed by a markdown at the start of the next period, reflecting a deterioration of expectations caused by the drying up of sales towards the end of the period. Whenever the length of the demand season is sufficiently long, the state of demand is eventually fully revealed. Even more surprisingly, under some additional regularity conditions, the final allocation and resulting overall equilibrium are efficient.

Our model relies on a number of assumptions should be relatively straightforward to relax. Instead of random arrivals in the queue, we could assume efficient arrivals in the queue. That is, in each round, any remaining consumers are assumed to arrive at the queue in the order of decreasing valuation. With efficient arrivals, the static equilibrium would surely be different. However, provided  $v_{i^*} \neq c$  there would still exist price dispersion, and a variant of Proposition 1 would continue to hold. All that would differ is the precise calculation of the probability of sales  $\pi(a)$  in equation (3). Furthermore, whenever  $v_n < c$ , the equilibrium of the static model would be inefficient, because there is a positive probability of unsold output, and efficiency could be improved by allocating remaining output to type  $n$  customers. More importantly, for the dynamic model considered in the main part of the paper, Proposition 3 shows that the successive revelation equilibria remain equilibria when the random arrival assumption is replaced with efficient arrivals.

It should also be straightforward to relax our assumption that production must occur entirely prior to the demand season. Assuming that firms have an incentive to smooth production over time, they will then build inventories to guard themselves against the possibility of being unable to serve the market when demand turns out to be unusually high. To the best of our knowledge, no one has analyzed such a model when consumers can intertemporally substitute, and optimize the timing of their purchases.<sup>21</sup>

Finally, it would be interesting to extend our analysis to an oligopolistic environment. The pioneering work of Dana (1999) has shown that such an extension is possible in the static context.

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<sup>21</sup>Judd (1996) develops a dynamic pricing model in which firms hold inventories. However, in his model there is neither demand uncertainty, nor intertemporal substitution in consumption.

With oligopolistic competition, the incentive to hold fire sales once the demand state is revealed will be dampened, but will not disappear entirely. Consumers will still allocate themselves optimally over time, but the overall equilibria will fail to be efficient because of the presence of market power.

We leave these extensions for future work.

## 6 Appendix

**Proof of Proposition 1.** First, we will show that each firm must make zero expected profits in any sequential equilibrium. No firm's expected profits can be strictly negative, since it can always decide not to produce. Furthermore, since  $M > C$ , there exists a set of firms of measure no lower than  $M - C$ , whose equilibrium profits must be zero. Suppose now that expected profits were strictly positive for some firm  $m$ . Such a firm must necessarily be active, and charge a price  $p > c$ . By undercutting  $p$  by an amount  $\varepsilon > 0$ , an inactive firm would experience a probability of sale no lower than that of firm  $m$ . Hence for  $\varepsilon$  sufficiently small, this deviation would yield positive expected profits, contradicting equilibrium. We conclude that for any active firm equation (1) must hold, and that any such firm must offer its product for sale, i.e. that  $a^{\max} = q^*$ .

Next, we argue that (2) must hold. Since a firm setting a price  $p < c$  would never be able to recoup its up front production cost, and since a firm pricing more than  $v_1$  would not sell, we must have  $c \leq p(a) \leq v_1$ , for all  $a \in [0, q^*]$ . If we had  $p(0) > c$ , then an inactive firm could profitably deviate by producing, and selling its output at a price in  $(c, p(0))$ , contradicting equilibrium. Equally, if we had  $v_1 > p(q^*)$ , then an inactive firm choosing to produce instead, and post the price  $v_1$ , would earn expected profits of  $v_1\pi(q^*) - c > p(q^*)\pi(q^*) - c = 0$ , again contradicting equilibrium.

It remains to show that (3) holds, and to compute the equilibrium function  $h$ . For this purpose, observe that since consumers only have a single chance to purchase, we must have  $\psi_i(a) = 1$  if  $p(a) < v_i$ , and  $\psi_i(a) = 0$  if  $p(a) > v_i$ . Note in particular that we have  $\psi_i \equiv 0$  for all  $i < i^*$ . Given this consumer behavior, we now claim the following:

**Lemma 1** *There exists a unique function  $h(a, \gamma_2, \dots, \gamma_{i^*})$ , strictly increasing in  $a$  and strictly decreasing in  $(\gamma_2, \dots, \gamma_{i^*})$ , such that unit  $a$  sells in the demand state  $(\gamma_1, \gamma_2, \dots, \gamma_n)$  if and only if  $\gamma_1 \geq h(a, \gamma_2, \dots, \gamma_{i^*})$ .*

**Proof.** Let  $q$  denote a consumer position in the queue of arrivals, so that  $0 \leq q \leq \sum_{i=1}^n \gamma_i$ . Also

let

$$\mu_i = \frac{\gamma_i}{\sum_{j=1}^n \gamma_j}$$

denote the fraction of type  $i$  customers when the state is  $\gamma = (\gamma_1, \dots, \gamma_n)$ . By the strong law of large numbers, the proportion of type  $i$  customers in any interval of length  $\Delta q$  in the queue when the state is  $\gamma$  equals  $\mu_i$ . Hence on any interval of  $a$  on which all  $\psi_i$  are constant, we have

$$\frac{\Delta a}{\Delta q} = \sum_{i=1}^n \mu_i \psi_i(a)$$

for  $\Delta q$  sufficiently small. Upon taking limits as  $\Delta q \rightarrow 0$  we obtain

$$\frac{da}{dq} = \sum_{i=1}^n \mu_i \psi_i(a)$$

Separating this differential equation by variables results in

$$dq = \frac{da}{\sum_{i=1}^n \mu_i \psi_i(a)} \quad (18)$$

Integrating (18) yields an implicit relationship between  $\bar{a}$ , aggregate sales, and the demand state  $\gamma$ :

$$\sum_{j=1}^n \gamma_j = \int_0^{\bar{a}} \frac{da}{\sum_{i=1}^n \mu_i \psi_i(a)}$$

Upon dividing by  $\sum_{j=1}^n \gamma_j$  we obtain

$$1 = \int_0^{\bar{a}} \frac{da}{\sum_{i=1}^n \gamma_i \psi_i(a)}$$

Differentiating this expression w.r.t.  $\gamma_j$ , for  $j \leq i^*$ , and solving for  $\frac{\partial \bar{a}}{\partial \gamma_j}$  yields:

$$\frac{\partial \bar{a}}{\partial \gamma_j} = \frac{1}{\sum_{i=1}^n \mu_i \psi_i(\bar{a})} \int_0^{\bar{a}} \frac{\psi_j(a)}{(\sum_{i=1}^n \mu_i \psi_i(a))^2} da > 0 \quad (19)$$

For  $j > i^*$  we have  $\psi_j(a) = 0$ , so  $\partial \bar{a} / \partial \gamma_j = 0$ . Now unit  $a$  sells in state  $\gamma$  if and only if  $\bar{a}(\gamma) \geq a$ . Therefore the graph of the function  $h$  is the level set  $\bar{a}(\gamma_1, \gamma_2, \dots, \gamma_{i^*}) = a$ . Since  $\frac{\partial \bar{a}}{\partial \gamma_1} > 0$ , the function  $\gamma_1 = h(a, \gamma_2, \dots, \gamma_{i^*})$  exists. Furthermore, the required monotonicity of  $h$  follows from the

implicit function theorem and (19). ■

We may extend the domain of the function to  $a < \bar{a}(\underline{\gamma}_1, \gamma_2, \dots, \gamma_{i^*})$  by setting  $h(a, \gamma_2, \dots, \gamma_{i^*}) = \underline{\gamma}_1$ . According to Lemma 1, the probability of selling unit  $a \in [0, q^*]$  associated with the price function  $p(\cdot)$  is then given by

$$\pi(a) = \int_{\{(\gamma_1, \dots, \gamma_n) : \gamma_1 \geq h(a, \gamma_2, \dots, \gamma_{i^*})\}} f(\gamma_1, \dots, \gamma_n) d\gamma_1 \cdots d\gamma_n$$

In particular, since we have  $h(a, \underline{\gamma}_2, \dots, \underline{\gamma}_{i^*}) = \underline{\gamma}_1$  for all  $a \leq \sum_{j=1}^{i^*} \underline{\gamma}_j$ , we have  $\pi(a) = 1$  and hence  $p(a) = c$  for all  $a \leq \sum_{j=1}^{i^*} \underline{\gamma}_j$ . Units that are sure to sell are priced at marginal cost. Furthermore, since  $\pi(a)$  is strictly decreasing in  $a$  for  $a > \sum_{j=1}^{i^*} \underline{\gamma}_j$ , the function  $p(a)$  is strictly increasing on  $[\sum_{j=1}^{i^*} \underline{\gamma}_j, q^*]$ . Thus the indeterminacy of  $\psi_i(a)$  at  $a$  such that  $p(a) = v_i$  for some  $i$  is immaterial. ■

**Proof of Proposition 2.** From (1) and (2), we have

$$1 - \pi(q^*) = 1 - \frac{c}{v_1} > 0.$$

For sufficiently small positive  $\varepsilon$ , and for all  $\gamma$  such that  $\bar{a}(\gamma) \in (q^* - \varepsilon, q^*)$ , it is the case that (i) there is unsold output in state  $\gamma$ , and (ii) the highest price at which output is sold is near  $v_1$  and therefore greater than  $v_n$ . As transactions prices rise above  $v_n$ , there are type  $n$  consumers who arrive to the head of the queue and choose not to purchase. However, if a planner were to allocate the unsold output to consumers who did not purchase in these states, surplus would be higher. ■

**Proof of Proposition 3.** Notice that for every demand state  $\gamma$ , in any period  $t$  only type  $t$  consumers purchase. Furthermore, either all output is allocated, or else all consumers receive a unit. In other words, we have

$$\phi_i(\gamma) = 1 \quad \text{if} \quad \sum_{j=i}^n \gamma_j \leq q^*, \quad (20)$$

$$\phi_i(\gamma) = 0 \quad \text{if} \quad \sum_{j=i+1}^n \gamma_j \geq q^*. \quad (21)$$

Therefore, given  $q^*$ , total surplus is maximized. It only remains to check that  $q^*$  maximizes total expected surplus.

From (20) and (21), we can write the surplus in state  $\gamma$  as

$$\sum_{i=1}^n \gamma_i \phi_i(\gamma) v_i = \int_{\hat{q}=0}^{q^*} P(\hat{q}, \gamma) d\hat{q}. \quad (22)$$

From (22), it follows that total surplus is given by

$$\int \left[ \int_{\hat{q}=0}^q P(\hat{q}, \gamma) d\hat{q} \right] f(\gamma) d\gamma - cq. \quad (23)$$

Therefore, a necessary condition for total surplus to be maximized is the first-order condition,

$$\int P(q, \gamma) f(\gamma) d\gamma - c = 0. \quad (24)$$

We now show that equation (24) has a unique solution, completing the proof. For any state  $\gamma$  define  $\tau_i = \sum_{j=1}^i \gamma_j$ . Here it is convenient to express  $P(\cdot, \gamma)$  in terms of valuations, as follows.

$$\begin{aligned} P(q; \gamma) &= v_1, & \text{if } q \leq \tau_1 \\ P(q; \gamma) &= v_i, & \text{if } \tau_{i-1} < q \leq \tau_i, \text{ and } 2 < i < n \\ P(q; \gamma) &= 0 & \text{if } \tau_n < q \end{aligned} \quad (25)$$

Let  $R(q) = \int P(q, \gamma) f(\gamma) d\gamma$ . From (25), we can express  $R(q)$  as<sup>22</sup>

$$v_1 + F(\tau_1^{-1}(q))(v_2 - v_1) + \cdots + F(\tau_{i-1}^{-1}(q))(v_{i+1} - v_i) + \cdots + F(\tau_n^{-1}(q))(-v_n). \quad (26)$$

Since the density,  $f$ , is positive over  $\prod_{i=1}^n [\underline{\gamma}_i, \bar{\gamma}_i]$ , we know that  $\partial R / \partial q$  exists and is strictly negative, for  $\underline{\gamma}_1 < q < \sum_{j=1}^n \bar{\gamma}_j$ . Evaluated at  $q = \underline{\gamma}_1$ , we have  $R = v_1$ . Evaluated at  $q = \sum_{j=1}^n \bar{\gamma}_j$ , we have  $R = 0$ . Since  $0 < c < v_n$  holds, there must exist a unique  $q^*$  solving  $c = R(q)$ . ■

#### Proof of Proposition 4.

We start by showing that every active firm makes zero profits in equilibrium. For  $a_1 \leq \gamma_1(\underline{\alpha})$ , the probability of sale in period 1 satisfies  $\pi(a_1) = 1$ , so firms posting the price  $p_1(a_1) = c$  receive zero profits. For  $\gamma_1(\underline{\alpha}) \leq a_1 \leq \min\{\gamma_1(\bar{\alpha}), q^*\}$ , unit  $a_1$  will sell in period 1 in state  $\alpha$  if and only if

<sup>22</sup>In expression (26), it is understood that  $F(\alpha) = 0$  holds for  $\alpha < \underline{\alpha}$ , and that  $F(\alpha) = 1$  holds for  $\alpha > \bar{\alpha}$ .

