Competing Mechanisms with Multi-Unit Consumer Demand

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Abstract

The competing mechanisms literature is extended to a market setting in which firms have fixed capacity, and there is a continuum of consumers who desire multiple units and can only purchase from one firm. Firms choose incentive compatible mechanisms in which consumers report their utility types; consumption of the good and payments of the numeraire are continuous functions of the reports. Uniform price auctions with reserve prices, reinterpreted as direct mechanisms, are not consistent with equilibrium. However, modified auctions without reserve prices but with type-specific entry fees do constitute an equilibrium of the competing mechanisms game under additional regularity assumptions. When all firms announce fixed prices at the perfectly competitive level, this profile also constitutes an equilibrium of the competing mechanisms game.

keywords: competing mechanisms, multi-unit demand, entry fees

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1. Introduction

This paper extends the competing mechanisms literature to a market setting in which consumers demand multiple units. Specifically, consider a market with a finite number of firms with fixed capacity. There is a continuum of consumers, each of whom demands multiple units and can only visit one firm during the market period.[1] There is no aggregate uncertainty about demand. In the Competing Mechanisms Game, Γ, firms simultaneously announce mechanisms, then consumers learn their utility type, choose a firm, and participate in that firm's mechanism. An allowable mechanism asks each of its consumers to report his type, and specifies the amount of the good received by each consumer and the amount of the numeraire each consumer pays to the firm, as a function of the measures of each type reported to the firm. Mechanisms are required to be feasible, incentive compatible, and continuous. Some indirect mechanisms that can be reinterpreted as allowable mechanisms include (i) fixed-price-per-unit mechanisms, in which the firm specifies a price and consumers are allocated their utility maximizing demands if resources permit, and a rationing rule clears the market if demand exceeds capacity, and (ii) uniform price auctions, possibly with reserve prices or entry fees.

The model and analysis provide several main takeaways. First, we show that Γ has a symmetric equilibrium in which every firm chooses a fixed-price-per-unit mechanism at the competitive equilibrium price, p^c. This result may come as somewhat of a surprise, given the IO literature on price competition with capacity constraints. The reason is that the rationing rule here differs from that of the literature. Here, consumers are exogenously restricted to choose a single firm and to abide by its mechanism, so all purchases must be made from the same firm. Also, in case of excess demand at a firm, the fixed-price-per-unit mechanism endogenously rations consumers with a maximum quantity per customer that clears the market. The reasons behind our competitive result and the connection to the IO literature are discussed further in the literature review section and after Proposition 3.

The second takeaway is the role of entry fees in surplus extraction. There is a symmetric equilibrium to Γ in which all firms choose a uniform price auction with type-specific entry fees. On the equilibrium path, each firm's auction price is p^c, and additional profits arising due to imperfect competition are extracted

[1] For example, suppose each of n firms owns a mountain for downhill skiing, with a chair lift that can accommodate a fixed number of ski runs during a particular day.

through entry fees. In contrast, uniform price auctions with no entry fees and reserve prices (binding or not), studied in the competing auctions literature, is not consistent with equilibrium to Γ. The connection to the auctions literature and the competing auctions literature is discussed in the literature review section.

The third takeaway is not a result, but it provides intuition for some of the results and provides an illustration of an important feature of competing mechanisms models with a finite number of firms: Given the mechanisms of the other firms, two mechanisms yielding the same allocation are not equivalent, because they can affect the incentives of other firms. For example, consider the two mechanisms, (i) a fixed-price-per-unit mechanism at p^c, and (ii) a uniform price auction with no reserve price and no entry fees. If all firms are choosing mechanism (i), then both mechanisms are best responses for a particular firm, f, and both mechanisms give rise to the competitive equilibrium allocation. It would seem that it does not matter whether firm f chooses mechanism (i) or mechanism (ii), but this is not the case. If firm f instead chooses the auction mechanism, then that might affect the best responses of the *other* firms. Indeed, we show that all firms choosing the fixed price mechanism (i) is consistent with equilibrium, but all firms choosing the auction mechanism (ii) is not consistent with equilibrium. In both cases, a firm considering a deviation that extracts more surplus from its customers will induce a new consumer equilibrium that loses some of its customers, and this outflow will lower the utility received by consumers visiting other firms. If other firms are choosing (i), there will be excess demand and rationing at the other firms, but the price will not change. If other firms are choosing (ii), the auction price at other firms will rise to clear the market.

The fourth takeaway is that this model of competing mechanisms combines a set of features that offers an unusual degree of tractability. Consumers report their demand types, rather than messages about the mechanisms chosen by other firms. There is a continuum of consumers of each type, and a single consumer's arrival choice or report has a negligible influence on any other agent in the economy. This structure provides a tractable way to study imperfect competition by the firms offering the mechanisms. The equilibrium mechanisms can turn out to be very simple, possibly involving no reports whatsoever.

Section 2 contains a literature review. Section 3 sets up the economic environment and defines the competing mechanisms game. Section 4 contains results about auction mechanisms. Section 5 contains results about price-per-unit mechanisms. Section 6 contains additional results for the case of one consumer type. In particular, it is shown that, when there is only one type of consumer and when mechanisms are monotonic in the sense of offering weakly lower utility when more consumers arrive at a firm, then equilibria are efficient in the sense of fully allocating capacity. Section 7 contains some brief concluding remarks. Proofs of all results are given in the Appendix, and an on-line Appendix contains some examples, illustrating the tractability of the model and the fact that results go through when the number of firms is as small as two. The on-line Appendix also contains details of certain claims made in the paper.

2. Literature Review

There is a considerable literature on competing mechanisms. McAfee (1993) provides a model in which sellers with one unit of a good choose efficient auctions in equilibrium, out of a general class of mechanisms. The number of sellers is assumed to be large enough that they can ignore their effect on the broader market. Peters and Severinov (1997) study competing sellers who choose second-price auctions with a reserve price, and show that reserve prices converge to zero as the number of sellers approaches infinity. Their solution concept specifies beliefs about the distribution of customers they will receive if they deviate from the common reservation price, and these beliefs are correct in the limit. Given these beliefs, firms maximize their expected profit. When the number of sellers approaches infinity, a second-price auction with a reserve price is always optimal, even if more general mechanisms are allowed. How should we reconcile this observation with the result from Proposition 1 in the present paper, that uniform price auctions with a reserve price are not consistent with equilibrium? The main difference is that, in the present paper, we assume that the number of firms, n, is finite, although it is "large enough" to establish the relevant second order conditions. If all other firms are selecting a uniform price auction with a zero reserve price, then the profitable deviation in the proof of Proposition 1 converges to no deviation in the limit as n approaches infinity.[2] Indeed, the equilibria considered in this paper all give rise to the competitive equilibrium allocation as n approaches infinity, with the exception of the full-extraction result in Proposition 5.

Burguet and Sákovics (1999) model a game with N buyers and two sellers

[2] A secondary difference is that here consumers demand multiple units. When n is large but finite, a deviation to a mechanism close to an auction with a zero reserve price is slightly more profitable. In the version of the model with unit demands, it is shown in the on-line Appendix that there is always an equilibrium where all firms set a zero reserve price, because an elasticity condition is satisfied for large n.

who choose second-price auctions with a reserve price. Although they cannot fully characterize the symmetric equilibrium, they show that it involves mixed strategies, and the support of the equilibrium reserve price is bounded above zero. Moreover, in their Proposition 4, Burguet and Sákovics (1999) allow for more general mechanisms and show that there cannot be an equilibrium in which all types participate (roughly corresponding to the competitive equilibrium allocation). The main reason that competitive allocations do not arise in Burguet and Sákovics (1999), while they do arise in the present model, is their finite buyer assumption. As N approaches infinity, it may be that the equilibria are becoming more and more competitive.

Pai (2014) models competition between two sellers, each with a single unit, who choose mechanisms from the space of "extended auctions" (which includes posted prices). It is shown that two forms of inefficiency exist in equilibrium: sellers sometimes withhold the good, and the good is sometimes allocated to an agent that does not have the highest valuation. Coles and Eeckhout (2003) consider a model with two sellers, each with one unit, and two identical buyers. When sellers can choose arbitrary anonymous mechanisms within this environment, it is shown that there is a continuum of equilibria, including price-posting and auctions with a reserve price. Virág (2007) generalizes Coles and Eeckhout (2003) by introducing multiple buyer types, so that price-posting is no longer efficient. Virág (2007) shows that, with two types, only ex-post efficient mechanisms such as auctions are consistent with equilibrium. Peters and Severinov (2006) consider a dynamic competing auctions setting, where buyers have multiple opportunities to place bids on any auction. A perfect Bayesian equilibrium is characterized in which buyers adopt the simple strategy of augmenting the lowest available standing bid by the minimum increment, as long as their bid does not exceed their value. It is shown that when the number of sellers is large, they choose a zero reserve price. All of the above-mentioned papers assume that sellers own one unit of the good and that buyers have unit demands.

A few papers study competing mechanisms in more general and abstract settings. Peters and Troncoso-Valverde (2013) prove a folk-theorem, showing that an allocation can be supported as an equilibrium outcome whenever it is incentive compatible and individually rational. The construction allows reports to depend on the agent's utility function and the mechanisms of the other sellers. See also Epstein and Peters (1999) and Peters and Szentes (2012). Pavan and Calzolari (2010) avoid the infinite regress problem by introducing extended direct mechanisms, where an agent reports his exogenous utility type and the endogenous payoff-relevant contracts chosen with the other principals. They focus on outcomes, sustained in pure-strategy equilibria in which the agent's behavior is Markovian (depending only on payoff-relevant information), but this is without loss under some conditions. Pavan and Calzolari (2009) study extended direct mechanisms with sequential contracting. The present paper also pursues a parsimonious approach, in a different setting, where multiple agents (the consumers) can only contract with one of the principals (the firms). In the present paper, consumers only report information about their utility functions. There is a continuum of consumers of each type, so a single consumer's arrival choice or report has a negligible influence on any other agent in the economy. It should be emphasized that consumers are able to convey market information to firms by their presence or absence. For example, if one firm deviates in an attempt to steal customers from other firms, the other firms will observe fewer than expected customers and detect that a deviation has occurred. This observation explains why multiple equilibrium is inevitable. However, the limited communication possibilities restrict the possible equilibrium outcomes relative to the folk theorem, as demonstrated by Proposition 6.

The literature on (single-seller) auctions with entry addresses the question of reserve prices vs. entry fees, although the role of entry fees is dramatically different from the present setting. Levin and Smith (1994) consider a model in which the seller announces an entry fee and a reserve price, after which buyers decide whether or not to receive private information and enter the auction. It is shown that the seller chooses a zero reserve price and a zero entry fee, because that induces the socially efficient amount of entry and all surplus goes to the seller. In Engelbrecht-Wiggans (1993), entry might be slightly profitable, just not profitable enough for an additional buyer to enter, so entry fees are used to extract this small amount of additional surplus due to the integer problem. In the present paper, entry fees are a manifestation of the additional surplus that can be extracted due to imperfect competition, akin to two-part pricing.

The present paper is closely related to several papers in the IO literature. The seminal study of supply function competition by Klemperer and Meyer (1989) shares similarities to the current model, yet there are significant differences. One similarity is that sellers supply multiple units and buyers can consume multiple units. Another similarity is that, for the version of their model without demand uncertainty, behavior off the equilibrium path affects the incentives of other firms to deviate and gives rise to multiple equilibria (the "third takeaway" described above).[3] A major difference is that Klemperer and Meyer (1989) model a centralized market clearing structure, while here each firm chooses its own mechanism, and market clearing at a single price is not imposed. Another difference is that firms produce to order in Klemperer and Meyer (1989), while firms are endowed with a fixed capacity here.

The result in Proposition 3, that all firms choosing a fixed-price-per-unit mechanism at p^c is consistent with equilibrium to Γ, is related to the IO literature on price competition. The closest paper is by Osborne and Pitchik (1986), who consider a duopoly model of price competition with capacity constraints and identical consumers who demand multiple units. For the version of their model in which capacities are fixed, they find that equilibrium is often in mixed strategies. The difference in results is due to the fact that consumers first purchase as much as they can/want from the lower price firm and then purchase the rest of their demand from the higher price firm, while we require that consumers can only purchase from one firm. Peters (1984) studies a model of price competition in which firms choose prices, following by a finite number of identical buyers choosing a single firm. In equilibrium, firms choose mixed strategies, although it is close to average cost pricing with a large number of firms. In Deneckere and Peck (1995), firms choose both price and capacity, then consumers choose a single firm. There is a continuum of consumers with unit demands. Because of aggregate uncertainty, prices are set above marginal cost, and both excess capacity and stockouts can arise.

In Dixon (1992), firms compete by choosing a price and a maximum quantity. Consumers demand multiple units, and purchase from the firm offering the lowest available price according to a first-come-first-served rationing rule. Because consumers are negligible, there would be no reason to purchase from more than one firm under this rationing rule. Production costs, which are not necessarily linear, are incurred only for the quantity demanded. Dixon (1992) shows that the only possible pure-strategy Nash equilibrium involves all firms setting the competitive equilibrium price, and that such an equilibrium exists under reasonable conditions. Intuitively, firms set the competitive price and offer to sell more than the profit maximizing quantity, which they are not required to do on the equilibrium path, but which deters any incentive for other firms to raise their price. Burguet and

[3] When Klemperer and Meyer introduce demand uncertainty, no prices are off the equilibrium path, so the scope for multiple equilibria is eliminated or greatly reduced. Unlike their setting with one-dimensional uncertainty facing firms (i.e., the price), introducing demand uncertainty in the present paper would involve multi-dimensional uncertainty (i.e., measures of each type).