$\gamma_1(\alpha) \geq a_1$ , so letting  $F_1$  be the marginal distribution of  $\gamma_1$ , we have

$$\pi(a_1) = 1 - F_1(\gamma_1^{-1}(a_1)) \quad \text{and} \quad \bar{a}_1(\alpha) = \gamma_1(\alpha). \quad (27)$$

From (27), we can write

$$E(p_2(\alpha) \mid a_1 \geq \bar{a}_1(\alpha))[1 - \pi(a_1)] = \frac{\int_{\underline{\alpha}}^{\gamma_1^{-1}(a_1)} P(q^*, \alpha) f(\alpha) d\alpha}{\int_{\underline{\alpha}}^{\gamma_1^{-1}(a_1)} f(\alpha) d\alpha} F_1(\gamma_1^{-1}(a_1)) = \int_{\underline{\alpha}}^{\gamma_1^{-1}(a_1)} P(q^*, \alpha) f(\alpha) d\alpha.$$

From (24), (8), and (27), the firm producing unit  $a_1$  receives the following profit:

$$\begin{aligned} \text{profit} &= p_1(a_1)\pi(a_1) + E(p_2(\alpha) \mid a_1 \geq \bar{a}_1(\alpha))[1 - \pi(a_1)] - c \\ &= \left[ \int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha) d\alpha \right] + \int_{\underline{\alpha}}^{\gamma_1^{-1}(a_1)} P(q^*, \alpha) f(\alpha) d\alpha - c = 0. \end{aligned} \quad (28)$$

Thus, all active firms posting a price in period 1 receive zero expected profits. Firms that reserve output until period 2 have an expected revenue of  $\int_{\underline{\alpha}}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha) d\alpha = c$ , and hence also receive nonnegative profits.

Let us now show that no inactive firm can profitably deviate by producing. There are two cases. First, if  $\gamma_1(\bar{\alpha}) \leq q^*$ , then from l'Hopital's rule and equation (8), we have  $\bar{p} = p_1(\gamma_1(\bar{\alpha})) = P(q^*, \bar{\alpha})$ , so the interval of prices at which period 1 transactions can occur in equilibrium is  $[c, P(q^*, \bar{\alpha})]$ . Second, if  $\gamma_1(\bar{\alpha}) > q^*$ , then  $P(q^*, \alpha) = v_n$  holds for  $\alpha \geq \gamma_n^{-1}(q^*)$ . Therefore,

$$\bar{p} = p_1(q^*) = \frac{\int_{\gamma_n^{-1}(q^*)}^{\bar{\alpha}} v_n f(\alpha) d\alpha}{1 - F(\gamma_n^{-1}(q^*))} = v_1,$$

so in this case the interval of prices at which period 1 transactions can occur in equilibrium is  $[c, v_1]$ . Certainly no firm could post a price below  $c$  in period 1. A firm posting a period 1 price between  $c$  and  $\bar{p}$ , and posting the market clearing price in period 2 if output is not sold in period 1, receives zero profits. A firm posting a period 1 price above  $\bar{p}$ , and posting the market clearing price in period 2, receives zero profits. Finally, a firm that (with positive probability) does not post the market clearing price in period 2, if output is not sold in period 1, receives negative profits.

We now show that the purchase policy for each type of consumer is optimal. Consider a type  $i$  consumer in period 1, whose lowest available price is  $p_1(a_1)$  for some  $a_1 \leq a_1^{\max}$ . Conditional on

being active and observing this price, the density function for  $\alpha$  can be calculated using Bayes' rule:

$$f(\alpha|\text{active type } i, \text{ observe } p_1(a_1)) = \frac{f(\alpha)\gamma_i(\alpha)[\gamma_1(\alpha)]^{-1}}{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} \gamma_i(\tilde{\alpha})[\gamma_1(\tilde{\alpha})]^{-1}f(\tilde{\alpha})d\tilde{\alpha}} \quad \text{for } \alpha \geq \gamma_1^{-1}(a_1), \quad (29)$$

$$f(\alpha|\text{active type } i, \text{ observe } p_1(a_1)) = 0 \quad \text{otherwise.}$$

From (29) and the specification  $p_2(\alpha) = P(q^*, \alpha)$ , we can calculate the expected price in period 2, conditional on being an active type 1 consumer and observing  $p_1(a_1)$  in period 1:

$$E(p_2(\alpha)|\text{active type 1, observe } p_1(a_1)) = \frac{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} P(q^*, \alpha)f(\alpha)d\alpha}{1 - F(\gamma_1^{-1}(a_1))}. \quad (30)$$

From (8) and (30), we have  $E(p_2(\alpha)|\text{active type } n, \text{ observe } p_1(a_1)) = p_1(a_1)$ . Thus, the expected price in period 2, conditional on being an active type 1 consumer and observing  $p_1(a_1)$  in period 1, is equal to  $p_1(a_1)$ . Consumers of type 1 are indifferent between purchasing in period 1 and waiting until period 2, for any observed price in period 1, because they are always willing to pay the market clearing price in period 2.

The expected price in period 2, conditional on being an active type  $i$  consumer and observing  $p_1(a_1)$  in period 1, is given by

$$E(p_2(\alpha)|\text{active type } i, \text{ observe } p_1(a_1)) = \frac{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} P(q^*, \alpha)\gamma_i(\alpha)[\gamma_1(\alpha)]^{-1}f(\alpha)d\alpha}{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} \gamma_i(\tilde{\alpha})[\gamma_1(\tilde{\alpha})]^{-1}f(\tilde{\alpha})d\tilde{\alpha}}. \quad (31)$$

A type  $i$  consumer prefers not to purchase if we have  $[p_1(a_1) - E(p_2(\alpha)|\text{active type } i, \text{ observe } p_1(a_1))] \geq 0$ . Since type 1 consumers are indifferent, a sufficient condition is

$$\frac{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} P(q^*, \alpha)f(\alpha)d\alpha}{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} f(\tilde{\alpha})d\tilde{\alpha}} \geq \frac{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} P(q^*, \alpha)\gamma_i(\alpha)[\gamma_1(\alpha)]^{-1}f(\alpha)d\alpha}{\int_{\gamma_1^{-1}(a_1)}^{\bar{\alpha}} \gamma_i(\tilde{\alpha})[\gamma_1(\tilde{\alpha})]^{-1}f(\tilde{\alpha})d\tilde{\alpha}} \quad (32)$$

From assumption (9) and the fact that  $P(q^*, \alpha)$  is weakly increasing and non-constant in  $\alpha$ , inequality (32) follows.<sup>23</sup>■

**Proof of Proposition 6.** We will prove a slightly stronger proposition:

<sup>23</sup>See Wang (1993, lemma 2). Since the ratio in assumption (9) is decreasing, our inequality is the reverse of Wang (1993, lemma 2). Also, since demand is weakly increasing, our result holds as a weak inequality.

**Proposition 11** Consider the dynamic model with a single demand parameter and  $T \geq 2$ . Then in any sequential equilibrium there exists a period  $S \leq T - 1$ , a sequence  $(\hat{a}_1^{\max}, \dots, \hat{a}_S^{\max})$  with  $\hat{a}_S^{\max} > 0$ , and for each  $t = 1, \dots, S$  a function  $\beta_t : [0, \hat{a}_t^{\max}] \rightarrow [\underline{\alpha}, \bar{\alpha}]$  with  $\beta_1(0) = \underline{\alpha}$ ,  $\beta_t(0) = \beta_{t-1}(\hat{a}_{t-1}^{\max})$ , and with  $\beta_t(\cdot)$  strictly increasing whenever  $\hat{a}_t^{\max} > 0$ , such that if the demand state equals  $\alpha$ , then:

(i)  $\alpha \in [\beta_t(0), \beta_t(\hat{a}_t^{\max})]$  implies  $(\bar{a}_1, \dots, \bar{a}_{t-1}) = (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})$ ,  $\bar{a}_t = \beta_t^{-1}(\alpha)$ , and any output remaining after period  $t$  is allocated efficiently, and all transactions after period  $t$  occur at the market clearing price  $P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha))$ ;

(ii)  $\alpha \in [\beta_S(\hat{a}_S^{\max}), \bar{\alpha}]$  implies  $(\bar{a}_1, \dots, \bar{a}_S) = (\hat{a}_1^{\max}, \dots, \hat{a}_S^{\max})$ , and any output remaining after period  $S$  is allocated efficiently, and all transactions after period  $S$  occur at the market clearing price  $P(q_{S+1}^*(\alpha), \gamma_{S+1}(\alpha))$ .

Furthermore, the function

$$P(\alpha) = P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha)), \text{ if } \alpha \in [\beta_t(0), \beta_t(\hat{a}_t^{\max})] \text{ and } t \in \{1, \dots, S\}$$

is non-decreasing in  $\alpha$ , and for any  $t \leq S$  we have

$$p_t(a_t; \hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}) = \frac{\int_{\beta_t(a_t)}^{\bar{\alpha}} P(\alpha) f(\alpha) d\alpha}{\int_{\beta_t(0)}^{\bar{\alpha}} f(\alpha) d\alpha}.$$

We will prove this proposition in a sequence of Lemmata.

**Lemma 2** In any sequential equilibrium of the  $T$ -period game, there exists a non-decreasing sequence of cutoffs  $\{\tilde{\alpha}_t\}_{t=1}^T$  with  $\tilde{\alpha}_1 = \underline{\alpha}$  and  $\underline{\alpha} < \tilde{\alpha}_T$  such that entering period  $t$  the beliefs are either that  $\alpha \geq \tilde{\alpha}_t$ , or else the state has been revealed in some period prior to  $t$ . Furthermore, we either have  $\tilde{\alpha}_T = \bar{\alpha}$  or else  $P(q_T^*(\alpha), \gamma_T(\alpha))$  is independent of  $\alpha$  for all  $\alpha \geq \tilde{\alpha}_T$ .

**Proof.** The proof is by induction. In period 1, the beliefs entering the period are the prior beliefs, so that the statement holds for  $t = 1$ . Next, suppose that the statement holds for all periods  $\tau = 1, \dots, t$ . We will now show that it holds for period  $t + 1$ . Define  $\hat{a}_t^{\max} = \max_{\alpha \geq \tilde{\alpha}_t} \bar{a}_t(\alpha)$ . If  $\hat{a}_t^{\max} > 0$ , then it follows from the proof of Lemma 3 below that  $\bar{a}_t(\alpha)$  is strictly increasing for all  $\alpha$  such that  $\bar{a}_t(\alpha) < \hat{a}_t^{\max}$ . Let  $\tilde{\alpha}_{t+1} = \min\{\alpha : \bar{a}_t(\alpha) = \hat{a}_t^{\max}\}$ . Then for  $\alpha \in [\tilde{\alpha}_t, \tilde{\alpha}_{t+1})$  the state is revealed in period  $t$ , and for  $\alpha < \tilde{\alpha}_t$  the state was revealed in some period prior to  $t$ . Furthermore, for  $\alpha \geq \tilde{\alpha}_{t+1}$  we have  $\bar{a}_t(\alpha) = \hat{a}_t^{\max}$ , so following the observation that  $\bar{a}_t = \hat{a}_t^{\max}$  market participants

will believe that  $\alpha \geq \tilde{\alpha}_{t+1}$ . If we have  $a_t^{\max} = 0$ , then we may set  $\tilde{\alpha}_{t+1} = \tilde{\alpha}_t$ .

It remains to show that we cannot have  $\tilde{\alpha}_T = \underline{\alpha}$ . If this were the case, then the equilibrium in period  $T$  would be described by the equilibrium of the static model. Since the period  $T$  equilibrium price function is nondegenerate, and since the lowest available price in period  $T$  then equals  $c$ , the expected utility of a consumer of type 1 would then be strictly less than  $v_1 - c$ . Thus there exists  $\varepsilon > 0$  such that if a unit of output priced at  $c + \varepsilon$  were made available in period  $T - 1$ , a consumer of type 1 would purchase it. But then any inactive firm could profitably deviate by producing a unit and posting this price in period 1. Since this contradicts equilibrium, we must have  $\tilde{\alpha}_T > \underline{\alpha}$ .

Finally, suppose that  $\tilde{\alpha}_T < \bar{\alpha}$  and that  $P(q_T^*(\alpha), \gamma_T(\alpha))$  is not constant in  $\alpha$ . Then if entering period  $T$  the beliefs are that  $\alpha \geq \tilde{\alpha}_T$ , the period  $T$  equilibrium price function will be nondegenerate, i.e. a positive measure of the remaining output  $q_T^*$  will be priced strictly higher than  $p_T(0; (\hat{a}_1^{\max}, \dots, \hat{a}_{T-1}^{\max}))$ . Suppose that the firm posting the price  $p_T(0; (\hat{a}_1^{\max}, \dots, \hat{a}_{T-1}^{\max}))$  in period  $T$  deviates to posting the price  $p_T(0; (\hat{a}_1^{\max}, \dots, \hat{a}_{T-1}^{\max})) + \varepsilon$  in period  $T - 1$ , for some  $\varepsilon > 0$ . Consider any consumer type  $j$  with  $v_j > p_T(0; (\hat{a}_1^{\max}, \dots, \hat{a}_{T-1}^{\max}))$ . If this consumer arrives at the market in period  $T - 1$  and faces  $p_T(0; (\hat{a}_1^{\max}, \dots, \hat{a}_{T-1}^{\max})) + \varepsilon$  as the lowest remaining price, she will surely purchase when  $\varepsilon$  is sufficiently small, since her expected payoff from waiting is strictly less than  $v_j - p_T(0; (\hat{a}_1^{\max}, \dots, \hat{a}_{T-1}^{\max}))$ . Consequently, the deviating firm would be sure to sell in period  $T - 1$ , thereby securing a net expected revenue strictly greater than  $p_T(0; (\hat{a}_1^{\max}, \dots, \hat{a}_{T-1}^{\max}))$ , which contradicts equilibrium. We conclude that whenever  $\tilde{\alpha}_T < \bar{\alpha}$  it must be that  $P(q_T^*(\alpha), \gamma_T(\alpha))$  is independent of  $\alpha$  for all  $\alpha \geq \tilde{\alpha}_T$ . ■

**Lemma 3** *Consider any sequential equilibrium in which the beliefs entering period  $t$  are that  $\alpha \geq \tilde{\alpha}_t$ , and suppose that  $\hat{a}_t^{\max} > 0$ . Then there exists a strictly increasing and differentiable function  $\bar{a}_t(\alpha)$  on  $[\tilde{\alpha}_t, \tilde{\alpha}_{t+1}]$  s.t.  $\bar{a}_t = \bar{a}_t(\alpha)$  for all  $\alpha \in [\tilde{\alpha}_t, \tilde{\alpha}_{t+1}]$  and  $\bar{a}_t(\tilde{\alpha}_{t+1}) = \hat{a}_t^{\max}$ .*

**Proof.** For every  $a_t < \hat{a}_t^{\max}$  there exists some  $i$  such that  $\psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) > 0$ , for otherwise sales in period  $t$  would equal zero when the measure of output sold in period  $t$  reached  $a_t$ , and hence we would never be able to reach the sales level  $\hat{a}_t^{\max}$ , contradicting the definition of  $\hat{a}_t^{\max}$ .

First, we prove the result if each  $\psi_t^i$  is a simple function, i.e. the range of  $\psi_t^i$  consists of finitely

many values. Let  $q$  denote a position in the period  $t$  queue, so that  $0 \leq q \leq \sum_{i=1}^n \gamma_t^i(\alpha)$ . Also let

$$\mu_t^i(\alpha) = \frac{\gamma_t^i(\alpha)}{\sum_{j=1}^n \gamma_t^j(\alpha)}$$

denote the fraction of type  $i$  customers present in period  $t$  when the state is  $\alpha \geq \tilde{\alpha}_t$ . By the strong law of large numbers, the proportion of type  $i$  customers in any interval of length  $\Delta q$  in the queue when the state is  $\alpha$  equals  $\mu_t^i(\alpha)$ . Hence on any interval of  $a_t$  on which all  $\psi_t^i$  are constant, we have

$$\frac{\Delta a_t}{\Delta q} = \sum_{i=1}^n \mu_t^i(\alpha) \psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) \quad (33)$$

for  $\Delta q$  sufficiently small. Upon taking limits as  $\Delta q \rightarrow 0$ , and separating the differential equation by variables, we obtain

$$dq = \frac{da_t}{\sum_{i=1}^n \mu_t^i(\alpha) \psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}))} \quad (34)$$

For any  $\alpha$  that results in  $\bar{a}_t < a_t^{\max}$ , integrating (34) and dividing  $\sum_{j=1}^n \gamma_t^j(\alpha)$  then yields

$$1 = \int_0^{\bar{a}_t} \frac{da_t}{\sum_{i=1}^n \gamma_t^i(\alpha) \psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}))}$$

Totally differentiating this expression w.r.t  $\alpha$  yields

$$\frac{1}{\sum_{i=1}^n \gamma_t^i(\alpha) \psi_t^i(\bar{a}_t(\alpha); (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}))} \frac{d\bar{a}_t(\alpha)}{d\alpha} - \int_0^{\bar{a}_t} \frac{\sum_{i=1}^n \psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) \frac{d\gamma_t^i(\alpha)}{d\alpha}}{(\sum_{i=1}^n \gamma_t^i(\alpha) \psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})))^2} da_t = 0 \quad (35)$$

Since  $\frac{d\gamma_t^i(\alpha)}{d\alpha} > 0$  (see Lemma 4, below), it follows that  $\bar{a}_t(\alpha)$  is strictly increasing in  $\alpha$ . Hence we may define  $\beta_t = \bar{a}_t^{-1}$ .

Next, for general measurable  $\psi_t^i$ , there exist simple functions  $\psi_{\nu t}^i \leq \psi_t^i$ , such that  $\psi_{\nu t}^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) \uparrow \psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}))$  for every  $a_t$ . Hence for each  $\nu$ , there exists a strictly increasing function  $\beta_t^\nu(a_t)$  such that  $\alpha = \beta_t^\nu(a_t)$  for every  $a_t$ . Furthermore, from (35) we have

$$\frac{d\beta_t^\nu}{da_t}(\bar{a}_{\nu t}) = \left( \sum_{i=1}^n \gamma_t^i(\alpha) \psi_{\nu t}^i(\bar{a}_{\nu t}; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) \int_0^{\bar{a}_t} \frac{\sum_{i=1}^n \psi_{\nu t}^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) \frac{d\gamma_t^i(\alpha)}{d\alpha}}{(\sum_{i=1}^n \gamma_t^i(\alpha) \psi_{\nu t}^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})))^2} da_t \right)^{-1}$$

Upon taking limits as  $v \rightarrow \infty$ , we therefore obtain

$$\frac{d\beta_t}{d\bar{a}_t}(\bar{a}_t) = \left( \sum_{i=1}^n \gamma_t^i(\alpha) \psi_t^i(\bar{a}_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) \int_0^{\bar{a}_t} \frac{\sum_{i=1}^n \psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) \frac{d\gamma_t^i(\alpha)}{d\alpha}}{(\sum_{i=1}^n \gamma_t^i(\alpha) \psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})))^2} da_t \right)^{-1}$$

so  $\alpha = \beta_t(\bar{a}_t)$ , where  $\beta_t$  is a strictly increasing function. We conclude that  $\bar{a}_t$  reveals the state  $\alpha$  whenever  $\bar{a}_t < \hat{a}_t^{\max}$ . ■

**Lemma 4** *Consider any pair of histories  $\alpha', \alpha''$  that are consistent with the history  $(\bar{a}_1, \dots, \bar{a}_{t-1})$ , for some  $t \leq T-1$ . Then  $\alpha'' > \alpha'$  and  $\gamma_t^i(\alpha'') > 0$  imply  $\gamma_t^i(\alpha'') > \gamma_t^i(\alpha')$ .*

**Proof.** In period 1, for  $j = 1, \dots, n$ , we arbitrarily select  $\gamma_j(\alpha')$  type- $j$  consumers and keep track of them by giving them black hats; we give white hats to the remaining  $\gamma_j(\alpha'') - \gamma_j(\alpha')$  type- $j$  consumers. Thus, the queue contains consumers with both color hats, but the queue of black-hat consumers looks exactly like the queue in state  $\alpha'$ . Purchases in period 1 by black-hat type- $j$  consumers in state  $\alpha''$  must be less than or equal to purchases in period 1 by type- $j$  consumers in state  $\alpha'$ . The reason is that purchases by white-hat consumers increase  $a_1$  and thereby crowd out purchases by black-hat consumers. Since  $\bar{a}_1$  is the same in states  $\alpha'$  and  $\alpha''$ , it follows that at the end of period 1, there are more type- $j$  consumers who have not yet purchased in state  $\alpha''$  than in state  $\alpha'$ . Then in period 2, we redistribute hats, where the measure of black hats given to type- $j$  consumers (in state  $\alpha''$ ) is the measure of type- $j$  consumers who have not yet purchased in state  $\alpha'$ . We know that there will be additional consumers of each type who receive white hats. Proceeding to period  $t$ , it follows that  $\gamma_t^i(\alpha'') > 0$  implies  $\gamma_t^i(\alpha'') > \gamma_t^i(\alpha')$ . ■

**Lemma 5** *If in any period  $t < T$  either (i) the demand state is revealed to equal  $\alpha$ , or (ii) it is revealed that the demand state is  $\alpha \geq \tilde{\alpha}_{t+1}$  and  $P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha))$  is independent of  $\alpha$  for all  $\alpha \geq \tilde{\alpha}_{t+1}$ , then the remaining output is allocated efficiently, and all future transactions take place at the price,  $P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha))$ .*

**Proof.** To prove the lemma, we first demonstrate the following two claims.

**Claim 1:** Any unsold output in period  $T$  must sell at the price  $p_T(\alpha) = P(q_T^*(\alpha), \gamma_T(\alpha))$ , and must be allocated efficiently.