Sákovics (2017) consider a model in which firms make personalized price offers to consumers, which they are obliged to honor, so that the allocation of consumers to firms occurs endogenously and there is no need for a rationing rule. Consumers are negligible and have unit demands. There is a unique equilibrium outcome, in which all trades take place at the competitive equilibrium price and all firms sell their competitive equilibrium quantity. One way that the present paper differs from Dixon (1992) and Burguet and Sákovics (2017) is that firms have a fixed capacity.[4] Another distinction is that the present paper allows for more general mechanisms than offering consumers a price.

3. The Competing Mechanisms Game

We consider a market with n firms selling a homogeneous good, and for simplicity, we assume that they all have the same capacity, normalized to 1, and no costs. There are I types of consumers, and a continuum of consumers of each type. Denote the measure of type i consumers as $r_i n$. Each consumer of type i has the quasilinear utility function $u_i(x_i) + M_i$, where x_i is the consumption of the (divisible) good and M_i is the consumption of the numeraire (or money). We assume that each consumer has a sufficiently large endowment of money to make any desired purchases, and that the utility function for each i satisfies $u_i'(x_i) > 0$ and $u_i''(x_i) < 0$ for all x_i.

Although we think of firms as being geographically separated, so that a consumer can visit at most one firm, the competitive-equilibrium benchmark will be useful. A consumer of type i facing price p will choose the quantity of the good satisfying

$$u_i'(x_i) = p,$$

whose solution we denote by the demand function, $d_i(p)$. Each firm inelastically supplies its capacity, so the competitive equilibrium price, denoted by p^c, is the unique solution to

$$\sum_{i=1}^{I} r_i n d_i(p^c) = n. \tag{3.1}$$

[4] One could consider the firms to have a cost function that is zero up until capacity, and infinite beyond the capacity. Such a cost function would violate the differentiability assumptions in Dixon (1992) and Burguet and Sákovics (2017).

We assume that types can be ranked in terms of willingness to pay, so that $i < h$ implies $d_i(p) > d_h(p)$ for all p.

In the Competing Mechanisms Game, denoted by Γ, firms simultaneously select a mechanism from a class of mechanisms, M, defined below. We restrict attention to incentive-compatible direct-revelation mechanisms, where consumers report their utility type. Let ρ_i^f denote the measure of agents participating in firm f's mechanism and reporting type i, and define $\rho^f = (\rho_1^f, ..., \rho_I^f)$. A mechanism for firm f, denoted by m^f, consists of continuous functions $x_i^f(\rho^f)$ and $P_i^f(\rho^f)$, satisfying for all ρ^f the feasibility condition,

$$\sum_{i=1}^{I} \rho_i^f x_i^f(\rho^f) \leq 1. \tag{3.2}$$

Given the reports ρ^f, $x_i^f(\rho^f)$ is the amount of the good received by a consumer reporting type i at firm f, and $P_i^f(\rho^f)$ is the non-negative money payment made by a consumer reporting type i at firm f. The profit or payoff to firm f is given by $\sum_{i=1}^{I} \rho_i^f P_i^f(\rho^f)$.

The timing of Γ is as follows. First, firms simultaneously choose a mechanism. Then consumers observe their type and the profile of mechanisms selected by the firms, denoted by $m = (m^1, ..., m^n)$. Finally, consumers choose which firm to visit, report a type, and participate in that firm's mechanism.[5] Our solution concept is subgame perfect Nash equilibrium in which all consumers of the same type choose the same mixed strategy.[6] That is, for any profile of mechanisms m, all consumers of the same type choose the same mixed strategy over arrivals. We denote an equilibrium by SPNE, and unless otherwise specified we will consider equilibria in which firms use pure strategies. Since we only consider type-symmetric equilibria and the relevant subgame will always be clear from the context, we can denote

[5] We do not impose individual rationality restrictions. However, individual rationality holds on the equilibrium path for all of the mechanisms considered in this paper. Also, individual rationality on and off the equilibrium path would hold if we were to impose the Inada condition, $u(0) = -\infty$.

[6] Strictly speaking, the nodes of the game tree, following the firms' mechanism choices and the observation of consumer types, do not initiate subgames. Consumers know the aggregate distribution of types, and that is all that matters for their decisions, but they do not observe individual types. We could adopt the PBE concept and specify beliefs about the type realizations of individual consumers, but there should be no confusion instead in simply referring to any realization of types following a choice of mechanisms as a subgame. Alternatively, we could model the contrived but essentially equivalent game in which consumers observe everyone's type after the firms choose their mechanisms.

the probability that a consumer of type i visits firm f as β_i^f.[7] Given a non-zero vector of arrival probabilities at firm f, $\beta^f = (\beta_1^f, ..., \beta_I^f)$, a mechanism is *incentive compatible* if reports satisfy the truth-telling condition,

$$\begin{aligned} u_i(x_i^f(\rho^f)) - P_i^f(\rho^f) &\geq u_i(x_h^f(\rho^f)) - P_h^f(\rho^f), \text{ for all } i, h, \text{ where} \\ \rho_i^f &= r_i n \beta_i^f \text{ holds.} \end{aligned} \tag{3.3}$$

We define M to be the set of continuous functions from reported types into a quantity consumed and payment by each type, satisfying (3.2) and (3.3) for all β^f.

Example: Fixed-Price-Per-Unit Mechanism.

If firm f chooses a fixed price per unit, p^f, then if there is no excess demand at firm f, each consumer receives his utility maximizing consumption and pays the per unit price p^f. If there is excess demand, then some consumers are rationed but each consumer continues to pay the per unit price p^f. We assume that there is a maximum quantity that any consumer can choose, $\overline{x}^f$, which clears the market as defined below. Consumers whose demand exceeds $\overline{x}^f$ consume at the maximum limit, and consumers whose demand is less than $\overline{x}^f$ consume their utility maximizing quantity.

Here is the mechanism in which firm f chooses a fixed price per unit, p^f.

$$\begin{aligned} \text{For} \sum_{i=1}^{I} \rho_i^f d_i(p^f) &\leq 1, \\ x_i^f(\rho^f) &= d_i(p^f) \quad \text{and} \quad P_i^f(\rho^f) = p^f d_i(p^f). \\ \text{For} \sum_{i=1}^{I} \rho_i^f d_i(p^f) &> 1, \\ x_i^f(\rho^f) &= \min[d_i(p^f), \overline{x}^f(\rho^f)] \quad \text{and} \\ P_i^f(\rho^f) &= p^f \min[d_i(p^f), \overline{x}^f(\rho^f)], \end{aligned} \tag{3.4}$$

where $\overline{x}^f(\rho^f)$ is the solution to

$$\sum_{i=1}^{I} \rho_i^f \min[d_i(p^f), \overline{x}^f(\rho^f)] = 1. \tag{3.5}$$

[7] We assume that the conclusion of the law of large numbers holds, so that if all type i consumers use the mixing probability β_i^f, then the measure of type i consumers visiting firm f is $r_i n \beta_i^f$.

From (3.5), it follows that whenever the consumption limit matters, $\sum_{i=1}^{I}\rho_i^f d_i(p^f) > 1$, then $\overline{x}^f(\rho^f)$ is uniquely defined. It is easy to see that this mechanism is continuous in ρ^f and is incentive compatible.

The class of allowable mechanisms, M, is fairly broad. However, M is far from completely general. Consumers are reporting their valuation-types, but not their full type which includes information about the other firms' mechanisms. See Epstein and Peters (1999) for an analysis of how to build a universal type space. It seems reasonable, when we are modeling competition by firms who set up their own markets to sell their capacity, to rule out mechanisms requiring higher order reports about the mechanisms of other firms.[8]

Whether or not a mechanism is incentive compatible can depend on β^f, and requiring that incentive compatibility holds for all β^f can be restrictive.[9] However, this restriction is only used to guarantee the existence of a Nash equilibrium of all subgames off the equilibrium path. The price-per-unit and price-per-share mechanisms (discussed below) have the nice property that no reports of any sort are required, and are obviously incentive compatible for any β^f.

Without the continuity assumption, we would typically have the problem that no SPNE exists. The reason is that, following a deviation in which firm f chooses a mechanism that is not continuous in ρ^f, the resulting consumer subgame often has no Nash equilibrium, due to the fact that we have a continuum of consumers.

For example, suppose there is only one type of consumer and all firms other than firm f choose a fixed price-per-unit of p^c. Suppose firm f chooses the following mechanism (for some positive ε), which is not continuous:

$$\begin{aligned}
x_1^f(\rho^f) &= \frac{1}{\rho_1^f} \\
P_1^f(\rho^f) &= (p^c+\varepsilon)d_1(p^c) \quad \text{if} \quad \rho_1^f \geq r_1 \\
P_1^f(\rho^f) &= 0 \quad \text{if} \quad \rho_1^f < r_1.
\end{aligned}$$

For this profile of mechanisms, the consumer subgame has no Nash equilibrium. There cannot be a NE with $\beta_i^f \geq \frac{1}{n}$, because consumers are not rationed at the

[8]Peters and Troncoso-Valverde (2013) develop the notion of sequential communication mechanism. The mechanisms allow players to report sequentially, first reports about types and then reports about the first-round reports of other players.

[9]See Peck (1997) for an example, in a different context, in which the revelation principle might fail when only valuation-types are reported.

other firms, and receive their competitive equilibrium utility; at firm f, consumers pay strictly more than what they would pay at other firms and receive less consumption, so consumers visiting firm f are not best responding. There cannot be a NE with $\beta_i^f < \frac{1}{n}$, because consumers visiting other firms are rationed and receive less than their competitive equilibrium utility; consumers at firm f receive well above their competitive equilibrium utility, so consumers visiting other firms are not best responding.

Fortunately, Lemma 1 shows that, for our class of allowable mechanisms, which requires continuity and incentive compatibility, the consumer subgame always has a Nash equilibrium.

Lemma 1: *For any profile of mechanisms, $m = (m^1, ..., m^n)$, where $m^f \in M$ for $f = 1, ..., n$, the resulting consumer subgame has a type-symmetric NE.*

4. Auction Mechanisms

4.1. Uniform Price Auctions with Reserve Prices

Many types of auctions are included within the set of allowable mechanisms, M. For example, if firm f holds a uniform price auction with reserve price R^f, its consumers submit demand schedules for prices greater than or equal to R^f, interpreted as follows. The "height" of the demand schedule, evaluated at quantity x, is his bid for the marginal unit, given that x units have been acquired. The firm collects the submitted demand schedules, and the auction price is the highest rejected bid. That is, the auction price is the price that clears the market if the measure of bids exceeding R^f is greater than the firm's supply of 1; otherwise, the auction price is R^f. Since consumers are negligible and cannot affect the auction price, in equilibrium a type i consumer will submit his true demand function, $d_i(p)$, for $p \geq R^f$.

Here is the mechanism corresponding to a uniform price auction with reserve price R^f. Given ρ^f, let p solve

$$\sum_{i=1}^{I} \rho_i^f d_i(p) = 1$$

and let $p^R \equiv \max[p, R^f]$. Then we have

$$\begin{aligned} x_i^f(\rho^f) &= d_i(p^R) \text{ and} \\ P_i^f(\rho^f) &= p^R d_i(p^R). \end{aligned} \tag{4.1}$$

Incentive compatibility and continuity are clearly satisfied.

In order to compare our setting to the competing auctions literature with single-unit demands, before considering whether profiles of auction mechanisms are equilibria to Γ, we consider the "reserve price game" Γ^R in which firms must choose a mechanism as specified in (4.1) for some R^f. In any equilibrium of the consumer subgame, the auction price at all firms attracting consumers must be the same. If not, a consumer could instead choose a firm with a lower auction price and receive his utility maximizing quantity at that price. For any firm f setting a reserve price that is non-binding in the ensuing consumer equilibrium, obviously it will sell its entire capacity. If there is a firm f attracting customers but whose reserve price, R^f, is binding in the ensuing consumer equilibrium, the auction price at all firms will be R^f, and it must be the case that $R^f > p^c$ holds.[10] Therefore, the firms setting lower (non-binding) reserve prices sell all their capacity, and firm f would not sell all of its capacity since there is excess capacity in the market at this price.[11]

As discussed in Section 2, Burguet and Sákovics (1999) show that symmetric equilibrium involves mixed strategies by firms, and that the support of the equilibrium distribution of reserve prices is bounded above zero. In the present setting, however, Γ^R could have a SPNE in pure strategies in which all firms choose a zero reserve price. To see this, consider the profit function of firm f, if all other firms choose a zero reserve price. For $R^f > p^c$, firm f can only make positive profits if R^f is close enough to p^c so that overall market demand at the price R^f exceeds the capacity of the other firms, $n-1$. In this case, $\{\beta_i^f\}_{i=1}^I$ satisfies

$$\begin{aligned} \sum_{i=1}^I r_i n(1-\beta_i^f) d_i(R^f) &= n-1, \text{ or} \\ \sum_{i=1}^I r_i n \beta_i^f d_i(R^f) &= \sum_{i=1}^I r_i n d_i(R^f) - (n-1). \end{aligned} \tag{4.2}$$

[10] A firm setting $R^f \leq p^c$ will have a non-binding reserve price in the ensuing consumer equilibrium, even if all other firms set a reserve price of zero.

[11] If several firms set the same (binding) reserve price as firm f, then it would be impossible for all of these firms to sell all of their capacity. In the most natural consumer equilibrium, these firms would be treated identically and all of them would have excess capacity.

From (4.2), we can write the profits of firm f as

$$\begin{aligned}\pi^f(R^f) &= \sum_{i=1}^{I} r_i n \beta_i^f d_i(p^f) R^f \\ &= R^f[\sum_{i=1}^{I} r_i n d_i(R^f) - (n-1)]. \end{aligned} \tag{4.3}$$

Differentiating (4.3) with respect to R^f and evaluating at $R^f = p^c$ yields

$$(\pi^f)'(p^c) = 1 + p^c \sum_{i=1}^{I} r_i n d_i'(p^c). \tag{4.4}$$

Whenever the right side of (4.4) is negative, and the second order condition,

$$2\sum_{i=1}^{I} r_i n d_i'(R^f) + R^f \sum_{i=1}^{I} r_i n d_i''(R^f) < 0$$

holds for all $R^f \geq p^c$, then there is a SPNE where all firms set a zero reserve price. It can be shown that the right side of (4.4) is negative if and only if the market price elasticity of demand (in absolute value) is greater than $1/n$ at the competitive price. Thus, unlike the literature that focuses on a finite number of consumers with unit demand, the competitive outcome of all reserve prices being zero can happen, but need not happen, in equilibrium.

Now let us consider the competing mechanisms game, Γ. It turns out that there is no SPNE in which all firms choose a uniform price auction with any reserve price, even allowing for mixed strategies by firms. This result, given in Proposition 1, below, is far from obvious, as evidenced by the fact that the proposition fails if consumers have unit demands. With unit demands, if each firm f is choosing $R^f = 0$, a firm cannot deviate (as the proof requires) by serving fewer customers and more consumption per customer.[12]

[12] With unit demands and discrete types, then when each firm f sets $R^f = 0$, the auction prices will correspond to one of the valuations, v_i. Especially with many firms, it can be the case that the auction price would remain at v_i even if firm f had no customers, so no profitable deviation is possible. With unit demands and a continuum of types, each firm f setting $R^f = 0$ is an equilibrium of Γ whenever the elasticity of demand at the competitive price exceeds $1/n$. For details, see the on-line appendix.

Proposition 1: *There is no SPNE of Γ in which all firms choose reserve price mechanisms, even allowing for mixed strategies by firms.*

Sketch of the proof: Suppose there is such an equilibrium. If firm f sets R^f at the upper support of equilibrium reserve prices and this reserve price is binding, it knows that some of its capacity will be wasted. There is a profitable deviation to another mechanism that allocates more of its capacity, while continuing to attract the same set of customers as before. For other vectors of reports at firm f, either profits are higher than before the deviation, or the reports cannot be consistent with a consumer equilibrium.

If the upper support of the equilibrium reserve prices is not binding, there is a profitable deviation to another mechanism. In the ensuing consumer equilibrium, the deviation by firm f raises the auction prices at other firms to $\widetilde{p} > p^c$. Here is a sketch of the mechanism, based on the desired $\widetilde{p}$. Consumers of type $i > 1$ at firm f are offered $d_i(\widetilde{p})$ units of consumption at the price $\widetilde{p}$, so they are indifferent between firm f and the other firms. Consumers of type 1 (with the highest demand per capita) consume $d_1(\widetilde{p})$ *plus* some of the excess capacity that would arise at the price $\widetilde{p}$. In exchange for the additional consumption, type 1 consumers increase their payment above $\widetilde{p}d_1(\widetilde{p})$ by enough to be indifferent between firm f and the other firms. Thus, for any $\widetilde{p}$ sufficiently close to p^c, the constructed deviation has a consumer equilibrium in which $\widetilde{p}$ is the induced auction price at the other firms. The resulting expression for firm f's profits is increasing in $\widetilde{p}$ when evaluated at p^c, so the deviation is profitable. The proof is complicated by the requirement to continously extend the mechanism off the equilibrium path of the consumer subgame, while maintaining incentive compatibility. Also, the mechanism must be such that every equilibrium of the consumer subgame makes the deviation profitable for firm f.

4.2. Uniform Price Auctions with Entry Fees

With additional structure on utility functions, the following proposition establishes that there is a SPNE to Γ in which firms choose a "modified auction with entry fees" ($\varepsilon - MAE$) mechanism. An "auction with entry fees" mechanism for firm f involves a zero reserve price and a set of entry fees, E_i^f, satisfying: (1) $x_i^f(\rho^f) = d_i(p(\rho^f))$, (2) $P_i^f(\rho^f) = p(\rho^f)d_i(p(\rho^f)) + E_i^f$, and (3) the auction price

$p(\rho^f)$ is determined by the market-clearing condition,

$$\sum_{i=1}^{I} \rho_i^f d_i(p(\rho^f)) = 1.$$

If E_i^f varies across types, the mechanism might not be incentive compatible, for example, when the measure of arriving consumers is small enough that the market clearing price is near zero. Hence, we modify the concept to $\varepsilon - MAE$, in which the entry fees fully apply in an ε neighborhood of p^c, and linearly drop to zero as the price reaches $p^c - 2\varepsilon$ or $p^c + 2\varepsilon$. Thus, in an $\varepsilon - MAE$, the entry fee is given by

$$\begin{array}{rlrl} E_i^f & \text{if} \quad p^c - \varepsilon & \leq & p(\rho^f) \leq p^c + \varepsilon, \\ \left(\dfrac{p(\rho^f) - p^c + 2\varepsilon}{\varepsilon}\right) E_i^f & \text{if} \quad p^c - 2\varepsilon & \leq & p(\rho^f) \leq p^c - \varepsilon, \\ \left(\dfrac{p^c + 2\varepsilon - p(\rho^f)}{\varepsilon}\right) E_i^f & \text{if} \quad p^c + \varepsilon & \leq & p(\rho^f) \leq p^c + 2\varepsilon, \\ & & & \text{and 0 otherwise.} \end{array}$$

Note that the profile of mechanisms given in Proposition 2 below would satisfy incentive compatibility on the equilibrium path without this modification, and there would be no incentive for a firm to deviate. The modification is only used to ensure that the mechanisms are within our allowable set, M.

Proposition 2: *Assume that for $i = 1, ..., I$, demand is of the form $d_i(p) = a_i d(p)$, where a_i is a positive parameter and $d'(p) < 0$. Then if n is sufficiently large and ε is sufficiently small, there is a SPNE of Γ in which, for $f = 1, ..., n$, firm f chooses the following $\varepsilon - MAE$ mechanism:*

$$E_i^* = -\frac{a_i[d(p^c)]^2}{(n-1)d'(p^c)} \quad \text{for } i = 1, ..., I.$$

Along the equilibrium path, we have $\beta_i^f = \frac{1}{n}$ for all i and f, each firm's auction price is p^c, and each firm's profit is

$$p^c - \frac{d(p^c)}{(n-1)d'(p^c)}.$$

Sketch of the proof and intuition for the result: First, we show that $\varepsilon - MAE$ mechanisms satisfy continuity and incentive compatibility. To show that there are no profitable deviations, we make deviations as attractive as possible by allowing a deviating firm f to choose its mechanism *and* the arrival vector of consumers, β^f, to maximize profits subject to being an allowable mechanism and the constraints that each type of consumer is indifferent between firm f and the other firms. To show that the $\varepsilon - MAE$ solves this problem, we use the necessary first-order conditions to show that marginal rates of substitution must be equated at firm f, giving rise to the shadow price, p^f. This implies that the solution must solve the simpler, unconstrained problem of choosing β^f to maximize profits, where the indifference constraints are substituted for P_i^f, and where p^f and the auction price at the other firms, $\widetilde{p}$, are functions of β^f. The interior first-order conditions can be expressed as functions of p^f only, where β^f drops out and $\widetilde{p}$ is a function of p^f. It is shown that $p^f = \widetilde{p} = p^c$ solves the first-order conditions, and that the solution is unique if n is sufficiently large. This pins down the best-response profits of firm f, and establishes that the $\varepsilon - MAE$ mechanism is a best response.

The economic intuition for Proposition 2 is that the best deviation involves allocating capacity based on a shadow price, p^f, and the optimal shadow price (given the other firms' mechanisms) is p^c, which occurs at the $\varepsilon - MAE$. Suppose firm f increases p^f above p^c. Then marginal rates of substitution are above p^c, so firm f is attracting more than "its share" of demand and consumers at firm f receive less than their competitive equilibrium consumption. This has the effect of pushing the auction price at other firms lower, so consumers at other firms receive higher utility. The per capita payment that firm f can extract must therefore be reduced, because per capita consumption is reduced and firm f must match the utility offered by other firms. The entry fees are just right to balance the tradeoff firm f faces between more (respectively, fewer if instead firm f reduces p^f) customers and lower (respectively, higher) payment per customer.

Note that, if other firms are choosing zero entry fees, the solution to the maximization problem in the proof would provide an incentive for firm f to reduce p^f below p^c, requiring an outflow of customers (in order for the marginal rates of substitution for the remaining customers to go down), higher auction prices and lower utility at other firms, but higher payments per customer at firm f. However, this line of reasoning could not be used to prove Proposition 1, by arguing that zero reserve prices (and zero entry fees) cannot be part of an equilibrium to Γ. In the proof of Proposition 2, we allow firm f to choose the equilibrium of the consumer subgame, but the proof of Proposition 1 must consider all possible equilibria of the consumer subgame following the deviation.

5. Fixed-Price-Per-Unit Mechanisms

The following proposition establishes that there is a SPNE of Γ in which all firms choose the fixed-price-per-unit mechanism with the competitive equilibrium price.

Proposition 3: *For sufficiently large n, there is a SPNE of Γ in which, for $f = 1, ..., n$, firm f chooses the price-per-unit mechanism defined in (3.4) with $p^f = p^c$, and $\beta_i^f = \frac{1}{n}$ for all i and f along the equilibrium path.*

Sketch of the proof and intuition for the result: The proof of Proposition 3 solves an optimization problem for a potential deviator, firm f, which chooses its mechanism, $x_i^f(\rho^f)$ and $P_i^f(\rho^f)$, and chooses consumer behavior, β, to maximize profits subject to its resource constraint and the consumer indifference condition necessary for β to be an equilibrium to the consumer subgame. First, it is shown that the solution to this problem entails allocating the good to equalize marginal rates of substitution across the types visiting firm f. Second, it is shown that the deviation cannot be profitable if the shadow price, λ, is greater than or equal to p^c. This is because $\lambda \geq p^c$ implies $x_i^f \leq d_i(p^c)$, so firm f must be attracting more than its share of consumers in order to allocate all of its capacity. It follows that firms other than f are not rationing their consumers, so firm f must be offering at least the competitive equilibrium utility, which it cannot do more profitably. Third, it is shown that the deviation cannot be profitable if $\lambda < p^c$ holds. This is because $\lambda < p^c$ implies $x_i^f > d_i(p^c)$, so firm f must be attracting less than its share of consumers and firms other than f are rationing their type 1 customers. It is shown that the best deviation of this form is to sell only to type 1 consumers. (For large enough n, only type 1 consumers can be rationed, and there will be enough type 1 consumers to consume all of firm f's capacity.) However, the first order conditions are satisfied at $x_1^f = d_1(p^c)$ and $P_1^f = p^c d_1(p^c)$, and the second order conditions are satisfied for large enough n. Therefore, the deviation cannot be profitable.

Focusing attention on fixed-price-per-unit mechanisms, here is the economic intuition for why firms in equilibrium choose the competitive equilibrium price. If firm f decided to raise its price slightly above p^c, the utility offered to its customers declines, necessitating a reduction in the measure of consumers visiting firm f and an increase in the measure of consumers visiting the other firms. Thus, type 1 consumers visiting other firms will be rationed. The resulting equilibrium of the consumer subgame will involve type 1 consumers being indifferent between buying as many units as they want at firm f at the higher price, vs. paying the price p^c and being rationed at one of the other firms. For a small price increase above p^c by firm f, the rate at which utility at firm f is reduced is approximately $d_1(p^c)$. For type 1 consumers choosing some other firm, j, the rate at which utility is reduced is approximately

$$-\frac{\partial x_1^j}{\partial p}[u_1'(x_1^j) - p^c],$$

where x_1^j is less than $d_1(p^c)$ due to excess demand and rationing. However, when the price increase is small, the term in brackets in the above expression is approximately zero, due to the envelope theorem. For the indifference condition to be satisfied, the above expression must equal $d_1(p^c)$, so $\frac{\partial x_1^j}{\partial p}$ must equal negative infinity. That is, we must have an infinite rate of outflow of demand from firm f at the margin. In other words, for a small increase in the price chosen by firm f, the reduction in the quantity sold is an order of magnitude greater than the increase in the price. Locally, raising or lowering the price from p^c strictly lowers profits. The assumption that n is sufficiently large is only needed as a sufficient condition to establish that a local optimum is a global optimum in the relevant maximization problem. For the examples presented in the on-line Appendix, the competitive pricing result holds for all $n > 1$.