**Proof of Claim 1:** If we have  $p_T(\alpha) = P(q_T^*(\alpha), \gamma_T(\alpha)) + \varepsilon$  for some positive  $\varepsilon$ , some firms will not be able to sell their output. Thus, firms sell with probability strictly less than one. Therefore,

by deviating to a price sufficiently close to but less than  $P(q_T^*(\alpha), \gamma_T(\alpha)) + \varepsilon$ , a firm sells for sure and increases its profits, a contradiction.

If we have  $p_T(\alpha) = P(q_T^*(\alpha), \gamma_T(\alpha)) - \varepsilon$  for some positive  $\varepsilon$ , a firm has a profitable deviation to post  $P(q_T^*(\alpha), \gamma_T(\alpha)) - \varepsilon/2$ , a contradiction. Therefore we have  $p_T(\alpha) = P(q_T^*(\alpha), \gamma_T(\alpha))$ . If some output does not sell in period  $T$ , then a firm's probability of sale is strictly less than one, so it could profitably deviate by undercutting and selling for sure. If a positive measure of consumers with valuation greater than  $P(q_T^*(\alpha), \gamma_T(\alpha))$  do not purchase, a firm would have a profitable deviation to post a higher price, contradicting the result that all firms post the same price in period  $T$ . It follows that the allocation is efficient.

**Claim 2:** In periods  $T-1$  and  $T$ , all transactions must occur at the price  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$ , and remaining output must be allocated efficiently.

**Proof of Claim 2:** If no output remains in period  $T$ , Claim 2 follows immediately from the argument of Claim 1, so assume that output remains in period  $T$ . We now show that  $p_T(\alpha) = P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  must hold. From claim 1,  $p_T(\alpha) = P(q_T^*(\alpha), \gamma_T(\alpha))$  will equal  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  if remaining output in period  $T-1$  is allocated efficiently, but could be higher otherwise. Thus,  $p_T(\alpha) \geq P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  must hold. Suppose we have  $p_T(\alpha) > P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$ . Then no firm would post at or below  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  in period  $T-1$ , and since no consumer would be willing to pay more than her valuation in period  $T-1$ , it follows that the market clearing price in period  $T$  must be  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$ , a contradiction.

Since  $p_T(\alpha) = P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  must hold, no firm will post a price lower than  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  in period  $T-1$ . Also, all consumers with valuation greater than  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  must purchase in either period  $T-1$  or  $T$ , because otherwise there would be a profitable deviation for a firm to raise its price in period  $T$ . Therefore, no consumer will purchase in period  $T-1$  at a price strictly greater than  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$ , because output will be available at a lower price in period  $T$ , with probability one. Thus, all transactions in periods  $T-1$  and  $T$  occur at the price  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$ . Clearly, no consumers with valuation below  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  purchase, and all consumers with valuation above  $P(q_{T-1}^*(\alpha), \gamma_{T-1}(\alpha))$  purchase. It follows that the allocation is efficient.

By induction, it follows that all transactions starting in period  $t+1$  must occur at the price  $P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha))$ , and the remaining output must be allocated efficiently. ■

By the previous Lemma, there is no loss of generality in assuming that when in any period  $t < T$

the demand state is revealed to equal  $\alpha$ , or it is revealed that the demand state is  $\alpha \geq \tilde{\alpha}_{t+1}$  and  $P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha))$  is independent of  $\alpha$  for all  $\alpha \geq \tilde{\alpha}_{t+1}$ , then all remaining output is sold in period  $t + 1$  at the price  $P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha))$ . We will make two further simplifications.<sup>24</sup> Define

$$S = \min\{t : P(q_t^*(\alpha), \gamma_t(\alpha)) \text{ is independent of } \alpha \text{ for all } \alpha \geq \tilde{\alpha}_t\} - 1.$$

Without loss of generality we may assume that once the market clearing price is revealed in period  $S$ , all inframarginal consumers (whose valuation exceeds the price) purchase with probability one. In period  $S + 1$ , purchases are made by all remaining inframarginal consumers (who arrived in period  $S$  before the price was revealed and did not purchase then), and by the right fraction of marginal consumers that clears the market. Hence we may assume that  $T = S + 1$ . Finally, since neither  $q_t^*(\alpha)$ ,  $\gamma_t(\alpha)$  or beliefs depend on sales in periods  $\tau < t$  whose measure equals zero, we can assume (by shortening  $T$  accordingly) that  $\hat{a}_t^{\max} > 0$  in every period  $t < T$ .

**Lemma 6** *In any period  $t$ , and following the equilibrium history  $(\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})$  the consumer type with the highest remaining valuation in any state  $\alpha > \tilde{\alpha}_t$  must be of type 1.*

**Proof.** If the beliefs entering period  $t$  are the non-degenerate interval  $[\tilde{\alpha}_t, \bar{\alpha}]$ , then it must be that  $\bar{a}_\tau(\alpha) = a_\tau^{\max}$  for all  $\tau \leq t - 1$ , for otherwise the state would have been revealed in period  $\tau$ . Furthermore, since for any  $\alpha < \tilde{\alpha}_t$  the sales level  $\bar{a}_{t-1}(\alpha)$  is reached in period  $t - 1$  only when the period  $(t - 1)$  queue is exhausted, and since  $\bar{a}_{t-1}(\alpha) = \hat{a}_{t-1}^{\max}$ , it must be that in any state  $\alpha > \tilde{\alpha}_t$  when the sales level  $\hat{a}_{t-1}^{\max}$  is reached in period  $t - 1$  the queue has not yet been exhausted, so that all consumer types who were present at the beginning of period  $t - 1$  must also be present at the beginning of period  $t$  when  $\alpha > \tilde{\alpha}_t$ . Using induction from period 1 on, it follows that when  $\alpha > \tilde{\alpha}_t$ , the consumer with the highest remaining valuation in period  $t$  when  $\alpha > \tilde{\alpha}_t$  is the consumer of type 1. ■

Define  $P(\alpha)$  to be the equilibrium final transaction price when the state equals  $\alpha$ :

$$P(\alpha) = P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha)), \text{ if } \alpha \in [\tilde{\alpha}_t, \tilde{\alpha}_{t+1}) \text{ and } t \in \{1, \dots, T - 1\}$$

Then we have:

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<sup>24</sup>This simplification is there to ease exposition. It would convert our successive revelation equilibrium to the equilibrium of a 2-period game.

**Lemma 7**  $P(\alpha)$  is nondecreasing in  $\alpha$  for all  $\alpha$ .

**Proof.** Let  $\alpha', \alpha'' \in [\tilde{\alpha}_t, \tilde{\alpha}_{t+1})$  be such that  $\alpha' < \alpha''$ . For  $\alpha \in [\alpha', \tilde{\alpha}_{t+1})$  let  $P^*(\alpha, \bar{a}_t(\alpha'))$  denote the market clearing price in state  $\alpha$ , based on remaining demand and remaining output, at the point at which sales reach  $\bar{a}_t(\alpha')$  in period  $t$ . Then we have  $P(\alpha') = P^*(\alpha', \bar{a}_t(\alpha')) \leq P^*(\alpha'', \bar{a}_t(\alpha')) \leq P(\alpha'')$ , where the last inequality holds as an equality if the output sold between  $\bar{a}_t(\alpha')$  and  $\bar{a}_t(\alpha'')$  is allocated efficiently, but could be strict otherwise. Thus  $P(\alpha)$  is nondecreasing on  $[\tilde{\alpha}_t, \tilde{\alpha}_{t+1})$ .

Next, we claim that  $p_{t+1}(0; (\hat{a}_1^{\max}, \dots, \hat{a}_t^{\max})) = p_t(\hat{a}_t^{\max}; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}))$ . Otherwise, if the price in period  $t + 1$  were higher, then the firm making the last sale in period  $t$  should deviate to post  $p_{t+1}(0; (\hat{a}_1^{\max}, \dots, \hat{a}_t^{\max}))$  in period  $t + 1$ ; if the price in period  $t$  were higher, then the firm selling first in period  $t + 1$  should deviate, to post  $p_t(\hat{a}_t^{\max}; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}))$  in period  $t$ . The result then follows because  $\lim_{\alpha \uparrow \tilde{\alpha}_{t+1}} P(\alpha) = p_t(\hat{a}_t^{\max}; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) = p_{t+1}(0; (\hat{a}_1^{\max}, \dots, \hat{a}_t^{\max})) = P(\tilde{\alpha}_{t+1})$ . ■

**Lemma 8 (Martingale Lemma):** For every  $t \in \{0, \dots, T - 1\}$  and every  $a_t \leq \hat{a}_t^{\max}$  we have

$$p_t(a_t; \hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}) = \frac{\int_{\beta_t(a_t)}^{\bar{\alpha}} P(\alpha) f(\alpha) d\alpha}{\int_{\bar{\alpha}_t}^{\bar{\alpha}} f(\alpha) d\alpha}$$

**Proof.** Consider a firm that sets a price  $p_t(a_t; \hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})$ . This firm sells its unit in period  $t$  if and only if  $\bar{a}_t(\alpha) \geq a_t$ . Hence its probability of selling in period  $t$  equals

$$\pi_t(a_t) = \frac{\int_{\beta_t(a_t)}^{\bar{\alpha}} f(\alpha) d\alpha}{\int_{\bar{\alpha}_t}^{\bar{\alpha}} f(\alpha) d\alpha}$$

When  $\bar{a}_t(\alpha) < a_t$ , this firm can sell its unit in period  $t + 1$  at the market clearing price  $P(\alpha)$ . Hence the expected revenue for the firm posting  $p_t(a_t; \hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})$  equals

$$p_t(a_t; \hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}) \pi_t(a_t) + E(P(\alpha) | \bar{a}_t(\alpha) < a_t) (1 - \pi_t(a_t))$$

Meanwhile, a firm that posts the price  $p_t(\hat{a}_t^{\max}; \hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})$  in period  $t$  does not get to sell in period  $t$  and hence has an expected revenue of

$$\int_{\bar{\alpha}_{t+1}}^{\bar{\alpha}} P(\alpha) f(\alpha) d\alpha$$

Since expected revenues must be equated across firms, it must be that

$$p_t(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max}) \pi_t(a_t) = E(P(\alpha) | \bar{a}_t(\alpha) \geq a_t) \pi_t(a_t),$$

from which the desired result follows. ■

**Proof of Proposition 7.** We prove this Theorem in a series of three auxiliary Lemmata.

**Lemma 9** (*Skimming Lemma*): Consider any period  $t \in \{1, \dots, T-1\}$  and the equilibrium history  $(\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})$ . Suppose that the best option for any consumer not purchasing in period  $t$  following this history is to wait purchase at the price  $P(\alpha)^{25}$ . Then for any  $a_t$  such that  $P(\beta_t(a_t)) < P(\bar{\alpha})$  we have:

$$\psi_i^t(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) > 0 \Rightarrow \psi_j^t(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) = 1 \text{ for every } j < i.$$

**Proof.** If  $\psi_i^t(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) > 0$  then we must have

$$v_i - p_t(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) \geq E[\max\{v_i - P(\alpha), 0\} | \text{active type } i, \alpha \geq \beta_t(a_t)] \quad (36)$$

Now consider any type  $j < i$ . We have:

$$E[\max\{v_j - P(\alpha), 0\} | \text{active type } j, \alpha \geq \beta_t(a_t)] = \int_{\beta_t(a_t)}^{\bar{\alpha}} \text{Max}\{v_j - P(\alpha), 0\} f(\alpha | \text{active type } j, \alpha \geq \beta_t(a_t)) d\alpha$$

Analogous to the derivation of equation (29), we may prove that for each type  $k$

$$f(\alpha | \text{active type } k, \alpha \geq \beta_t(a_t)) = \frac{\gamma_k(\alpha) f(\alpha) g_t(\alpha)}{\int_{\beta_{T-1}(a_{T-1})}^{\bar{\alpha}} \gamma_k(\alpha) f(\alpha) g_t(\alpha) d\alpha}, \quad (37)$$

for some function  $g_t(\alpha)$ . Hence if  $\alpha'' > \alpha'$  then

$$\frac{f(\alpha'' | \text{active type } k, \alpha \geq \beta_t(a_t))}{f(\alpha' | \text{active type } k, \alpha \geq \beta_t(a_t))} = \frac{\gamma_k(\alpha'') f(\alpha'') g_t(\alpha'')}{\gamma_k(\alpha') f(\alpha') g_t(\alpha')} \quad (38)$$

Hence by assumption 14 it follows that

$$\frac{f(\alpha'' | \text{active type } j, \alpha \geq \beta_t(a_t))}{f(\alpha' | \text{active type } j, \alpha \geq \beta_t(a_t))} > \frac{f(\alpha'' | \text{active type } i, \alpha \geq \beta_t(a_t))}{f(\alpha' | \text{active type } i, \alpha \geq \beta_t(a_t))}$$

<sup>25</sup>Thus the consumer waits to purchase until either  $\alpha$  has been revealed, or until it has been revealed that  $\alpha \geq \bar{\alpha}_T$ .

for all  $j < i$ . Thus the distribution  $F(\alpha''|\text{active type } j, \alpha \geq \beta_t(a_t))$  strictly dominates the distribution  $F(\alpha'|\text{active type } i, \alpha \geq \beta_t(a_t))$  in the monotone likelihood ratio order. It is well known (see e.g. Shaked and Shanthikumar (1994)), that this implies that  $F(\alpha''|\text{active type } j, \alpha \geq \beta_t(a_t))$  first order stochastically dominates  $F(\alpha'|\text{active type } j, \alpha \geq \beta_t(a_t))$ .

Now by Lemma 7  $P(\alpha)$  is a non-decreasing function. Moreover, since we have  $P(\beta_t(a_t)) < P(\bar{\alpha})$  it follows that

$$E[\max\{v_j - P(\alpha)|\text{active type } j, \alpha \geq \beta_t(a_t)\}] < E[\max\{v_j - P(\alpha)|\text{active type } i, \alpha \geq \beta_t(a_t)\}] \quad (39)$$

Furthermore, we have

$$E[\max\{v_j - P(\alpha)|\text{active type } i, \alpha \geq \beta_t(a_t)\}] - E[\max\{v_i - P(\alpha)|\text{active type } i, \alpha \geq \beta_t(a_t)\}] \leq v_j - v_i \quad (40)$$

Hence we obtain

$$\begin{aligned} E[\max\{v_j - P(\alpha)|\text{active type } j, \alpha \geq \beta_t(a_t)\}] &< E[\max\{v_i - P(\alpha)|\text{active type } i, \alpha \geq \beta_t(a_t)\}] \\ &\leq v_j - v_i + E[\max\{v_i - P(\alpha)|\text{active type } i, \alpha \geq \beta_t(a_t)\}] \\ &\leq v_j - p_t(a_t; (a_1^{\max}, \dots, a_{t-1}^{\max})) \end{aligned}$$

where the first inequality follows from (39), the second from (40), and the last from (36). We conclude that type  $j$  strictly prefers to purchase at the price  $p_t(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max}))$ , i.e. that  $\psi_j^t(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) = 1$  for every  $j < i$ , as was to be demonstrated. ■

**Lemma 10** *Consider any period  $t \in \{1, \dots, T-1\}$  and the equilibrium history  $(\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})$ . Suppose that the best option for any consumer not purchasing in period  $t$  following this history is to wait purchase at the price  $P(\alpha)$ . Then for every  $a_t \in (0, \hat{a}_t^{\max}]$  we have  $\psi_t^i(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) = 0$  for all  $i > 1$ , and type 1 is indifferent between purchasing in period  $t$  and waiting for the price  $P(\alpha)$ .*

**Proof.** Suppose that contrary to the statement of the Lemma, we had  $i = \max\{j : \psi_t^j(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) > 0\} > 1$ . Since by Lemma 9 we must have  $\psi_t^j(a_t; (\hat{a}_1^{\max}, \dots, \hat{a}_{t-1}^{\max})) = 1$  for all  $j < i$ , it follows from

equation (38) that

$$\frac{f(\alpha''|\text{active type } i, \alpha \geq \beta_t(a_t))}{f(\alpha'|\text{active type } i, \alpha \geq \beta_t(a_t))} = \frac{f(\alpha'') \psi_t^i(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) + \sum_{j < i} \frac{\gamma_j(\alpha')}{\gamma_i(\alpha')}}{f(\alpha') \psi_t^i(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) + \sum_{j < i} \frac{\gamma_j(\alpha'')}{\gamma_i(\alpha'')}}}$$

By Assumption 14 we have

$$\frac{\psi_t^i(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) + \sum_{j < i} \frac{\gamma_j(\alpha')}{\gamma_i(\alpha')}}{\psi_t^i(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) + \sum_{j < i} \frac{\gamma_j(\alpha'')}{\gamma_i(\alpha'')}} < 1$$

so

$$\frac{f(\alpha''|\text{active type } i, \alpha \geq \beta_t(a_t))}{f(\alpha'|\text{active type } i, \alpha \geq \beta_t(a_t))} < \frac{f(\alpha'')}{f(\alpha')}$$

i.e.  $F(\alpha|\alpha \geq \beta_t(a_t))$  strictly dominates  $F(\alpha|\text{active type } i, \alpha \geq \beta_t(a_t))$  in the monotone likelihood ratio order, and hence in order of first order stochastic dominance. Hence it follows that

$$\int_{\beta_t(a_t)}^{\bar{\alpha}} P(\alpha) f(\alpha|\text{active type } i, \alpha \geq \beta_t(a_t)) d\alpha < \frac{\int_{\beta_t(a_t)}^{\bar{\alpha}} P(\alpha) f(\alpha) d\alpha}{\int_{\beta_t(a_t)}^{\bar{\alpha}} f(\alpha) d\alpha} \quad (41)$$

Now by Lemma 8 we have

$$p_t(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max}) = \frac{\int_{\beta_t(a_t)}^{\bar{\alpha}} P(\alpha) f(\alpha) d\alpha}{\int_{\bar{\alpha}_t}^{\bar{\alpha}} f(\alpha) d\alpha}$$

Combining the two previous equations thus results in

$$E[P(\alpha)|\text{active type } i, \alpha \geq \beta_t(a_t)] < p_t(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max}). \quad (42)$$

But since

$$E[\max\{v_i - P(\alpha)|\alpha \geq \beta_t(a_t)\}] \geq v_i - E[P(\alpha)|\alpha \geq \beta_t(a_t)] \quad (43)$$

we have

$$v_i - p_t(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max}) < v_i - E[P(\alpha)|\text{active type } i, \alpha \geq \beta_t(a_t)] \quad (44)$$

$$\leq E[\max\{v_i - P(\alpha)|\alpha \geq \beta_t(a_t)\}], \quad (45)$$

where the first inequality follows from (42) and the second from (43). By (44), type  $i$  strictly prefers to postpone purchasing when facing the price  $p_t(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})$ , contradicting the presumption that  $\psi_t^i(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max}) > 0$ . We conclude that we must have  $\psi_t^i(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max}) = 0$  for all  $i > 1$  and  $a_t \in (0, \widehat{a}_t^{\max})$ . This in turn implies that

$$f(\alpha | \text{active type } i, \alpha \geq \beta_t(a_t)) = \frac{f(\alpha)}{\int_{\beta_t(a_t)}^{\bar{\alpha}} f(\alpha) d\alpha} = f(\alpha | \alpha \geq \beta_t(a_t)).$$

Therefore we have

$$E[P(\alpha) | \text{active type } i, \alpha \geq \beta_t(a_t)] = E[P(\alpha) | \alpha \geq \beta_t(a_t)].$$

Since  $P(\alpha) \leq v_1$  for all  $\alpha$ , it then follows that

$$\begin{aligned} v_1 - p_t(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max}) &= v_1 - E[P(\alpha) | \alpha \geq \beta_t(a_t)] \\ &= v_1 - E[P(\alpha) | \text{active type } i, \alpha \geq \beta_t(a_t)] \\ &= E[\max\{v_1 - P(\alpha) | \text{active type } i, \alpha \geq \beta_t(a_t)\}] \end{aligned}$$

so type 1 is indifferent between purchasing at  $p_t(a_t; \widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})$  and waiting for  $P(\alpha)$ . ■

**Lemma 11** *In any period  $t \in \{1, \dots, T-1\}$ , following the equilibrium history  $(\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})$ , and for any  $a_t$  s.t.  $P(\beta_t(a_t)) < P(\bar{\alpha})$  we have  $\psi_t^i(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) = 0$  for all  $i \neq 1$ , and  $\psi_1^t(a_t; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{t-1}^{\max})) > 0$ .*

**Proof.** Consider first period  $T-1$ , following the history  $(\widehat{a}_1^{\max}, \dots, \widehat{a}_{T-2}^{\max})$ . Then any consumer not purchasing at the price  $p_{T-1}(a_{T-1}; (\widehat{a}_1^{\max}, \dots, \widehat{a}_{T-2}^{\max}))$  in period  $(T-1)$  will only be able to purchase at the price  $P(\alpha)$  for some  $\alpha \geq \beta_{T-1}(a_{T-1})$  in period  $T$ , so the assumption of Lemma 10 holds. It then inductively follows from this lemma that the desired statement holds. ■

**Lemma 12** *For every  $\alpha$  we have  $P(\alpha) = P(q^*, \gamma(\alpha))$ .*

**Proof.** It follows from Lemma 11 that for  $\alpha \in [\tilde{\alpha}_t, \tilde{\alpha}_{t+1})$  we have  $q_{t+1}^*(\alpha) = q^* - \sum_{\tau=1}^t \bar{a}_t(\alpha)$  and  $\gamma_{t+1}(\alpha) = (\gamma_1(\alpha) - \sum_{\tau=1}^t \bar{a}_t(\alpha), \gamma_2(\alpha), \dots, \gamma_n(\alpha))$ . Since only consumers of type 1 purchased prior to the start of period  $t+1$ , we have  $P(\alpha) = P(q_{t+1}^*(\alpha), \gamma_{t+1}(\alpha)) = P(q^* - \sum_{\tau=1}^t \bar{a}_t(\alpha), (\gamma_1(\alpha) - \sum_{\tau=1}^t \bar{a}_t(\alpha), \gamma_2(\alpha), \dots, \gamma_n(\alpha))) = P(q^*, \gamma(\alpha))$ . ■

**Lemma 13** *In any sequential equilibrium of the  $T$ -period game with  $T \geq 2$ , given the equilibrium  $q^*$ , the allocation of output to consumers is efficient.*

**Proof.** It follows from Lemma 11 that some type 1 consumers and no lower valuation consumers will purchase before the demand state has been revealed, and from Lemma 2 that the remaining output is allocated efficiently once the demand state has been revealed. An identical statement holds if demand is never revealed, for then only type 1 consumers and no lower valuation consumers will purchase before trade occurs at the price  $P(\alpha) = P(q^*, \alpha)$ . Therefore, the allocation of output to consumers is efficient. ■