Other consumer equilibria exist, yielding the same allocation, but a different pattern of consumer types across firms. Rather than all firms seeing the same distribution of consumer types, as in Proposition 3, any mixed strategy profile by consumers is consistent with equilibrium, as long as the total demand at each firm, at the price p^c, is exactly equal to the total capacity at each firm, 1.

A corollary of Proposition 3 is that competitive pricing is an equilibrium of the game in which firms are restricted to choose a price per unit. This form of price competition, where the rationing rule requires consumers to purchase from only one firm, has not been discussed before in the IO literature. It is worth emphasizing that the reason for the perfectly competitive outcome is completely different from that of the Bertrand model without capacity constraints. True, when all firms choose a price per unit, p, where $p > p^c$ holds, then firms will not sell all of their capacity, and a firm that deviates to a slightly lower price can profitably sell more or all of its capacity. However, when all firms choose a price per unit where $p < p^c$ holds, then a firm that deviates to a higher or lower price *does not* face an infinite rate of outflow or inflow of customers, even at the margin; there would be a profitable deviation to a higher price. The actual case in which all firms set $p = p^c$ is a borderline case. All capacity is used, so a firm could never benefit from lowering its price. However, a firm does not want to raise its price either, due to the envelope theorem logic explained above.

We finish this section with a discussion of the robustness of the competitive pricing result to the rationing rule.[13] Details of the claims made in this discussion are available in the on-line Appendix. Consider the (symmetric) n firm version of Osborne and Pitchik (1986), who assume one type of consumer and a rationing rule in which consumers can purchase from multiple firms, but where there is a maximum quantity per customer at any given firm. The competing mechanisms game where consumers choose a subset of firms is beyond the scope of this paper, but we can consider the restriction to price setting (where consumers can purchase as much as desired up to the limit). Then there is a pure strategy equilibrium with $p = p^c$ if and only if the elasticity of demand in absolute value is greater than $1/n$.

What about the endogenous feature of the rationing rule, specifying a maximum quantity per customer that clears the market at each firm? The set M requires alternative rationing rules to be continuous and type-symmetric, although there is flexibility to ration some or all consumer types. Thus, the intuition based on the envelope theorem continues to apply. A small price increase by firm f induces an outflow of consumers of a higher order of magnitude than the price increase, at the margin. Large deviations must also be ruled out, but a reasonable conjecture is that Proposition 3 is robust to the specifics of the rationing rule. However, if we enlarge M to allow fixed-price-per-unit mechanisms with first-come-first-served rationing,[14] competitive pricing with first-come-first-served rationing is never consistent with equilibrium. See the on-line Appendix for details.

[13] For a classic example of the importance of the rationing rule, see Kreps and Scheinkman (1983) and Davidson and Deneckere (1986).

[14] Under M, consumers of the same type must receive the same consumption. If firm f chooses the price p^c with first-come-first-served rationing, then if there is excess demand, we have $x_i^f(\rho^f) = x_i(p^c)$ w.p. μ, and $x_i^f(\rho^f) = 0$ w.p. $1 - \mu$. The parameter μ is chosen to exactly allocate capacity.

6. Additional Results for $I = 1$

6.1. Fixed-Price-Per-Share Mechanisms

With a fixed-price-per-share mechanism, rather than setting a price for each unit and letting consumers decide how many units to buy, here a firm sets a price for the right to consume an equal share of the firm's capacity.[15] Here is the mechanism in which firm f chooses a fixed price-per-share, P^f.

$$\begin{aligned} x_i^f(\rho^f) &= \frac{1}{\sum_{h=1}^{I} \rho_h^f} \\ P_i^f(\rho^f) &= P^f. \end{aligned} \tag{6.1}$$

In (6.1), capacity is divided evenly across all consumers at firm f, and each consumer makes a total payment equal to P^f, independent of how many consumers choose firm f. This mechanism implies infinite consumption when the measure of consumers is zero, so we will have to modify it slightly to avoid a discontinuity at zero. A modified fixed-price-per-share mechanism caps consumption at X, with the idea that X is so large that it cannot bind in equilibrium.

$$\begin{aligned} x_i^f(\rho^f) &= \min[\frac{1}{\sum_{h=1}^{I} \rho_h^f}, X] \\ P_i^f(\rho^f) &= P^f. \end{aligned} \tag{6.2}$$

This mechanism is continuous and incentive compatible. Proposition 4, below, shows that for the special case of one consumer type there is a SPNE of Γ in which all firms choose a modified fixed-price-per-share mechanism. In this equilibrium, consumers receive the same consumption of the good as in the competitive equilibrium, but their total payment is higher than in the competitive equilibrium. The cap, $X = \frac{n}{r_1}$, is only needed to ensure that the mechanism is continuous everywhere and within the allowable set, M; it is not needed for the result itself.

Proposition 4: *If we have $I = 1$, then for sufficiently large n, there is a SPNE of Γ in which all firms choose a modified fixed-price-per-share mechanism defined*

[15] For example, under ski lift competition, a fixed-price-per-share mechanism is simply a lift ticket. The customer would pay for the right to go on multiple runs, with the lift queue guaranteeing that all customers receive the same quantity, as determined by the lift capacity and the number of customers. See Barro and Romer (1987).

in (6.2). That is, each firm f chooses the share price

$$P^f = P^* \equiv \frac{np^c}{(n-1)r_1}$$

and any (non-binding) cap, $X > \frac{n}{r_1}$. Consumers (on the equilibrium path) choose a mixed strategy that assigns probability $\frac{1}{n}$ to each firm.

What is the intuition for why firms raise its effective price[16] above p^c with price-per-share competition, but not with price-per-unit competition? The answer is that, in the two equilibria, there is a different effect of a price increase on the utility received in the consumer subgame. In both equilibria, a price increase by firm f sends consumers to the other firms, so that the quantity they consume at other firms in the consumer subgame falls. With price-per-unit competition, the payment consumers make at other firms also falls. However, with price-per-share competition, when firm f increases its price and consumers shift to the other firms, the payment consumers make at other firms does not fall. It turns out that this softening of competition provides an incentive to raise the share price above p^c. In Proposition 4, the assumption that n is "sufficiently large" is only used as a convenient way of demonstrating the second order conditions. Examples indicate that n can be as small as 2.

When $I > 1$ holds, there is no hope for a SPNE in which all firms choose the same fixed-price-per-share mechanism, because that would entail the same consumption by all consumers. Barro and Romer (1987) assume that firms are perfectly competitive, and argue that, with heterogenous consumers, there will be an equilibrium in which firms specialize in serving one consumer type. Translated into the current notation, a firm serving type i will choose a share price (they call it a lift-ticket price) equal to $p^c d_i(p^c)$. The number of firms serving type i, denoted by n_i, would be determined by the condition that per capita consumption is $d_i(p^c)$, so $n_i = r_i n d_i(p^c)$ would hold. Is there a similar result for Γ with imperfect competition? The answer, generically, is no, due to integer constraints.[17] To prevent a local deviation targeting the same consumer type, the consumption of type i consumers will generally have to differ from $d_i(p^c)$. As a result, marginal rates of

[16] The effective price of firm f with price-per-share competition (Proposition 4) is defined to be P^f/x^f.

[17] Barro and Romer (1987) ignore integer constraints, presumably because they imagine that n is large.

substitution will not be equated across consumer types, creating the incentive for a firm to adopt a more complicated mechanism that attracts multiple consumer types.[18]

6.2. Full Surplus Extraction with $I = 1$

When there is only one consumer type, $I = 1$, there are other equilibria to Γ in which firms receive even higher profits than the equilibria of Propositions 3 and 4, based on mechanisms that become very generous to consumers when fewer than expected arrive at the firm. Proposition 5 shows that full surplus extraction is possible in equilibrium. See Virag (2011) for a similar full-extraction result.

Proposition 5: *Assume that there is only one consumer type, $I = 1$, and that $u_1(0)$ is finite. Then for a sufficiently large parameter (of the mechanism), A, there is a SPNE of Γ in which, for $f = 1, ..., n$, firm f chooses the following mechanism:*

$$\begin{aligned}
x_1^f(\rho^f) &= \frac{1}{\rho_1^f} \\
P_1^f(\rho^f) &= u_1(\frac{1}{\rho_1^f}) - u_1(0) \quad \text{if} \quad \rho_1^f \geq r_1 \\
P_1^f(\rho^f) &= \max[0, A(\rho_1^f - r_1) + \; u_1(\frac{1}{r_1}) - u_1(0)] \;\; \text{if} \;\; \rho_1^f < r_1.
\end{aligned}$$

On the equilibrium path, we have $\beta_1^f = \frac{1}{n}$ for all f, and firms extract full surplus.

The intuition for full surplus extraction in Proposition 5 is that firms drastically reduce the payment consumers make when they receive fewer customers than "expected." The mechanism is continuous, but no firm will want to steal any consumers at all from the other firms. However, if other firms receive at least as many customers as expected, they leave consumers with zero surplus. Therefore, the best response is to split the market but extract all surplus. With more than one type of consumer, a consumer could pretend to be a lower-demand type, so incentive compatibility precludes full surplus extraction.

[18] Details were in an earlier version of the paper, and are available upon request.

6.3. Monotonic Mechanisms and Full Capacity Allocation

Does the model place restrictions on the set of equilibrium outcomes, due to the fact that consumers cannot report the mechanisms chosen by other firms? It would be tempting to claim that all firms must fully allocate their capacity, according to the following argument. Consider an equilibrium of Γ in which firm f does not allocate its entire capacity. Firm f can deviate to a mechanism that, if the set of consumers visiting the firm is held fixed, the remaining capacity is allocated and the payments made by consumers is increased. This can be done in such a way that the utility received by each type of consumer is unchanged and incentive compatibility is maintained. Thus, there continues to be a consumer equilibrium with the arrival choices of consumers unchanged, and firm f receives higher profits. The problem with this argument is that there may be multiple consumer equilibria, and in response to the deviation, consumers select a different consumer equilibrium that is unfavorable to firm f. The on-line Appendix provides an example of an equilibrium of Γ in which one firm does not allocate its entire capacity. However, it is shown in Proposition 6 below that, with one type of consumer, all firms must allocate their entire capacity in any equilibrium of Γ in which all firms use monotonic mechanisms.

Definition 1: A *monotonic* mechanism for firm f is a mechanism, $m^f \in M$, such that $u_i(x_i^f(\rho^f)) - P_i^f(\rho^f)$ is weakly decreasing in ρ_j^f for all i, j, and ρ^f.

Proposition 6: *Assume that there is only one consumer type, $I = 1$, and consider a SPNE of Γ in which all firms choose monotonic mechanisms. Then, along the equilibrium path, each firm allocates its entire capacity.*

Sketch of the Proof: Suppose there is a SPNE of Γ in which, in the consumer equilibrium on the equilibrium path, firm f receives reports $\widehat{\rho}_1^f$ and we have $\widehat{\rho}_1^f x_1^f(\widehat{\rho}_1^f) < 1$. Firm f has a profitable deviation, to a mechanism that fully allocates capacity and increases the payment made by its customers, such that their utility remains unchanged and their arrival choices continue to form a consumer equilibrium. If this consumer equilibrium is selected, profits for firm f increase. Here is how to extend the mechanism to other values of ρ_1^f. In all cases, fully allocate capacity, and keep the (per capita) payment made by its customers the same as it was for $\widehat{\rho}_1^f$. For $\rho_1^f > \widehat{\rho}_1^f$, the firm receives more payments than it did for $\widehat{\rho}_1^f$, so again the deviation is profitable. For $\rho_1^f < \widehat{\rho}_1^f$, firm f offers strictly higher utility than it did for $\widehat{\rho}_1^f$. Because firm f receives fewer customers, more consumers are visiting the other firms. Since these firms are choosing monotonic mechanisms, the utility they offer their customers cannot be higher. Therefore, consumers would receive higher utility at firm f than at the other firms, which is inconsistent with a consumer equilibrium.

Notice that a monotonic mechanism is consistent with a firm offering its customers higher utility when fewer than expected show up, thereby punishing another firm for poaching its customers. Under this additional condition, firms efficiently allocate their output with one consumer type, so clearly the folk theorem does not hold. Equilibria can only differ based on the measure of consumers at each firm, and the distribution of the surplus at each firm.

7. Concluding Remarks

This paper develops a framework for studying competing mechanisms in an economic environment where firms sell and consumers demand multiple units. There is a literature in which firms selling a single unit compete by choosing auctions with a reserve price. In the present setting, where consumers are negligible and there is no aggregate uncertainty, we find that these reserve-price mechanisms are not used in equilibrium. Under certain assumptions, equilibrium exists in which firms choose auctions with type-specific entry fees but no reserve price.