**Lemma 14** *In any sequential equilibrium of the  $T$ -period game with  $T \geq 2$ , the equilibrium output level  $q^*$  is efficient, i.e. we have*

$$\int_{\alpha}^{\bar{a}} P(q^*, \alpha) f(\alpha) d\alpha = c. \quad (46)$$

**Proof.** It follows from the martingale Lemma and Lemma 12 that

$$p_1(0) = \int_{\alpha}^{\bar{a}} P(q^*, \alpha) f(\alpha) d\alpha.$$

Suppose instead that  $q^*$  is greater than the solution to (46). Then any firm posting the price  $p_1(0)$  would be sure to sell, and hence receive an expected revenue strictly lower than its cost of production  $c$ . Hence this firm could profitably deviate by not producing. Suppose instead that  $q^*$  is less than the solution to (46). Then a nonproducing firm could post the price  $p_1(0)$  and be sure to sell. Since  $p_1(0) > c$ , this deviation is profitable, contradicting equilibrium. ■

**Proof of Proposition 8.** From (4) we have

$$p_i(a_i; \bar{a}_1, \dots, \bar{a}_{i-1}) = \frac{\int_{a_i}^{\bar{\gamma}_i} \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \dots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_n \dots d\gamma_{i+1} \right) d\gamma_i}{\int_{a_i}^{\bar{\gamma}_i} \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \dots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_n \dots d\gamma_{i+1} \right) d\gamma_i}.$$

It follows that the sign of  $\frac{\partial p_i(a_i; \bar{a}_1, \dots, \bar{a}_{t-1})}{\partial a_i}$  equals the sign of

$$\begin{aligned}
& - \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1} \right) \\
& \int_{a_i}^{\bar{\gamma}_i} \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1} \right) d\gamma_i \\
& + \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1} \right) \\
& \int_{a_i}^{\bar{\gamma}_i} \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1} \right) d\gamma_i,
\end{aligned}$$

which is just the integral from  $a_i$  to  $\bar{\gamma}_i$  of the following expression

$$\begin{aligned}
& \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1} \right) \\
& \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1} \right) \\
& - \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1} \right) \\
& \left( \int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1} \right)
\end{aligned}$$

Thus we will have  $\frac{\partial p_i(a_i; \bar{a}_1, \dots, \bar{a}_{i-1})}{\partial a_i} \geq 0$  if

$$\begin{aligned}
& \frac{\int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1}}{\int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1}} \\
& \geq \frac{\int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) f(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1}}{\int_{\underline{\gamma}_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} f(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1}}
\end{aligned} \tag{47}$$

for all  $\gamma_i \geq a_i$ . Let  $F(\gamma_{i+1}, \dots, \gamma_n; \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i) = \int_{\gamma_{i+1}}^{\bar{\gamma}_{i+1}} \cdots \int_{\gamma_n}^{\bar{\gamma}_n} f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \gamma_{i+1}, \dots, \gamma_n) d\gamma_n \cdots d\gamma_{i+1}$ . Since  $P(q^*, \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n)$  is non-decreasing in  $\gamma_i$ , inequality (47) will hold if for each  $\bar{a}_1, \dots, \bar{a}_{i-1}$  and each  $\gamma_i \geq a_i$  the distribution function  $F(\gamma_{i+1}, \dots, \gamma_n; \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i)$  first order stochastically dominates the distribution function  $F(\gamma_{i+1}, \dots, \gamma_n; \bar{a}_1, \dots, \bar{a}_{i-1}, a_i)$ .

Suppose now that Assumption (15) holds. Then the distributions  $F(\gamma_{i+1}, \dots, \gamma_n; \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i)$

and  $F(\gamma_{i+1}, \dots, \gamma_n; \bar{a}_1, \dots, \bar{a}_{i-1}, a_i)$  are totally positive, and the likelihood ratio

$$\frac{f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma'_i, \dots, \gamma_n)}{f(\bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i, \dots, \gamma_n)} \text{ is increasing in } (\gamma_{i+1}, \dots, \gamma_n).$$

It follows from Whitt (1982) that  $F(\gamma_{i+1}, \dots, \gamma_n; \bar{a}_1, \dots, \bar{a}_{i-1}, \gamma_i)$  dominates  $F(\gamma_{i+1}, \dots, \gamma_n; \bar{a}_1, \dots, \bar{a}_{i-1}, a_i)$  in the strong monotone likelihood ratio order, and hence in the order associated with first order stochastic dominance, as needed to be shown. ■

Before presenting the proof of Proposition 9, we need to establish an auxiliary result:

**Proof of Proposition 9.** First, we will show that the equilibrium expected revenues for every active firm equal

$$E(P(q^*, \gamma)) = \int_{\underline{\gamma}_1}^{\bar{\gamma}_1} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \gamma_1, \dots, \gamma_n) f(\gamma_1, \dots, \gamma_n) d\gamma_n \cdots d\gamma_1 \quad (48)$$

Thus by Definition 2 each active firm makes zero profits.

First consider period  $n$ , and suppose that some output remains available for sale, i.e. suppose that  $\sum_{i=1}^{n-1} \gamma_i < q^*$ . If  $\bar{\gamma}_n \leq q^* - \sum_{i=1}^{n-1} \gamma_i$  then we have  $P(q^*, \gamma_1, \dots, \gamma_n) = 0$  for all  $\gamma_n$ , and so  $p_n(\cdot; \gamma_1, \dots, \gamma_{n-1}) \equiv 0$ . If  $\bar{\gamma}_n > q^* - \sum_{i=1}^{n-1} \gamma_i$  then a firm that posts the price  $p_n(a_n; \gamma_1, \dots, \gamma_{n-1})$  will sell in period  $n$  if and only if  $\gamma_n \geq a_n$ , so we have

$$\pi_n(a_n; \gamma_1, \dots, \gamma_{n-1}) = 1 - F_n(a_n; \gamma_1, \dots, \gamma_{n-1}) \quad \text{and} \quad \bar{a}_n(\gamma) = \gamma_n$$

where  $F_n$  is the marginal distribution of type  $n$ , conditional on the realization  $(\gamma_1, \dots, \gamma_{n-1})$ , i.e.

$$F_n(a_n; \gamma_1, \dots, \gamma_{n-1}) = \int_{a_n}^{\bar{\gamma}_n} f(\gamma_1, \dots, \gamma_{n-1}, \gamma_n) d\gamma_n$$

If unit  $a_n$  does not sell in period  $n$ , which happens with probability  $1 - \pi_n(a_n; \gamma_1, \dots, \gamma_{n-1}) = F_n(a_n; \gamma_1, \dots, \gamma_{n-1})$ , then no further revenue will be collected in the future. Equivalently, the firm then collects  $P(q^*, \gamma_1, \dots, \gamma_{n-1}, \gamma_n) = 0$ . The expected revenue of a firm posting the price  $p_n(a_n; \gamma_1, \dots, \gamma_{n-1})$  therefore equal

$$\text{profit} = p_n(a_n; \gamma_1, \dots, \gamma_{n-1}) \pi_n(a_n; \gamma_1, \dots, \gamma_{n-1}) + E(P(q^*, \gamma_1, \dots, \gamma_{n-1}, \gamma_n) \mid a_n \geq \gamma_n) [1 - \pi_n(a_n; \gamma_1, \dots, \gamma_{n-1})]$$

Now

$$E(P(q^*, \gamma_1, \dots, \gamma_{n-1}, \gamma_n) \mid a_n \geq \gamma_n)[1 - \pi_n(a_n; \gamma_1, \dots, \gamma_{n-1})] = \int_{\underline{\gamma}_n}^{\alpha_n} P(q^*, \gamma_1, \dots, \gamma_{n-1}, \gamma_n) f(\gamma_1, \dots, \gamma_{n-1}, \gamma_n) d\gamma_n$$

and, using the martingale property,

$$p_n(a_n; \gamma_1, \dots, \gamma_{n-1}) \pi_n(a_n; \gamma_1, \dots, \gamma_{n-1}) = \int_{a_n}^{\bar{\gamma}_n} P(q^*, \gamma_1, \dots, \gamma_{n-1}, \gamma_n) f(\gamma_1, \dots, \gamma_{n-1}, \gamma_n) d\gamma_n$$

$$\text{profit} = \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \gamma_1, \dots, \gamma_{n-1}, \gamma_n) f(\gamma_1, \dots, \gamma_{n-1}, \gamma_n) d\gamma_n$$

We conclude that each firm posting a price in period  $n$  makes the same expected profits as the firm that posts  $p_n(0; \gamma_1, \dots, \gamma_{n-1})$ .

Using entirely parallel arguments, we may recursively prove that each firm that posts a price  $p_t(a_t; \gamma_1, \dots, \gamma_{t-1})$  in period  $t$  earns expected profits of

$$\int_{\underline{\gamma}_t}^{\bar{\gamma}_t} \cdots \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \gamma_1, \dots, \gamma_n) f(\gamma_1, \dots, \gamma_n) d\gamma_n \cdots d\gamma_t$$

and so by setting  $t = 1$  equation (48) follows. The argument that no inactive firm can gain by producing is similar to the one in the proof of Proposition 4.

It remains to be shown that consumer purchase strategies are optimal, i.e. that it is optimal for a consumer of type  $t \leq n$  to purchase in period  $t$ . First, let us consider  $t = n$ , a history in which output remains available in period  $n$ , i.e.  $\sum_{i=1}^{n-1} \gamma_i < q_n^*$ , and a lowest available price of  $p_n(a_n; \gamma_1, \dots, \gamma_{n-1})$ . If type  $n$  postpones purchasing until period  $n + 1$ , then she will face the price  $p_{n+1}(\gamma) = P(q^*, \gamma)$ . Indeed, if  $\gamma_n$  is such that  $\sum_{i=1}^n \gamma_i > q^*$ , then no supply remains in period  $n + 1$ , and so  $p_{n+1}$  can be taken to equal the highest remaining valuation,  $v_n$ . On the other hand, if  $\sum_{i=1}^n \gamma_i < q^*$ , then no demand remains in period  $n + 1$ , and so we will have  $p_{n+1} = 0$ . Thus by waiting until period  $n + 1$ ,

or any later periods, type  $n$  will face an expected price of

$$\begin{aligned}
& E(P(q^*, \bar{a}_1, \dots, \bar{a}_{n-1}, \gamma_n) | \bar{a}_1 = \gamma_1, \dots, \bar{a}_{n-1} = \gamma_{n-1}, \gamma_n \geq a_n, \text{ type } n \text{ alive}) \\
&= \frac{\int_{a_n}^{\bar{\gamma}_n} P(q^*, \gamma) \frac{\gamma_n}{C_n} \frac{1}{\gamma_n} f(\gamma) d\gamma_n}{\int_{a_n}^{\bar{\gamma}_n} \frac{\gamma_n}{C_n} \frac{1}{\gamma_n} f(\gamma) d\gamma_n} \\
&= p_n(a_n; \gamma_1, \dots, \gamma_{n-1})
\end{aligned}$$

Thus if period  $n$  arrives, and output remains, a consumer of type  $n$  is indifferent between purchasing in period  $n$ , and waiting to purchase in a later period. Purchasing in period  $n$  is therefore an optimal strategy for type  $n$ . Similarly, if a single consumer of type  $i < n$  (who must have deviated in period  $i$  by not purchasing) remained in period  $n$ , purchasing in period  $n$  is an optimal strategy for her (as is waiting to purchase until period  $n + 1$ ).

Next, let us consider period  $n - 1$ , a history in which output remains available in period  $n - 1$ , i.e.  $\sum_{i=1}^{n-2} \gamma_i < q_n^*$ , and a lowest available price of  $p_{n-1}(a_{n-1}; \gamma_1, \dots, \gamma_{n-2})$ . First, let us argue that it is optimal for a consumer of type  $(n - 1)$  to purchase in period  $n - 1$ . If type  $n - 1$  deviated by not purchasing, then as shown above, it would be an optimal strategy for her to postpone purchasing until period  $n + 1$ . The expected price type  $n - 1$  would face in period  $n + 1$  equals

$$\begin{aligned}
& E(p_{n+1}(\bar{a}_1, \dots, \bar{a}_{n-2}, \gamma_{n-1}, \gamma_n) | \bar{a}_1 = \gamma_1, \dots, \bar{a}_{n-2} = \gamma_{n-2}, \gamma_{n-1} \geq a_{n-1}, \text{ type } n - 1 \text{ alive}) \\
&= \frac{\int_{a_{n-1}}^{\bar{\gamma}_{n-1}} \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \gamma) f(\gamma) d\gamma_n d\gamma_{n-1}}{\int_{a_{n-1}}^{\bar{\gamma}_{n-1}} \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} f(\gamma) d\gamma_n d\gamma_{n-1}} \\
&= p_{n-1}(a_{n-1}; \gamma_1, \dots, \gamma_{n-2})
\end{aligned}$$

Therefore if period  $n$  arrives, and output remains, a consumer of type  $n - 1$  is indifferent between purchasing in period  $n$ , and waiting to purchase in a later period. Purchasing in period  $n - 1$  is therefore an optimal strategy for type  $n - 1$ . Similarly, if a single consumer of type  $i < n$  who deviated in period  $i$  by not purchasing remained in period  $n$ , purchasing in period  $n$  is also an optimal strategy for her (as is waiting until period  $n$  or  $n + 1$ ).

It remains to be shown that in period  $n - 1$ , after a history in which output remains available in period  $n - 1$ , and the lowest available price is  $p_{n-1}(a_{n-1}; \gamma_1, \dots, \gamma_{n-2})$ , type  $n$  optimally postpones purchasing until period  $n$ . Since such a consumer is indifferent between purchasing in

period  $n$  and period  $n + 1$ , it will suffice to show that type  $n$  optimally postpones purchasing until period  $n + 1$ . From the viewpoint of type  $n$ , the expected period  $n + 1$  price equals

$$\begin{aligned} E[P(q^*, \bar{a}_1, \dots, \bar{a}_{n-1}, \gamma_{n-1}, \gamma_n) | \bar{a}_1 = \gamma_1, \dots, \bar{a}_{n-2} = \gamma_{n-2}, \gamma_{n-1} \geq a_{n-1}, \text{ type } n \text{ alive}] \\ = \frac{\int_{a_{n-1}}^{\gamma_{n-1}} \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} P(q^*, \gamma) \frac{\gamma_n}{\gamma_{n-1}} f(\gamma) d\gamma_n d\gamma_{n-1}}{\int_{a_{n-1}}^{\gamma_{n-1}} \int_{\underline{\gamma}_n}^{\bar{\gamma}_n} \frac{\gamma_n}{\gamma_{n-1}} f(\gamma) d\gamma_n d\gamma_{n-1}} \\ \geq p_{n-1}(a_{n-1}; \gamma_1, \dots, \gamma_{n-2}), \end{aligned}$$

where the inequality follows from assumption (16). Thus it is optimal for type  $n$  to postpone purchasing.

The remainder of the proof then follows by an induction argument. ■

To establish Proposition 10, we first need an auxiliary result. Define

$$p_1^2(a_1) \equiv E[P(q^*, \gamma_1, \gamma_2) | \gamma_1 \geq a_1, \text{ type 2 alive}] = \frac{\int_{a_1}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} P(q^*, \gamma_1, \gamma_2) \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}{\int_{a_1}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}$$

**Lemma 15** *The inequality  $p_1^2(a_1) < p_1(a_1)$  holds for  $\gamma_1$  in a left neighborhood of  $\gamma_1^H$  if and only if the following condition holds:*

$$\frac{\underline{\gamma}_2(\gamma_1^H)}{\gamma_1^H} > \frac{\int_{\gamma_1^H}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}{\int_{\gamma_1^H}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1} \quad (49)$$

**Proof.** Let  $\Delta(a_1) = p_1^2(a_1) - p_1(a_1)$ . Then we have  $\Delta(a_1) = \Delta(\gamma_1^H) - \Delta'(\gamma_1^H)(\gamma_1^H - a_1) + \frac{1}{2}\Delta''(\gamma_1^H)(\gamma_1^H - a_1)^2 + o((\gamma_1^H - a_1)^2)$ . We will show below that  $\Delta(\gamma_1^H) = \Delta'(\gamma_1^H) = 0$ , and that  $\Delta''(\gamma_1^H) < 0$  holds if and only if (49) holds, implying the desired result.

First, we establish that  $\Delta(a_1) = 0$  for all  $a_1 \geq \gamma_1^H$ . To simplify notation, define

$$\begin{aligned} Z(\gamma_1) &= \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} P(q^*, \gamma_1, \gamma_2) \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 \\ z(\gamma_1) &= \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 \end{aligned}$$

so that we have

$$p_1^2(a_1) = \frac{\int_{a_1}^{\bar{\gamma}_1} Z(\gamma_1) d\gamma_1}{\int_{a_1}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1} \quad (50)$$

Since  $P(q^*, \gamma_1, \gamma_2) = v_1$  for all  $\gamma_1 \geq \gamma_1^H$  and all  $\gamma_2$ , we have

$$Z(\gamma_1) = v_1 \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 = v_1 z_1(\gamma_1), \text{ for all } \gamma_1 \geq \gamma_1^H. \quad (51)$$

It follows that

$$\int_{a_1}^{\bar{\gamma}_1} Z(\gamma_1) d\gamma_1 = v_1 \int_{a_1}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1, \text{ for all } a_1 \geq \gamma_1^H$$

Hence from (50) we obtain  $p_1^2(a_1) = v_1$ , for all  $a_1 \geq \gamma_1^H$ . An entirely parallel argument establishes that  $p_1(a_1) = v_1$  for all  $a_1 \geq \gamma_1^H$ . We conclude that  $\Delta(a_1) = 0$  for all  $a_1 \geq \gamma_1^H$ .

Next, let us establish that  $\Delta'(\gamma_1^H) = 0$ . Differentiating (50) yields

$$\frac{dp_1^2}{da_1}(a_1) = \frac{-Z(a_1) \left( \int_{a_1}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \right) + z(a_1) \left( \int_{a_1}^{\bar{\gamma}_1} Z(\gamma_1) d\gamma_1 \right)}{\left( \int_{a_1}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \right)^2} \quad (52)$$

Therefore

$$\lim_{a_1 \uparrow \gamma_1^H} \frac{dp_1^2}{da_1}(a_1) = \frac{-Z(\gamma_1^H) \left( \int_{\gamma_1^H}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \right) + z(\gamma_1^H) \left( \int_{\gamma_1^H}^{\bar{\gamma}_1} Z(\gamma_1) d\gamma_1 \right)}{\left( \int_{\gamma_1^H}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \right)^2} \quad (53)$$

It now follows from (51) that

$$\int_{\gamma_1^H}^{\bar{\gamma}_1} Z(\gamma_1) d\gamma_1 = v_1 \int_{\gamma_1^H}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \text{ and } Z(\gamma_1^H) = v_1 z(\gamma_1^H) \quad (54)$$

Thus from (53) we have

$$\lim_{a_1 \uparrow \gamma_1^H} \frac{dp_1^2}{da_1}(a_1) = 0 \quad (55)$$

An entirely parallel argument also establishes that

$$\lim_{a_1 \uparrow \gamma_1^H} \frac{dp_1}{da_1}(a_1) = 0$$

so we indeed do have  $\Delta'(\gamma_1^H) = 0$ .

Finally, we establish that  $\Delta''(\gamma_1^H) < 0$  if and only if (49) holds. Differentiating (52) yields

$$\frac{d^2 p_1^2}{da_1^2}(a_1) = \frac{-Z'(a_1) \left( \int_{a_1}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \right) + z'(a_1) \left( \int_{a_1}^{\bar{\gamma}_1} Z(\gamma_1) d\gamma_1 \right)}{\left( \int_{a_1}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \right)^2} + \frac{2z(a_1)}{\int_{a_1}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1} \frac{d^2 p_1^2}{da_1^2}(a_1)$$

Therefore we have

$$\begin{aligned} \lim_{a_1 \uparrow \gamma_1^H} \frac{d^2 p_1^2}{da_1^2}(a_1) &= \frac{-Z'(\gamma_1^H) \left( \int_{\gamma_1^L}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \right) + z'(\gamma_1^H) \left( \int_{\gamma_1^H}^{\bar{\gamma}_1} Z(\gamma_1) d\gamma_1 \right)}{\left( \int_{\gamma_1^H}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1 \right)^2} \\ &= \frac{-Z'(\gamma_1^H) + v_1 z'(\gamma_1^H)}{\int_{\gamma_1^H}^{\bar{\gamma}_1} z(\gamma_1) d\gamma_1} \end{aligned}$$

where the first equality follows from (55), and the second equality from (51). Let  $\gamma_1^L$  be the solution to  $P(q^*, \gamma_1, \bar{\gamma}_2(\gamma_1)) = v_1$ . Then for  $\gamma_1 \in (\gamma_1^L, \gamma_1^H)$  we have

$$P(q^*, \gamma_1, \gamma_2) = \begin{cases} v_1 & \text{if } \gamma_2 \geq q^* - \gamma_1 \\ 0 & \text{if } \gamma_2 < q^* - \gamma_1 \end{cases},$$

and so

$$Z(\gamma_1) = v_1 \int_{q^* - \gamma_1}^{\bar{\gamma}_2(\gamma_1)} \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2, \text{ for } \gamma_1 \in (\gamma_1^L, \gamma_1^H).$$

Hence we have

$$Z'(\gamma_1) = v_1 \frac{q^* - \gamma_1}{\gamma_1} f(\gamma_1, q^* - \gamma_1) + v_1 \frac{d\bar{\gamma}_2}{d\gamma_1}(\gamma_1) \frac{\bar{\gamma}_2(\gamma_1)}{\gamma_1} f(\gamma_1, \bar{\gamma}_2(\gamma_1)) + v_1 \int_{q^* - \gamma_1}^{\bar{\gamma}_2(\gamma_1)} \frac{d}{d\gamma_1} \left( \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) \right) d\gamma_2$$

and

$$z'(\gamma_1) = -\frac{d\gamma_2}{d\gamma_1}(\gamma_1) \frac{\gamma_2(\gamma_1)}{\gamma_1} f(\gamma_1, \gamma_2(\gamma_1)) + \frac{d\bar{\gamma}_2}{d\gamma_1}(\gamma_1) \frac{\bar{\gamma}_2(\gamma_1)}{\gamma_1} f(\gamma_1, \bar{\gamma}_2(\gamma_1)) + \int_{\gamma_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{d}{d\gamma_1} \left( \frac{\gamma_2}{\gamma_1} f(\gamma_1, \gamma_2) \right) d\gamma_2$$

Now at  $\gamma_1 = \gamma_1^H$  we have  $q^* - \gamma_1^H = \gamma_2(\gamma_1^H)$ , and so

$$-Z'(\gamma_1^H) + v_1 z'(\gamma_1^H) = -v_1 \frac{\gamma_2(\gamma_1^H)}{\gamma_1} f(\gamma_1, \gamma_2(\gamma_1^H)) - v_1 \frac{d\gamma_2}{d\gamma_1}(\gamma_1) \frac{\gamma_2(\gamma_1)}{\gamma_1} f(\gamma_1, \gamma_2(\gamma_1))$$

$$-Z'(\gamma_1^H) + v_1 z'(\gamma_1^H) = -v_1 f(\gamma_1^H, \gamma_2(\gamma_1^H)) \frac{\gamma_2(\gamma_1^H)}{\gamma_1^H} \left( 1 + \frac{d\gamma_2}{d\gamma_1}(\gamma_1^H) \right)$$