We also show that Γ has an equilibrium in which all firms choose a fixed-price-per-unit mechanism with the price equal to the competitive equilibrium price. While this result is a contribution to the literature on price competition with capacity constraints, I can confidently conjecture that fixed-price-per-unit mechanisms will not be consistent with equilibrium of the competiting mechanisms game when there is aggregate demand uncertainty. The reason is that, if the price at firm f clears the market in one demand state, then there will be excess demand and rationing in higher demand states, and wasteful excess supply in lower demand states. Holding fixed the probabilities with which consumers choose firm f, the firm could deviate to a mechanism that maintains the expected utility of each consumer type at firm f, while efficiently allocating the good and generating higher total surplus. Such a deviation would have to be profitable. Introducing demand uncertainty is a subject for future work.

Much of the tractability of this model stems from the fact that individual consumers are unable to affect anyone's allocation other than their own. Besides enhancing tractability, the assumption of negligible consumers may be a desirable description of certain markets. Although the model is quite tractable, some of the proofs are difficult, owing to the requirement that mechanisms be continuous and incentive compatible off the equilibrium path. For example, if all firms chose auctions with entry fees as specified in Proposition 2, but where the entry fee was not "modified" to depend on the reports of other consumers, this profile would satisfy incentive compatibility on the equilibrium path, but it might not be incentive compatible if the measure of arriving consumers is very small.

Throughout the paper, we assume that consumers must commit to a single mechanism. This rules out more complicated environments in which a consumer could contact a firm, attempt to arrange a transaction, and contact a different firm if a favorable transaction could not be completed. See Peters and Severinov (2006) and Peters (2015) for important steps in this direction. It might be both useful and tractable to combine the possibility of consumers contacting multiple firms with the framework of a continuum of consumers who cannot individually affect the market.

8. Appendix: Proofs

Proof of Lemma 1. Fix a profile of mechanisms, m. For $f = 1, ..., n$, let $U_i^f(\beta)$ denote the utility received by type i consumers who choose firm f when consumers mix across firms according to β. Note that $U_i^f(\beta)$ depends on β only through β^f. Given that all mechanisms are incentive compatible for any β, it follows that $U_i^f(\beta)$ is continuous in β. Consider the mapping $g : \triangle^{nI} \to \triangle^{nI}$, where

$$g_i^f(\beta) = \frac{\beta_i^f + \max[0, U_i^f(\beta) - \sum_{j=1}^n \beta_i^j U_i^j(\beta)]}{1 + \sum_{j'=1}^n \max[0, U_i^{j'}(\beta) - \sum_{j=1}^n \beta_i^j U_i^j(\beta)]}.$$

Because each $U_i^f(\beta)$ is continuous, it follows that g is a continuous function. The simplex is a compact, convex set. Applying Brouwer's fixed point theorem, we have at the fixed point,

$$\begin{aligned} & \beta_i^f \sum_{j'=1}^n \max[0, U_i^{j'}(\beta) - \sum_{j=1}^n \beta_i^j U_i^j(\beta)] \\ = \quad & \max[0, U_i^f(\beta) - \sum_{j=1}^n \beta_i^j U_i^j(\beta)] \ \text{ for all } i, f. \end{aligned} \tag{8.1}$$

Consider the possible cases. From (8.1), if

$$U_i^f(\beta) < \sum_{j=1}^{n} \beta_i^j U_i^j(\beta) \tag{8.2}$$

holds, then the right side of (8.1) is zero. Then either $\beta_i^f = 0$ holds, or we have $U_i^{j'}(\beta) \leq \sum_{j=1}^{n} \beta_i^j U_i^j(\beta)$ for all j'. But the latter cannot occur, because it would imply $U_i^{j'}(\beta) = U_i^j(\beta)$ for all j, j', in contradiction to (8.2). Thus, for all f, i, we have either $\beta_i^f = 0$ or

$$U_i^f(\beta) \geq \sum_{j:\beta_i^j > 0} \beta_i^j U_i^j(\beta). \tag{8.3}$$

Applying (8.3) to all j' such that $\beta_i^{j'} > 0$, it follows that $U_i^f(\beta) = U_i^{j'}(\beta)$ for all i and for all f, j' such that $\beta_i^f > 0$ and $\beta_i^{j'} > 0$. Therefore, all consumer choices are best responses, and the fixed point is a Nash equilibrium of the subgame. ■

Proof of Proposition 1. Suppose that there is a SPNE of Γ in which all firms choose reserve price mechanisms, possibly in mixed strategies. Let $\overline{R}$ denote the supremum of reserve prices in the support of the equilibrium profile of mechanisms.

Case 1. We have $\overline{R} > p^c$.

First, it cannot be the case that two or more firms have a mass point at $\overline{R}$. If so, there would be a positive probability that all of the firms with a mass point at $\overline{R}$ choose that reserve price. Total sales by firms setting the reserve price $\overline{R}$ must be positive in this situation, or else $\overline{R}$ would always yield zero revenue. Let firm f be a firm such that, of all the firms setting the reserve price $\overline{R}$ in this situation, firm f is selling the least of its capacity. Firm f must be selling strictly less than all of its capacity, since $\overline{R} > p^c$ holds. However, firm f could slightly reduce its reserve price, selling strictly more of its capacity, thereby increasing its profits.

Second, suppose that $\overline{R}$ is in the support of reserve prices chosen by firm f.[19] Since no firm, other than possibly firm f, can have a mass point at $\overline{R}$, when firm f chooses the reserve price $\overline{R}$, it knows that it is the only firm with a reserve price that high. If firm f receives zero revenue in any consumer equilibrium, this

[19] For the third situation, where $\overline{R}$ is not in the support of reserve prices chosen by any firm, then for R^f sufficiently close to $\overline{R}$, firm f knows that its reserve price is highest with probability arbitrarily close to one, and the argument mirrors the second situation of $\overline{R}$ in the support.

contradicts the fact that $\overline{R}$ is a best response to the mixed strategies of the other firms. Therefore, in the ensuing consumer equilibrium, the auction price at all firms is $\overline{R}$ and all firms other than firm f sell all their capacity. It follows that, no matter what reserve prices the other firms choose, the consumer equilibrium β satisfies the market clearing condition

$$\sum_{i=1}^{I} r_i n(1-\beta_i^f) d_i(\overline{R}) = n-1, \tag{8.4}$$

and the profits of firm f are

$$\sum_{i=1}^{I} r_i n \beta_i^f d_i(\overline{R})\overline{R}.$$

Substituting (8.4) into the profit expression yields profits of

$$\left[\sum_{i=1}^{I} r_i n d_i(\overline{R}) - (n-1)\right]\overline{R}. \tag{8.5}$$

Furthermore, any consumer mixed strategy profile β satisfying (8.4) is an equilibrium of the consumer subgame, yielding the profits given in (8.5). Let β^* be a consumer equilibrium satisfying $\beta_1^{*f} > 0$.

We will construct a profitable deviation for firm f. Intuitively, the new mechanism is constructed so that utility is unaffected if consumers continue to mix according to β^*, so this remains an equilibrium of the consumer subgame. If reports are consistent with β^*, then some of the capacity that was not utilized under the mechanism $\overline{R}$ is allocated to type 1 consumers in exchange for an additional payment. Then the mechanism is extended to other reports to maintain continuity, incentive compatibility, and profitability.

Here is the deviation mechanism for Case 1, denoted by m^f.

If there is no excess demand at the price $\overline{R}$ based on the reports, so $\sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) \leq$

1 holds, then we have

$$\begin{aligned}
x_i^f(\rho^f) &= d_i(\overline{R}) \quad \text{for } i > 1 \\
P_i^f(\rho^f) &= \overline{R}d_i(\overline{R}) \quad \text{for } i > 1 \\
x_1^f(\rho^f) &= d_1(\overline{R}) + \frac{1 - \sum_{i=1}^I \rho_i^f d_i(\overline{R})}{A + \rho_1^f} \qquad (8.6)\\
P_1^f(\rho^f) &= \overline{R}d_1(\overline{R}) + u_1(x_1^f(\rho^f)) - u_1(d_1(\overline{R})) \\
&\quad + \varepsilon[1 - \sum_{i=1}^I \rho_i^f d_i(\overline{R})] \max[0, r_1 n \beta_1^{*f} - \rho_1^f].
\end{aligned}$$

The positive parameters A and ε are chosen as part of the mechanism. The purpose of A is to guarantee that type 1 consumption is well defined even if $\rho_1^f = 0$ holds. Larger values of A mean that less of the excess supply is allocated to type 1 consumers. The term involving ε is used below to ensure that there cannot be an equilibrium to the consumer subgame in which too few consumers report type 1.

If there is excess demand at the price $\overline{R}$ based on the reports, so $\sum_{i=1}^I \rho_i^f d_i(\overline{R}) > 1$ holds, then we implement a uniform price auction. Defining p^f as the solution to$\sum_{i=1}^I \rho_i^f d_i(p^f) = 1$, we have

$$\begin{aligned}
x_i^f(\rho^f) &= d_i(p^f) \\
P_i^f(\rho^f) &= p^f d_i(p^f).
\end{aligned}$$

This completes the definition of m^f.

m^f **is allowable for large A and small ε, satisfying continuity and IC:**

Simple algebra calculations establish that firm f always has enough capacity to provide the promised consumption. It is immediate that m^f is continuous in ρ^f. Incentive compatibility holds for sufficiently large A and sufficiently small ε, which follows from the fact that, when there is excess supply at the price $\overline{R}$, a type 1 consumer receives utility close to $u_1(d_1(\overline{R})) - \overline{R}d_1(\overline{R})$, which is higher than utility from reporting any other type. Also, if a type $i > 1$ were to report type 1, his net increase in utility is given by

$$\begin{aligned}
& u_i(x_1^f(\rho^f)) - P_1^f(\rho^f) - u_i(d_i(\overline{R})) + \overline{R}d_i(\overline{R}) \\
= \; & u_i(x_1^f(\rho^f)) - \overline{R}d_1(\overline{R}) - u_1(x_1^f(\rho^f)) + u_1(d_1(\overline{R})) \qquad (8.7)\\
& - u_i(d_i(\overline{R})) + \overline{R}d_i(\overline{R}) - \varepsilon[1 - \sum_{i=1}^I \rho_i^f d_i(\overline{R})] \max[0, r_1 n \beta_1^{*f} - \rho_1^f].
\end{aligned}$$

In the limit, as $A \to \infty$ and $\varepsilon \to 0$, (8.7) approaches

$$\left[u_i(d_1(\overline{R})) - \overline{R}d_1(\overline{R})\right] - \left[u_i(d_i(\overline{R})) - \overline{R}d_i(\overline{R})\right] < 0.$$

The mechanism m^f yields higher profits than the mechanism $\overline{R}$: First, if the consumer equilibrium following the deviation to m^f satisfies

$$\sum_{i=1}^{I} r_i n \beta_i^{*f} d_i(\overline{R}) \leq \sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) \leq 1,$$

then we must have $\rho_1^f \geq r_1 n \beta_1^{*f} > 0$, because otherwise the auction price at firms other than f is at most $\overline{R}$ and type 1 consumers would be receiving strictly lower utility from firm f, due to the ε term in (8.6). Therefore, the profits are greater than the profits of the mechanism $\overline{R}$ by at least

$$\left[u_1(x_1^f(\rho^f)) - u_1(d_1(\overline{R}))\right] \rho_1^f,$$

which is strictly positive.

Second, if the consumer equilibrium following the deviation to m^f satisfies

$$\sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) < \sum_{i=1}^{I} r_i n \beta_i^{*f} d_i(\overline{R}),$$

then the auction price at firms other than f is strictly greater than $\overline{R}$, so consumers of type $i > 1$ are strictly better off at firm f, contradicting consumer equilibrium.[20]

Third, if the consumer equilibrium following the deviation to m^f satisfies

$$\sum_{i=1}^{I} \rho_i^f d_i(\overline{R}) > 1,$$

then firm f sells all of its capacity at a price greater than $\overline{R}$, so again the mechanism m^f yields higher profits than the mechanism $\overline{R}$.

Case 2. We have $\overline{R} \leq p^c$.

[20] If there is only one type, $I = 1$, then set $\varepsilon = 0$ and the argument goes through since incentive compatibility is not an issue.

In this case, consumers distribute themselves across firms so that none of the reserve prices is binding, and all of the auction prices are equal to p^c. We now show that there is a profitable deviation to a mechanism that, roughly speaking, raises the auction price at the other firms to some price $\widetilde{p}$, allocates $d_i(\widetilde{p})$ to its type i customers at price $\widetilde{p}$, and allocates additional capacity as extra consumption to type 1 customers. Then the mechanism is extended continuously to satisfy feasibility and incentive compatibility off the equilibrium path.

Here is the deviation mechanism for Case 2, denoted by $\widetilde{m}^f$.

Below, we treat $\widetilde{A}$, ε, and $\widetilde{p}$ as parameters of the mechanism. First, let $C(\rho^f)$ be defined by

$$C(\rho^f) = \sum_{i=1}^{I} \rho_i^f d_i(\widetilde{p}) - \left[\sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1) \right].$$

The economic interpretation of $C(\rho^f)$ is the amount by which the actual demand at firm f at price $\widetilde{p}$, based on reported types, exceeds the anticipated demand, based on the residual market demand faced by firm f. We will suppress the dependence on ρ^f and refer to $C(\rho^f)$ as C.

For $C \leq 0$, $\widetilde{m}^f$ is given by[21]

$$\begin{aligned}
x_i^f(\rho^f) &= d_i(\widetilde{p}) \quad \text{for } i > 1 \\
P_i^f(\rho^f) &= \widetilde{p} d_i(\widetilde{p}) \quad \text{for } i > 1 \\
x_1^f(\rho^f) &= d_1(\widetilde{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{\widetilde{A} + \rho_1^f} \\
P_1^f(\rho^f) &= \widetilde{p} d_1(\widetilde{p}) + u_1(x_1^f(\rho^f)) - u_1(d_1(\widetilde{p})) - \varepsilon \max[0, r_1 - \rho_1^f].
\end{aligned}$$

For $C > 0$, $\widetilde{m}^f$ is given by

[21] If we have only one type, $I = 1$, then $\widetilde{m}^f$ must be modified, but a simpler proof along the same lines is available, which eliminates the term, $\varepsilon \max[0, r_1 - \rho_1^f]$. We omit the details to save space.

$$
\begin{aligned}
x_i^f(\rho^f) &= d_i(\widehat{p}) \quad \text{for } i > 1 \\
P_i^f(\rho^f) &= \widehat{p} d_i(\widehat{p}) \quad \text{for } i > 1 \\
x_1^f(\rho^f) &= d_1(\widehat{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{\widetilde{A} + \rho_1^f} \\
P_1^f(\rho^f) &= \widehat{p} d_1(\widehat{p}) + u_1(x_1^f(\rho^f)) - u_1(d_1(\widehat{p})) - \varepsilon \max[0, r_1 - \rho_1^f].
\end{aligned}
$$

where $\widehat{p}$ is the unique solution to

$$
\sum_{i=1}^{I} \rho_i^f d_i(\widehat{p}) = \sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1).
$$

Note that $C > 0$ implies $\widehat{p} > \widetilde{p}$.

$\widetilde{m}^f$ is feasible for small ε, satisfying continuity and IC:

Because we have $\widetilde{A} > 0$, and as C approaches 0 from above $\widehat{p}$ approaches $\widetilde{p}$, it is immediate that $\widetilde{m}^f$ is continuous. For small enough ε, the consumption of each type is arbitrarily close to the utility maximizing consumption at price $\widetilde{p}$ or $\widehat{p}$, depending on C. Thus, there is no incentive for a consumer to report a different type. Also, simple algebra calculations establish that firm f always has enough capacity to provide the promised consumption.

The mechanism $\widetilde{m}^f$ yields higher profits than the mechanism $\overline{R}$:

We claim that for $\widetilde{p}$ close enough to p^c, there is a consumer equilibrium, β, satisfying $\beta_1^f = \frac{1}{n}$, in which the auction price at other firms will be $\widetilde{p}$. The price at other firms is $\widetilde{p}$ if and only if we have

$$
\begin{aligned}
\sum_{i=1}^{I} r_i n (1 - \beta_i^f) d_i(\widetilde{p}) &= (n-1), \quad \text{or} \\
\sum_{i=1}^{I} r_i n \beta_i^f d_i(\widetilde{p}) &= \sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1). \qquad (8.8)
\end{aligned}
$$

For $\widetilde{p}$ close enough to p^c, the right side of (8.8) is positive. Then any β that satisfies (8.8) gives rise to $C = 0$ under truthful reporting. The mechanism $\widetilde{m}^f$ then delivers the same utility to each type as they would receive at other firms if $\beta_1^f = \frac{1}{n}$ and therefore $r_1 = \rho_1^f$ holds, ensuring that β is an equilibrium of the consumer subgame.[22]

The profits for firm f are then given by

$$\sum_{i=1}^{I} r_i n \beta_i^f d_i(\widetilde{p})\widetilde{p} + r_1[u_1(d_1(\widetilde{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{\widetilde{A} + r_1}) - u_1(d_1(\widetilde{p}))], \tag{8.9}$$

which, from (8.8), can be written as

$$\begin{aligned} \pi^f(\widetilde{p}) &= \widetilde{p}\left[\sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1)\right] \\ &\quad + r_1[u_1(d_1(\widetilde{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{\widetilde{A} + r_1}) - u_1(d_1(\widetilde{p}))]. \end{aligned} \tag{8.10}$$

Differentiating (8.10) yields

$$\begin{aligned} (\pi^f)'(\widetilde{p}) &= \sum_{i=1}^{I} r_i n d_i(\widetilde{p}) - (n-1) + \widetilde{p}\sum_{i=1}^{I} r_i n d_i'(\widetilde{p}) \\ &\quad + r_1 u_1'(x_1^f(\rho^f))\left[d_1'(\widetilde{p}) - \frac{\sum_{i=1}^{I} r_i n d_i'(\widetilde{p})}{\widetilde{A} + r_1}\right] \\ &\quad - r_1 u_1'(d_1(\widetilde{p}))d_1'(\widetilde{p}). \end{aligned} \tag{8.11}$$

Evaluating (8.11) at $\widetilde{p} = p^c$, which implies $\sum_{i=1}^{I} r_i n d_i(\widetilde{p}) = n$ and $u_1'(x_1^f(\rho^f)) = u_1'(d_1(\widetilde{p})) = p^c$, yields

$$(\pi^f)'(p^c) = 1 + \left(1 - \frac{r_1}{\widetilde{A} + r_1}\right) p^c \sum_{i=1}^{I} r_i n d_i'(p^c).$$

For $\widetilde{A}$ sufficiently close to zero, $(\pi^f)'(p^c)$ is positive, so for $\widetilde{p}$ slightly greater than p^c.

[22] If we have $I = 1$, then there is a unique consumer equilibrium, but β_1^f will be slightly less than $\frac{1}{n}$ for $\widetilde{p}$ close enough to p^c. With $I > 1$, there will be a consumer equilibrium with $\beta_1^f = \frac{1}{n}$ for $\widetilde{p}$ close enough to p^c, because types $i > 1$ can go to other firms.

What if a different consumer equilibrium is selected in response to this deviation? If we have $C = 0$ and $\rho_1^f > r_1$, then profits are given by

$$\sum_{i=1}^{I} r_i n \beta_i^f d_i(\widetilde{p})\widetilde{p} + \rho_1^f[u_1(d_1(\widetilde{p}) + \frac{n - \sum_{i=1}^{I} r_i n d_i(\widetilde{p})}{\widetilde{A} + r_1}) - u_1(d_1(\widetilde{p}))],$$

which is greater than the expression in (8.9), so once again the deviation is profitable.

If we have $C = 0$ and $\rho_1^f < r_1$, then type 1 consumers receive higher utility at firm f than at the other firms, which is inconsistent with equilibrium of the consumer subgame. If we have $C < 0$, then the auction price at the other firms will be greater than $\widetilde{p}$, so all consumers receive higher utility at firm f than at the other firms, which is inconsistent with equilibrium of the consumer subgame.

Now consider the possibility of $C > 0$. If somehow this is consistent with equilibrium of the consumer subgame, this would imply $\widehat{p} > \widetilde{p}$, and also that the auction price at other firms is less than $\widetilde{p}$. However, a consumer of type $i > 1$ is worse off at firm f, which is inconsistent with equilibrium of the consumer subgame. ■

Proof of Proposition 2. First, notice that the $\varepsilon - MAE$ mechanism in the statement of Proposition 2 is within the class of available mechanisms. Continuity follows from the facts that the market clearing price is continuous in β^f and competitive equilibrium consumption is continuous in price. When the auction price is outside the 2ε neighborhood of p^c, entry fees are zero, and incentive compatibility follows immediately. When the auction price is inside the 2ε neighborhood of p^c, incentive compatibility follows from the fact that when n is sufficiently large, entry fees are close to zero, and any difference in entry fees across types is swamped by the loss of utility associated with misreporting and receiving consumption that does not maximize utility given the equilibrium price.

For any profile of mechanisms, the ensuing consumer subgame has a type-symmetric Nash equilibrium, which follows from Lemma 1. Select an arbitrary type-symmetric Nash equilibrium following a deviation by two or more firms.

On the equilibrium path, since all firms are choosing the same mechanism, it is clear that $\beta_i^f = \frac{1}{n}$ for all i, f forms an equilibrium of the subgame. The auction price at each firm is therefore p^c, so the profits of each firm are given by

$$\sum_{i=1}^{I} r_i a_i d(p^c) p^c - \sum_{i=1}^{I} \frac{r_i a_i d(p^c)^2}{(n-1)d'(p^c)}.$$

Because the competitive equilibrium price satisfies $\sum_{i=1}^{I} r_i a_i d(p^c) = 1$, the profit expression simplifies to

$$p^c - \frac{d(p^c)}{(n-1)d'(p^c)}.$$

Now consider a potential deviation by a single firm, f. An upper bound to the profits available is the solution to the following optimization problem, in which firm f chooses its mechanism and its arrival vector, β^f, subject to its capacity constraint and the constraint that consumers are indifferent between firm f and the other firms[23]:

$$\max \sum_{i=1}^{I} r_i n \beta_i^f P_i^f$$

subject to

$$\begin{aligned} \sum_{i=1}^{I} r_i n \beta_i^f x_i^f &= 1 \qquad (8.12) \\ u_i(x_i^f) - P_i^f &= u_i(a_i d(\widetilde{p}(\beta^f))) - \widetilde{p}(\beta^f) a_i d(\widetilde{p}(\beta^f)) - E_i^*, \quad \text{for } i = 1, ..., I \\ 1 &\geq \beta_i^f \geq 0 \quad \text{for } i = 1, ..., I. \end{aligned}$$

In (8.12), $\widetilde{p}(\beta^f)$ is defined to be the auction price at other firms when the arrival vector at firm f is β^f. Then, suppressing the dependence on β^f, $\widetilde{p}$ solves

$$1 = \sum_{h=1}^{I} \frac{r_h n(1-\beta_h^f) a_h d(\widetilde{p})}{n-1}. \qquad (8.13)$$

Letting λ denote the Lagrange multiplier on the capacity constraint and λ_i denote the multiplier on the indifference constraint for type i, some of the *necessary* first-order conditions are, for $i = 1, ..., I$, the equality constraints in (8.12) and

$$\begin{aligned} \lambda_i u_i'(x_i^f) &= \lambda r_i n \beta_i^f \qquad (8.14) \\ r_i n \beta_i^f &= \lambda_i \qquad (8.15) \end{aligned}$$

[23] It is without loss of generality to impose the indifference constraint for each type in (8.12), because if some type i strictly prefers to visit other firms and $\beta_i^f = 0$ holds for some type at the solution, there is another solution in which type i consumers are indifferent and $\beta_i^f = 0$ holds.

In particular, we have $u_i'(x_i^f) = \lambda \equiv p^f$ for all i such that $1 > \beta_i^f > 0$. Equivalently, we have $x_i^f = a_i d(p^f)$ for all i, which implies

$$\sum_{i=1}^{I} r_i n \beta_i^f a_i = \frac{1}{d(p^f)}. \tag{8.16}$$

Based on the above necessary conditions, it follows that for any solution to (8.12), β^f must solve (suppressing the dependence of p^f and $\widetilde{p}$ on β^f through (8.13) and (8.16))

$$\max_{0 \le \beta_i^f \le 1, i=1,...,I} \sum_{i=1}^{I} r_i n \beta_i^f [u_i(a_i d(p^f)) - u_i(a_i d(\widetilde{p})) + \widetilde{p} a_i d(\widetilde{p}) + E_i^*]. \tag{8.17}$$

Because a continuous function on a compact set has a maximum, we know that there is a solution to (8.17). An interior solution must satisfy the first order conditions, simplified by the condition that $u_i'(a_i d(p)) = p$ holds and the convenient notation $P_i^f(\beta^f) = u_i(a_i d(p^f)) - u_i(a_i d(\widetilde{p})) + \widetilde{p} a_i d(\widetilde{p}) + E_i^*$, given by

$$r_i n P_i^f(\beta^f) + (\sum_{h=1}^{I} r_h n \beta_h^f a_h) d(\widetilde{p}) \frac{\partial \widetilde{p}}{\partial \beta_i^f} + (\sum_{h=1}^{I} r_h n \beta_h^f a_h) p^f d'(p^f) \frac{\partial p^f}{\partial \beta_i^f} = 0 \tag{8.18}$$

for $i = 1, ..., I$.

The first order conditions (8.18) can be simplified further. Using (8.16), we have

$$r_i n P_i^f(\beta^f) + \frac{d(\widetilde{p})}{d(p^f)} \frac{\partial \widetilde{p}}{\partial \beta_i^f} + \frac{p^f d'(p^f)}{d(p^f)} \frac{\partial p^f}{\partial \beta_i^f} = 0. \tag{8.19}$$

Differentiating (8.16), we derive

$$\frac{\partial p^f}{\partial \beta_i^f} = -\frac{d(p^f)^2 r_i n a_i}{d'(p^f)}. \tag{8.20}$$

Also, we can differentiate (8.13) to derive

$$\frac{\partial \widetilde{p}}{\partial \beta_i^f} = \frac{r_i n a_i d(\widetilde{p})^2}{(n-1) d'(\widetilde{p})}. \tag{8.21}$$

Substituting (8.20) and (8.21) into (8.19) yields

$$r_i n \left[P_i^f(\beta^f) + \frac{a_i d(\widetilde{p})^3}{(n-1) d(p^f) d'(\widetilde{p})} - a_i p^f d(p^f) \right] = 0. \tag{8.22}$$

We now argue that adopting the same $\varepsilon - MAE$ mechanism adopted by the other firms is a solution to (8.12). This mechanism corresponds to $\beta_i^f = \frac{1}{n}$ for all i, which implies $p^f = \widetilde{p} = p^c$. Since the corresponding value of P_i^f is

$$a_i d(p^c)p^c - \frac{a_i d(p^c)^2}{(n-1)d'(p^c)},$$

it follows that (8.22) is satisfied.