We therefore have

$$\lim_{a_1 \uparrow \gamma_1^H} \frac{d^2 p_1^2}{da_1^2}(a_1) = - \frac{v_1 f(\gamma_1^H, \underline{\gamma}_2(\gamma_1^H))}{\int_{\gamma_1^H}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{\underline{\gamma}_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1} \frac{\underline{\gamma}_2(\gamma_1^H)}{\gamma_1^H} \left( 1 + \frac{d\underline{\gamma}_2}{d\gamma_1}(\gamma_1^H) \right)$$

An entirely parallel argument also establishes that

$$\lim_{a_1 \uparrow \gamma_1^H} \frac{d^2 p_1}{da_1^2}(a_1) = - \frac{v_1 f(\gamma_1^H, \underline{\gamma}_2(\gamma_1^H))}{\int_{\gamma_1^H}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1} \left( 1 + \frac{d\underline{\gamma}_2}{d\gamma_1}(\gamma_1^H) \right)$$

We therefore may conclude that

$$\begin{aligned} \lim_{a_1 \uparrow \gamma_1^H} \left( \frac{d^2 p_1^2}{da_1^2}(a_1) - \frac{d^2 p_1}{da_1^2}(a_1) \right) &= \frac{v_1 f(\gamma_1^H, \underline{\gamma}_2(\gamma_1^H))}{\int_{\gamma_1^H}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{\underline{\gamma}_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1} \left( 1 + \frac{d\underline{\gamma}_2}{d\gamma_1}(\gamma_1^H) \right) \\ &\quad \left( \frac{\int_{\gamma_1^H}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} \frac{\underline{\gamma}_2}{\gamma_1} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1}{\int_{\gamma_1^H}^{\bar{\gamma}_1} \int_{\underline{\gamma}_2(\gamma_1)}^{\bar{\gamma}_2(\gamma_1)} f(\gamma_1, \gamma_2) d\gamma_2 d\gamma_1} - \frac{\underline{\gamma}_2(\gamma_1^H)}{\gamma_1^H} \right) \end{aligned}$$

and the desired result follows. ■

**Proof of Proposition 10:** Suppose that the dispersion in  $\gamma_2$  collapses, so that  $\underline{\gamma}_2(\gamma_1)$  and  $\bar{\gamma}_2(\gamma_1)$  both converge to a strictly decreasing function  $\gamma_2(\gamma_1)$ . Then both  $\mu(\gamma_1)$  and  $\lambda(\gamma_1)$  converge to  $\gamma_2(\gamma_1)/\gamma_1$ . Consider first the limiting version of condition (49):

$$\frac{\gamma_2(\gamma_1^H)}{\gamma_1^H} > \frac{\int_{\gamma_1^H}^{\bar{\gamma}_1} \frac{\gamma_2(\gamma_1)}{\gamma_1} f(\gamma_1, \gamma_2(\gamma_1)) d\gamma_1}{\int_{\gamma_1^H}^{\bar{\gamma}_1} \frac{\gamma_2(\gamma_1)}{\gamma_1} f(\gamma_1, \gamma_2(\gamma_1)) d\gamma_1}$$

which holds with strict inequality since  $\gamma_2(\gamma_1)/\gamma_1$  is strictly decreasing. It follows that condition (49) is satisfied when the dispersion in  $\gamma_2$  is sufficiently small, and hence that there exists  $\varepsilon > 0$  such that  $p_1^2(a_1) < p_1(a_1)$  for all  $a_1 \in (\gamma_1^H - \varepsilon, \gamma_1^H)$ . Meanwhile, for  $a_1 \leq \gamma_1^H - \varepsilon$ , the limiting version of condition (17) becomes

$$\frac{\int_{a_1}^{\bar{\gamma}_1} \frac{\gamma_2(\gamma_1)}{\gamma_1} P(q^*, \gamma_1, \gamma_2(\gamma_1)) f(\gamma_1, \gamma_2(\gamma_1)) d\gamma_1}{\int_{a_1}^{\bar{\gamma}_1} \frac{\gamma_2(\gamma_1)}{\gamma_1} f(\gamma_1, \gamma_2(\gamma_1)) d\gamma_1} \leq \frac{\int_{a_1}^{\bar{\gamma}_1} P(q^*, \gamma_1, \gamma_2(\gamma_1)) f(\gamma_1, \gamma_2(\gamma_1)) d\gamma_1}{\int_{a_1}^{\bar{\gamma}_1} f(\gamma_1, \gamma_2(\gamma_1)) d\gamma_1} \quad (56)$$

It follows from the reasoning given in the proof of Proposition 4 that when  $\gamma_2(\gamma_1)/\gamma_1$  is strictly decreasing, (56) holds with strict inequality. It follows that condition (17) when the dispersion in  $\gamma_2$  is sufficiently small, condition (17) holds for all  $a_1 \leq \gamma_1^H - \varepsilon$  as well. ■

## References

- [1] Ausubel, L. and R. Deneckere (1989), "Reputation in Bargaining and Durable Goods Monopoly," *Econometrica*, 57(3), pp. 511-531.
- [2] Bryant, J. (1980), "Competitive Equilibrium with Price-Setting Firms and Stochastic Demand," *International Economic Review*, 21 (1), pp. 519-526.
- [3] Caplin, A. (1985), "The Variability of Aggregate Demand with (S,s) Inventory Policies," *Econometrica*, 53(6), pp. 1395-1409.
- [4] Crew, M. A. and P. Kleindorfer (1986), *The Economics of Public Utility Regulation*, MIT Press, Cambridge, Massachusetts.
- [5] Dana, J. D. Jr. (1999), "Equilibrium Price Dispersion under Demand Uncertainty: the Roles of Market Structure and Costly Capacity," *Rand Journal of Economics*, pp. 632-660.
- [6] Dana, J. D. Jr. (1998), "Advanced Purchase Discounts and Price Discrimination in Competitive Markets," *Journal of Political Economy*, 106, pp. 395-422.
- [7] Dana, J. D. Jr. (1999), "Using Yield Management to Shift Demand When the Peak Time is Unknown," *Rand Journal of Economics*, 30, pp. 456-474.
- [8] Danthine, J.-P. (1977), "Martingale, Market Efficiency and Commodity Prices," *European Economic Review*, 10(1), pp. 1-17.
- [9] Deneckere, R., H. P. Marvel, and J. Peck (1996), "Demand Uncertainty, Inventories, and Resale Price Maintenance," *Quarterly Journal of Economics*, August, pp. 885-913.
- [10] Eden, B. (1990), "Marginal Cost Pricing when Spot Markets are Complete," *Journal of Political Economy* 98, December, pp. 1293-1306.
- [11] Eden, B. (2009), "Efficient Barriers to Trade: A Sequential Trade Model With Heterogeneous Agents," *Journal of International Economics* 77(2), pp. 234-244.
- [12] Fudenberg, D., D. Levine, and J. Tirole (1985), "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information," in Alvin Roth (Ed.), *Game Theoretic Models of Bargaining*, Cambridge, England: Cambridge University Press.

- [13] Gul, F., H. Sonnenschein, and R. Wilson (1986), "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory*, 39, pp. 155-190.
- [14] Harris, M. and A. Raviv (1981), "A Theory of Monopoly Pricing Schemes with Demand Uncertainty," *American Economic Review*, 71 (3), pp. 347-365.
- [15] Judd, K. L. (1996), "Cournot Versus Bertrand: A Dynamic Resolution," Working Paper, Hoover Institution, Stanford University.
- [16] Lazear, E. (1986), "Retail Pricing and Clearance Sales," *The American Economic Review*, 76(1), pp. 14-32.
- [17] Lucas, R. E. and M. Woodford (1993), "Real Effects of Monetary Shocks in an Economy with Sequential Purchases," NBER Working Paper No. 4250, January.
- [18] Pashigian, B. P. (1988), "Demand Uncertainty and Sales: A Study of Fashion and Markdown Pricing," *American Economic Review*, 78 (5), pp. 936-953.
- [19] Shaked, M. and J. Shantikumar (1994), "Stochastic Orders and their Applications," Academic Press, San Diego.
- [20] Prescott, E. C., "Efficiency of the Natural Rate," *Journal of Political Economy*, 1975, 83(6), pp. 1229-1236.
- [21] Stokey, N. (1981), "Rational Expectations and Durable Goods Pricing," *Bell Journal of Economics*, 12, pp. 112-128.
- [22] Wang, R. (1993), "Auctions versus Posted-Price Selling," *American Economic Review*, 83(4), pp. 838-851.
- [23] Warner, E. J. and R. B. Barsky (1995), "The Timing and Magnitude of Retail Store Markdowns: Evidence from Weekends and Holidays," *The Quarterly Journal of Economics*, 110(2), pp. 321-352.
- [24] Whitt, W., (1982), "Multivariate Monotone Likelihood Ratio Order and Uniform Conditional Stochastic Order," *Journal of Applied Probability*, 19(3), pp. 695-701.

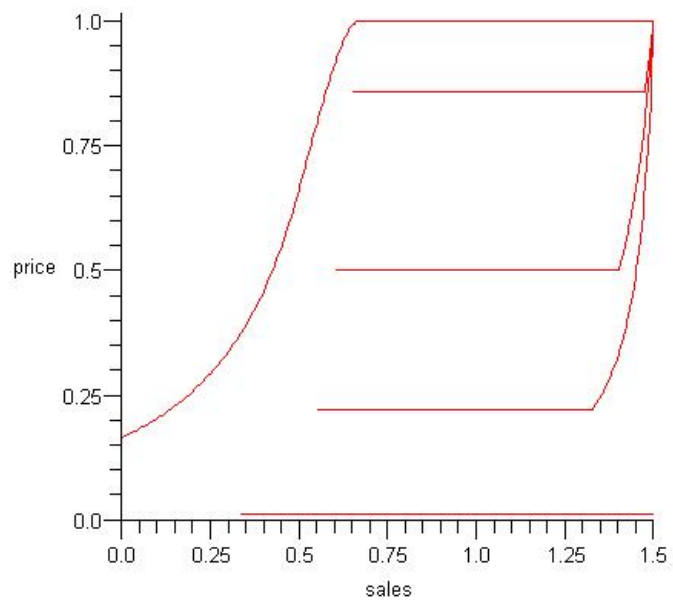


Figure 1: Price vs. Cumulative Sales

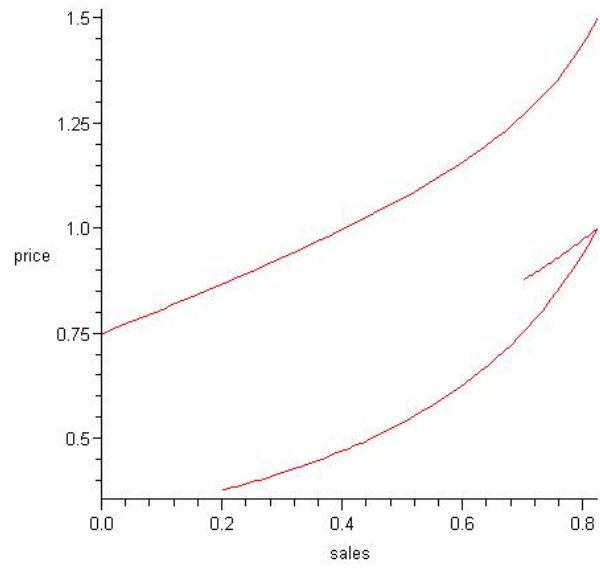


Figure 2: Price vs. Cumulative Sales