This establishes that the $\varepsilon - MAE$ mechanism corresponds to β^f that solves (8.22) for each i, the necessary first order conditions to (8.12). To complete the proof, we first show that ignoring corner solutions is without loss of generality. Then we show any interior solution to (8.12) requires $p^f = \widetilde{p} = p^c$, which pins down the profits of firm f and establishes the $\varepsilon - MAE$ mechanism as a best response.

Claim: For any corner solution to (8.17), β^{*f}, there is an interior solution yielding the same payoff.

Proof of Claim: Any β^f satisfying

$$\sum_{i=1}^{I} r_i n \beta_i^f a_i = \sum_{i=1}^{I} r_i n \beta_i^{*f} a_i \tag{8.23}$$

yields the same payoff as β^{*f}. Since $\beta^{*f} = 0$ cannot be optimal, we must have $\beta_i^{*f} > 0$ for some i. For any $\beta_h^{*f} = 0$, we can reduce β_i^{*f} and increase β_h^{*f}, such that (8.23) holds. Similarly, if we have $\beta_i^{*f} < 1$ for some i, then for any $\beta_h^{*f} = 1$, we can increase β_i^{*f} and reduce β_h^{*f}, such that (8.23) holds.

The only remaining case is $\beta_i^{*f} = 1$ for all i. But this implies $\widetilde{p} = 0$. One can show that profits are strictly less than the profits from $\beta_i^f = \frac{1}{n}$ for all i, so this case is impossible, thereby proving the claim.

Claim: Any interior solution to (8.17) must yield $p^f = \widetilde{p} = p^c$, and all such solutions yield the same profits.

Proof of Claim: Substituting

$$P_i^f(\beta^f) = u_i(a_i d(p^f)) - u_i(a_i d(\widetilde{p})) + \widetilde{p} a_i d(\widetilde{p}) + E_i^*$$

into (8.22) implies the necessary condition,

$$\begin{aligned} 0 &= u_i(a_i d(p^f)) - u_i(a_i d(\widetilde{p})) + \widetilde{p} a_i d(\widetilde{p}) + E_i^* \\ &\quad + \frac{a_i d(\widetilde{p})^3}{(n-1)d(p^f)d'(\widetilde{p})} - a_i p^f d(p^f). \end{aligned} \tag{8.24}$$

Using (8.13) and (8.16), we can express $\widetilde{p}$ in terms of p^f, given by

$$d(\widetilde{p}) = \frac{(n-1)d(p^f)}{d(p^f)\sum_{h=1}^{I} r_h n a_h - 1}, \tag{8.25}$$

so the right side of (8.24) depends only on p^f, taking into account the dependence of $\widetilde{p}$ on p^f, based on (8.25). Differentiating (8.24) with respect to p^f, and simplifying using the condition, $u_i'(a_i d(p)) = p$, yields the expression,

$$a_i d(\widetilde{p})\frac{\partial \widetilde{p}}{\partial p^f} - a_i d(p^f) + \frac{\partial}{\partial p^f}[\frac{a_i d(\widetilde{p})^3}{(n-1)d(p^f)d'(\widetilde{p})}]. \tag{8.26}$$

Also, differentiating (8.25) yields

$$\frac{\partial \widetilde{p}}{\partial p^f} = -\frac{(n-1)d'(p^f)}{[d(p^f)\sum_{h=1}^{I} r_h n a_h - 1]^2 d'(\widetilde{p})}. \tag{8.27}$$

For sufficiently large n, the last term in (8.26) is negligible, and it is clear from (8.27) that the first term in (8.26) is negative. Therefore, the entire expression is strictly negative. Thus, there is a unique p^f that solves (8.24), so the only value of p^f that satisfies (8.24) is p^c. ■

Proof of Proposition 3. First, for any profile of prices, the ensuing consumer subgame has a type-symmetric equilibrium, by Lemma 1. Select an arbitrary type-symmetric equilibrium following a deviation by two or more firms.

Now consider a potential deviation by a single firm, f. To show that the deviation is not profitable, we will show that there is no profitable deviation, even if firm f could choose any equilibrium of the subgame. Since it is without loss of generality to restrict attention to mechanisms that fully allocate capacity, thereby allowing a higher total payment, an upper bound to the profits available is the solution to the following optimization problem, in which firm f chooses its mechanism and its arrival vector, β^f, subject to its capacity constraint and the constraint that consumers are indifferent between firm f and the other firms:[24]

[24] It is an upper bound because we do not impose incentive compatibility, but as it turns out, incentive compatibility is not binding. It is without loss of generality to impose the consumer indifference condition. Also, $x_i(\beta^f)$ is the consumption level at other firms (utility maximizing demand or rationing level, whichever is smaller), which depends on firm f's choice of β^f.

$$\max \sum_{i=1}^{I} r_i n \beta_i^f P_i^f$$
subject to
$$\begin{aligned} \sum_{i=1}^{I} r_i n \beta_i^f x_i^f &= 1 \qquad (8.28) \\ u_i(x_i^f) - P_i^f &= u_i(x_i(\beta^f)) - p^c x_i(\beta^f), \quad \text{for } i = 1, ..., I \\ \beta_i^f &\geq 0 \quad \text{for } i = 1, ..., I. \end{aligned}$$

Letting λ denote the Lagrange multiplier on the capacity constraint and λ_i denote the multiplier on the indifference constraint, the necessary first-order conditions with respect to x_i^f and P_i^f are

$$\begin{aligned} \lambda_i u_i'(x_i^f) &= \lambda r_i n \beta_i^f \quad \text{and} \\ r_i n \beta_i^f &= \lambda_i, \end{aligned}$$

implying $u_i'(x_i^f) = \lambda$ for all i. Therefore, we have $x_i^f = d_i(\lambda)$ for all i.

We now rule out a profitable deviation in which $\lambda \geq p^c$ holds. If so, we would have $x_i^f \leq d_i(p^c)$ for all i, which implies

$$\sum_{i=1}^{I} r_i n \beta_i^f d_i(p^c) \geq 1.$$

Therefore, demand at the other firms satisfies

$$\sum_{i=1}^{I} r_i n (1 - \beta_i^f) d_i(p^c) \leq \sum_{i=1}^{I} r_i n d_i(p^c) - 1 = n - 1,$$

where the equality follows from the definition of the market clearing price. Therefore, firms other than f have excess capacity, so consumers at other firms receive their competitive equilibrium bundle. Denoting firm f's deviation profits by π^f, the total surplus offered by firm f to its customers is

$$\begin{aligned} & \sum_{i=1}^{I} r_i n \beta_i^f [u_i(d_i(\lambda)) - P_i^f] \\ = \; & \sum_{i=1}^{I} r_i n \beta_i^f u_i(d_i(\lambda)) - \pi^f. \qquad (8.29) \end{aligned}$$

The total surplus these consumers would receive by consuming their competitive equilibrium bundle is

$$\sum_{i=1}^{I} r_i n \beta_i^f [u_i(d_i(p^c)) - p^c d_i(p^c)]. \tag{8.30}$$

The consumer indifference condition requires the expression in (8.29) to equal the expression in (8.30), implying

$$\pi^f = \sum_{i=1}^{I} r_i n \beta_i^f [u_i(d_i(\lambda)) - u_i(d_i(p^c)) + p^c d_i(p^c)].$$

If we suppose $\lambda \geq p^c$ holds, then we have $u_i(d_i(\lambda)) \leq u_i(d_i(p^c))$, which implies that the deviation cannot be profitable.

We now consider the remaining possibility of a profitable deviation with $\lambda < p^c$. This implies $x_i^f > d_i(p^c)$ for all i. An argument similar to that of the previous paragraph, but with the inequalities reversed, yields the conclusion that there is excess demand at firms other than f, so type 1 consumers are being rationed there. If n is sufficiently large, then only type 1 consumers are rationed at the other firms, even if firm f sends away all of its customers. For type $i > 1$, the contribution to the profit of firm f, per unit of capacity allocated to them, can then be calculated from the indifference condition as

$$\frac{P_i^f}{x_i^f} = \frac{u_i(d_i(\lambda)) - u_i(d_i(p^c)) + p^c d_i(p^c)}{d_i(\lambda)}. \tag{8.31}$$

Differentiating the right side of (8.31) with respect to λ yields

$$\frac{d_i(\lambda)\lambda - [u_i(d_i(\lambda)) - u_i(d_i(p^c)) + p^c d_i(p^c)]d_i'(\lambda)}{d_i(\lambda)^2}.$$

Since $\lambda < p^c$ holds, we have $u_i(d_i(\lambda)) > u_i(d_i(p^c))$, so the above expression is strictly positive. Therefore, the right side of (8.31), is greater when evaluated at $\lambda = p^c$ than when evaluated at $\lambda < p^c$. Thus, we have $\frac{P_i^f}{x_i^f} < p^c$, so profits per unit allocated to type i are less than under the competitive equilibrium. It follows that, if there is a solution to (8.28) with $\lambda < p^c$, it must entail $\beta_i^f = 0$ for all $i > 1$. Otherwise, higher profits would be possible by adjusting β_i^f to hold $\beta_i^f x_i^f$ constant, but with $x_i^f = d_i(p^c)$ and $P_i^f = p^c d_i(p^c)$.

Thus, we can simplify the optimization problem of firm f as follows.

$$\begin{aligned} & \max_{\beta_1^f, P_1^f} r_1 n \beta_1^f P_1^f \\ & \text{subject to} \\ u_1(\frac{1}{r_1 n \beta_1^f}) - P_1^f &= u_1(\overline{x}) - p^c \overline{x}, \end{aligned} \tag{8.32}$$

where $\overline{x}$ is the rationing level offered by other firms, which depends on β_1^f. Therefore, $\overline{x}$ must be at least as high as what would obtain if type 1 consumers visit each of the other firms with probability $(1-\beta_1^f)/(n-1)$, other consumers visit each of the other firms with probability $1/(n-1)$, and only type 1 consumers are rationed. In this case, a lower bound for $\overline{x}$ (which represents an upper bound to firm f's profit) is the solution to

$$\frac{(1-\beta_1^f) r_1 n \overline{x}}{n-1} + \frac{\sum_{i=2}^{I} r_i n d_i(p^c)}{n-1} = 1. \tag{8.33}$$

From (3.1), we have

$$\sum_{i=2}^{I} r_i n d_i(p^c) = n - r_1 n d_1(p^c). \tag{8.34}$$

Combining (8.33) and (8.34), we have a lower bound for $\overline{x}$ given by

$$\overline{x} = \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n}.$$

Since the solution to (8.32) will have the constraint hold with equality, we can substitute the constraint into the objective, so an upper bound to the firm's profit is the solution to

$$\max_{\beta_1^f} r_1 n \beta_1^f \left[u_1(\frac{1}{r_1 n \beta_1^f}) - u_1(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n}) + p^c \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n} \right]. \tag{8.35}$$

Differentiating the profit expression in (8.35) with respect to β_1^f yields

$$\begin{aligned} & r_1 n \left[u_1(\frac{1}{r_1 n \beta_1^f}) - u_1(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n}) + p^c \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n} \right] \\ & + \beta_1^f \left[u_1'(\frac{1}{r_1 n \beta_1^f})(-\frac{1}{(\beta_1^f)^2}) - u_1'(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n})(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^2}) + p^c \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^2} \right]. \end{aligned} \tag{8.36}$$

Evaluating the first order condition at $\beta_1^f = \frac{1}{r_1 n d_1(p^c)}$, which implies $x_1^f = \overline{x} = d_1(p^c)$, (8.36) becomes

$$r_1 n p^c d_1(p^c) + [-u_1'(d_1(p^c))(r_1 n d_1(p^c)) + (p^c - u_1'(d_1(p^c)))\frac{1}{r_1 n d_1(p^c)}\frac{r_1 n d_1(p^c) - 1}{(1 - \frac{1}{r_1 n d_1(p^c)})^2}],$$

which is zero, due to the fact that $u_1'(d_1(p^c)) = p^c$. Thus, as long as the second order condition is satisfied, firm f can do no better than to offer the price p^c.

The derivative of (8.36) with respect to β_1^f is

$$\begin{aligned}
&2\left[u_1'(\frac{1}{r_1 n \beta_1^f})(-\frac{1}{(\beta_1^f)^2}) - u_1'(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n})(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^2}) + p^c \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^2}\right] \\
&+\beta_1^f \left[u_1''(\frac{1}{r_1 n \beta_1^f})(\frac{1}{r_1 n (\beta_1^f)^4}) + u_1'(\frac{1}{r_1 n \beta_1^f})(\frac{2}{(\beta_1^f)^3})\right] \\
&-\beta_1^f \left[u_1''(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n})(\frac{r_1 n d_1(p^c) - 1}{r_1 n})(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^4})\right] \\
&-\beta_1^f \left[2u_1'(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n})(\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^3})\right] + \beta_1^f \left[p^c \frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f)^3}\right],
\end{aligned}$$

which, after simplifying and substituting $\overline{x}$ for $\frac{r_1 n d_1(p^c) - 1}{(1-\beta_1^f) r_1 n}$, becomes

$$r_1 n \left[\frac{2\overline{x}}{(1-\beta_1^f)^3}([p^c - u_1'(\overline{x})] + u_1''(\frac{1}{r_1 n \beta_1^f})(\frac{1}{(r_1 n)^2 (\beta_1^f)^3}) - u_1''(\overline{x})\frac{\beta_1^f \overline{x}^2}{(1-\beta_1^f)^4}\right]. \tag{8.37}$$

Because type 1 consumers are rationed at the other firms, we have $p^c < u_1'(\overline{x})$. Because n is large, firm f must choose β_1^f that is small enough that $n\beta_1^f$ is bounded from above, or else it would be impossible to satisfy the constraint in (8.32). Therefore, the first term in (8.37) is negative, the second term becomes unboundedly negative as n gets large, and the third term is positive but becomes negligible as n gets large. We conclude that the second order conditions are satisfied. ■

Proof of Proposition 4. First, for any profile of share prices, the ensuing consumer subgame has a type-symmetric equilibrium, which follows from Lemma 1. Select an arbitrary type-symmetric equilibrium following a deviation by two or more firms. On the equilibrium path, it is obvious that consumers are best responding to each other in the ensuing consumer subgame, by choosing each firm with probability $\frac{1}{n}$.

Now suppose that all firms, except possibly firm f, choose the share price $P^* = \frac{np^c}{(n-1)r_1}$. Consider a potential deviation by a single firm, f. To show that a deviation is not profitable, we will show that firm f cannot increase its profits, even if it could choose the equilibrium of the consumer subgame following its deviation. To show that there is no profitable deviation, we can restrict attention to mechanisms that fully allocate capacity, thereby allowing a higher total payment. Thus, the optimal deviation can be seen as choosing P_1^f and β_1^f to maximize profits, subject to the constraint of making type 1 consumers indifferent between firm f and the other firms (i.e., that consumers adjust their arrival probabilities to form a Nash equilibrium of the subgame):[25]

$$\begin{aligned} & \max r_1 n \beta_1^f P_1^f \\ & \text{subject to} \\ u_1\left(\frac{1}{r_1 n \beta_1^f}\right) - P_1^f &= u_1\left(\frac{n-1}{r_1 n(1-\beta_1^f)}\right) - P^*. \end{aligned}$$

Substituting the constraint into the objective, we have the equivalent unconstrained problem of choosing β_1^f to maximize

$$r_1 n \beta_1^f \left[u_1\left(\frac{1}{r_1 n \beta_1^f}\right) - u_1\left(\frac{n-1}{r_1 n(1-\beta_1^f)}\right) + P^* \right]. \tag{8.38}$$

The necessary first order conditions are given by

$$\begin{aligned} 0 &= r_1 n \left[u_1\left(\frac{1}{r_1 n \beta_1^f}\right) - u_1\left(\frac{n-1}{r_1 n(1-\beta_1^f)}\right) + P^* \right] \qquad (8.39) \\ &\quad + \beta_1^f \left[u_1'\left(\frac{1}{r_1 n \beta_1^f}\right)\left(\frac{-1}{(\beta_1^f)^2}\right) - u_1'\left(\frac{n-1}{r_1 n(1-\beta_1^f)}\right)\left(\frac{n-1}{(1-\beta_1^f)^2}\right) \right]. \end{aligned}$$

Differentiating the right side of (8.39) with respect to β_1^f and simplifying, the

[25] For a deviation that attracts so many consumers that the consumption cap, X, is reached at the other firms, the utility received by consumers is so high that the deviation cannot be profitable. Therefore, it suffices to consider this simplified optimization problem.

second derivative of profits is given by

$$-2u_1'(\frac{n-1}{r_1n(1-\beta_1^f)})(\frac{n-1}{(1-\beta_1^f)^3})+u_1''(\frac{1}{r_1n\beta_1^f})(\frac{1}{r_1n(\beta_1^f)^3})$$
$$-u_1''(\frac{n-1}{r_1n(1-\beta_1^f)})(\frac{(n-1)^2\beta_1^f}{r_1n(1-\beta_1^f)^4}). \tag{8.40}$$

The second order conditions are satisfied if (8.40) is negative, which must be the case if the sum of the first and third terms is negative, given by

$$-(\frac{n-1}{(1-\beta_1^f)^3})\left[2u_1'(\frac{n-1}{r_1n(1-\beta_1^f)})+u_1''(\frac{n-1}{r_1n(1-\beta_1^f)})(\frac{n-1}{r_1n(1-\beta_1^f)})(\beta_1^f)\right]. \tag{8.41}$$

When n is sufficiently large, β_1^f must be close to zero, or else type 1 consumers would prefer one of the other firms even if firm f chose a share price of zero. Also, the consumption offered by other firms, $\frac{n-1}{r_1n(1-\beta_1^f)}$, is bounded from above and below. Therefore, the expression in brackets in (8.41) must be positive, so the second order conditions are satisfied.

Substituting $\beta_1^f=\frac{1}{n}$ and $P^*=\frac{np^c}{(n-1)r_1}=u_1'(\frac{1}{r_1})\frac{1}{r_1}\frac{n}{n-1}$ into (8.39), we see that the first order conditions are satisfied, so firm f has no profitable deviation. From the constraint, the corresponding value of P_1^f is P^*, so the mechanism chosen by firm f is the same fixed-price-per-share mechanism chosen by the other firms. ■

Proof of Proposition 5. For any profile of mechanisms, the ensuing consumer subgame has a type-symmetric equilibrium, which follows from Lemma 1. Select an arbitrary type-symmetric equilibrium following a deviation by two or more firms.

Consider a potential deviation by a single firm, f. To show that there is no profitable deviation, we can restrict attention to mechanisms that fully allocate capacity, $x_1=\frac{1}{r_1n\beta_1^f}$, thereby allowing a higher total payment. Thus, the optimal deviation can be seen as choosing P_1^f and β_1^f to maximize profits, subject to the constraint of making type 1 consumers indifferent between firm f and the other firms (i.e., that consumers adjust their arrival probabilities to form a Nash equilibrium of the subgame).

If $\beta_1^f\leq\frac{1}{n}$ holds, consumers at other firms receive zero surplus, so the optimization problem for firm f is given by

$$\begin{aligned} & \max r_1 n \beta_1^f P_1^f \\ & \text{subject to} \\ u_1(\frac{1}{r_1 n \beta_1^f}) - P_1^f \quad = \quad & u_1(0). \end{aligned}$$

Substituting the constraint into the objective, we equivalently have the unconstrained problem of maximizing

$$\beta_1^f [u_1(\frac{1}{r_1 n \beta_1^f}) - u_1(0)].$$

The derivative of this function is

$$\begin{aligned} & u_1(\frac{1}{r_1 n \beta_1^f}) - u_1(0) - \beta_1^f u_1'(\frac{1}{r_1 n \beta_1^f}) \frac{1}{r_1 n (\beta_1^f)^2} \\ = \quad & u_1(x_1) - u_1(0) - u_1'(x_1) x_1, \end{aligned}$$

which is strictly positive due to the strict concavity of $u_1(x_1)$. Thus, the objective is increasing in β_1^f, and the highest payoff within this range is to choose $\beta_1^f = \frac{1}{n}$.

If $\beta_1^f \geq \frac{1}{n}$ holds, we have $\rho_1^j = \frac{(1-\beta_1^f) r_1 n}{n-1} < r_1$ at firms $j \neq f$, so the optimization problem for firm f is given by

$$\begin{aligned} & \max r_1 n \beta_1^f P_1^f \\ & \text{subject to} \\ u_1(\frac{1}{r_1 n \beta_1^f}) - P_1^f \quad = \quad & u_1(\frac{n-1}{(1-\beta_1^f) r_1 n}) - A r_1 \left[\frac{1 - n\beta_1^f}{n-1} \right] - u_1(\frac{1}{r_1}) + u_1(0), \end{aligned}$$

which is equivalent to the unconstrained problem of maximizing

$$\beta_1^f \left[u_1(\frac{1}{r_1 n \beta_1^f}) - u_1(\frac{n-1}{(1-\beta_1^f) r_1 n}) + A r_1 \left[\frac{1 - n\beta_1^f}{n-1} \right] + u_1(\frac{1}{r_1}) - u_1(0) \right]. \quad (8.42)$$

Differentiating (8.42) with respect to β_1^f yields

$$\begin{aligned} & -A r_1 \left[\frac{(n+1)\beta_1^f - 1}{n-1} \right] + u_1(\frac{1}{r_1 n \beta_1^f}) - u_1(\frac{n-1}{(1-\beta_1^f) r_1 n}) + u_1(\frac{1}{r_1}) \quad (8.43) \\ & -u_1(0) + \beta_1^f \left[-u_1'(\frac{1}{r_1 n \beta_1^f}) \frac{1}{r_1 n (\beta_1^f)^2} - u_1'(\frac{n-1}{(1-\beta_1^f) r_1 n}) \frac{n-1}{(1-\beta_1^f)^2 r_1 n} \right]. \end{aligned}$$

Since $\beta_1^f \geq \frac{1}{n}$ holds, the first expression in brackets in (8.43) is greater than $\frac{1}{n(n-1)}$, so the first overall term can be made arbitrarily negative for sufficiently large A. Also, the optimal β_1^f must be bounded well below 1 for sufficiently large A, or else satisfying the indifference constraint would require P_1^f to be negative. Therefore, (8.43) is strictly negative. Since profits for firm f are decreasing in β_1^f, it follows that the optimal choice within this range is $\frac{1}{n}$, and the indifference constraint implies $P_1^f = u_1(\frac{1}{r_1}) - u_1(0)$. Firm f receives the same profit as it would by adopting the mechanism specified in the statement of Proposition 5, so there is no profitable deviation. ■

Proof of Proposition 6. Suppose there is a SPNE of Γ such that some firm, f, does not allocate all of its capacity on the equilibrium path. That is, in the consumer equilibrium on the equilibrium path, firm f receives a measure of arrivals (everyone must report type 1) of $\widehat{\rho}_1^f$, and we have $\widehat{\rho}_1^f x_1^f(\widehat{\rho}_1^f) < 1$. We will show, by contradiction, that firm f has a profitable deviation to some mechanism, $\widetilde{m}^f \in M$ (which also happens to be monotonic). The mechanism is very simple, satisfying continuity and satisfying incentive compatibility by definition, providing consumers with an equal share of full capacity, up to some finite maximum, $X > \frac{1}{\widehat{\rho}_1^f}$ in order to ensure continuity at $\rho_1^f = 0$, and charging the same payment no matter how many consumers arrive.

$$\begin{aligned}\widetilde{x}_1^f(\rho_1^f) &= \min[\frac{1}{\rho_1^f}, X] \\ \widetilde{P}_1^f(\rho_1^f) &= u_1(\frac{1}{\widehat{\rho}_1^f}) - u_1(x_1^f(\widehat{\rho}_1^f)) + P_1^f(\widehat{\rho}_1^f).\end{aligned} \tag{8.44}$$

The payment made by the customers of firm f, given by (8.44), is constructed such that the utility offered by firm f under the deviation is the same as the utility offered under the original mechanism, if the measure of customers is $\widehat{\rho}_1^f$. Therefore, since $(\widehat{\rho}_1^1, ..., \widehat{\rho}_1^f, ...\widehat{\rho}_1^n)$ was a consumer equilibrium before the deviation,[26] it is a consumer equilibrium after the deviation. The profits of firm f are strictly higher after the deviation, since the right side of (8.44) is strictly greater than $P_1^f(\widehat{\rho}_1^f)$, due to the fact that $\frac{1}{\widehat{\rho}_1^f}$ exceeds $x_1^f(\widehat{\rho}_1^f)$.

We must allow for the possibility that a different consumer equilibrium is selected in response to the deviation. If $\rho_1^f > \widehat{\rho}_1^f$ occurs, the profits of firm f are

[26] That is, the mixing probabilities are such that $r_1\beta_1^f = \widehat{\rho}_1^f$.

even higher than they would have been if $\widehat{\rho}_1^f$ occurred, because the payment per customer is fixed but the firm receives more customers. Thus, if a new consumer equilibrium of this sort occurs, the deviation remains profitable.

If $\rho_1^f < \widehat{\rho}_1^f$ occurs, then consumers at firm f receive strictly higher utility than they would have received if $\widehat{\rho}_1^f$ occurred, since consumption is higher and the payment is the same. The total measure of consumers choosing other firms, $nr_1 - \rho_1^f$, is strictly more than it would have been if $\widehat{\rho}_1^f$ occurred, so there must be some firm, f', such that $\rho_1^{f'} > \widehat{\rho}_1^{f'}$ holds. Since firm f' is choosing a monotonic mechanism, the utility firm f' offers its customers must be less than or equal to the utility it would have offered if $(\widehat{\rho}_1^1, ..., \widehat{\rho}_1^n)$ occurred. Thus, $(\rho_1^1, ..., \rho_1^n)$ cannot be a consumer equilibrium. ■

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