Mixed Strategy Nash Equilibrium

Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game. Preferences must be specified over lotteries on A, which we assume are represented by the expectation of $u_i(a)$.

Let $\Delta(A_i)$ be the set of probability distributions over A_i.

$\alpha_i \in \Delta(A_i)$ is called a **mixed strategy** and $a_i \in A_i$ is called a **pure strategy**.

It is assumed that randomizations according to $(\alpha_1, ..., \alpha_n)$ are performed independently.

Thus, $(\alpha_1, ..., \alpha_n)$ induces a lottery on A, where $a \in A$ has probability

$$\prod_{j \in N} \alpha_j(a_j)$$

where $\alpha_j(a_j)$ is the probability of action a_j under the mixed strategy α_j.

Definition 32.1: The **mixed extension** of the strategic game $\langle N, (A_i), (u_i) \rangle$ is the strategic game $\langle N, (\Delta(A_i)), (U_i) \rangle$, where $U_i : \times_{j \in N} \Delta(A_j) \to \mathbb{R}$ assigns each $\alpha = (\alpha_1, ..., \alpha_n)$ the expected value under u_i of the lottery induced by α.

For finite games, we have

$$U_i(\alpha) = \sum_{a \in A} \left(\prod_{j \in N} \alpha_j(a_j) u_i(a) \right).$$

Also, letting $e(a_i)$ denote the degenerate mixed strategy assigning probability one to a_i, we can write

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) U_i(\alpha_{-i}, e(a_i)). \tag{1}$$

Definition 32.3: A mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.

Claim: If α^* is a mixed strategy Nash equilibrium of G such that $\alpha_i^* = e(a_i^*)$ for all $i \in N$, then a^* is a (pure strategy) Nash equilililibrium of G, and conversely.

$\Rightarrow$ Since there is no element of $\Delta(A_i)$ that yields higher expected utility than $e(a_i^*)$, then no $e(a_i')$ can yield higher utility. Thus, a^* is a (pure strategy) Nash equilililibrium.

$\Leftarrow$ From (1), we can write

$$U_i(\alpha_i, a_{-i}^*) = \sum_{a_i \in A_i} \alpha_i(a_i) u_i(a_{-i}^*, a_i).$$

Because $u_i(a_{-i}^*, a_i) \leq u_i(a_{-i}^*, a_i^*)$ for all $a_i \in A_i$, we have $U_i(\alpha_i^*, a_{-i}^*) \geq U_i(\alpha_i, a_{-i}^*)$ for all $\alpha_i \in \Delta(A_i)$.

Proposition 33.1: Every finite strategic game has a mixed strategy Nash equilibrium.

The strategy set is now convex, and payoff is linear in probabilities, so apply Proposition 20.3.

What if the game is not finite?

Lemma 33.2: Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game. Then α^* is a mixed strategy Nash equilibrium of G if and only if for all $i \in N$, every pure strategy in the support of α_i^* is a best response to α^*_{-i}.

In other words, every action that is played with positive probability yields that player the same payoff. This fact can be used to compute mixed strategy Nash equilibria.

Example: A simple entry game. Two firms must choose whether or not to enter a market. The fixed cost of entry for firm i is c_i. Monopoly revenues are 1, and duopoly revenues are 0.

		firm 2 enter	firm 2 don't enter
firm 1	enter	$-c_1, -c_2$	$1 - c_1, 0$
	don't enter	$0, 1 - c_2$	$0, 0$

To compute the mixed strategy Nash equilibrium, let q_i denote the probability that firm i enters. If firm 1 enters, its payoff is

$$q_2(-c_1) + (1 - q_2)(1 - c_1).$$

For firm 1 to be indifferent between entering and not, both payoffs must be zero, yielding $q_2 = 1 - c_1$. Similarly, $q_1 = 1 - c_2$.

Notice that the firm with the higher cost is more likely to enter! Mixed strategy NE often yields unintuitive comparative statics, because mixing probabilities must make the **other** player(s) indifferent.

Interpretations.

1. Players intentionally introduce randomness into their play, as in bluffing in poker.

2. In recurring games, where players ignore strategic links that could exist between plays, mixed strategy NE captures the idea of stochastic regularity. Players will choose a best response to the observed relative frequency of actions chosen, so a steady state must correspond to a mixed strategy NE. We can predict long run frequencies, but an individual play can be arbitrary.

3. Uncertainty is in the eye of the beholder. Players could be choosing actions deterministically, based on external factors that are impossible for the other player to know. For example, a poker player might bet aggressively in certain situations with certain bad hands, or when the clock is at a certain time. An Olympic wrestler might deterministically choose which moves to try, as long as no one has studied his previous matches. Once he becomes well known, he might plan his choice of moves the night before the match (no time to think during the match), as long as he is not being predictable.

4. If payoffs are slightly uncertain, then this "perturbed" Bayesian game has a Nash equilibrium in pure strategies arbitrarily close to the mixed strategy NE of the original game, and vice versa.

5. A mixed strategy NE is a commonly held set of beliefs, β, where β_i is what everyone except i believes about the distribution of i's actions. Each action in the support of β_i must be optimal, given β_{-i}.

Correlated Equilibrium

If players are basing their actions on external signals received from nature, these actions could be correlated. What new types of equilibria are possible?

Just as in Bayesian games, each player receives a signal that partitions the set of states of nature. However, we further assume that signals (types) and states do not affect the "rules of the game." They only matter if actions depend on the signal realizations (just like an external factor used for mixing).

An information partition for player i is a set of disjoint elements whose union is Ω. That is, $\wp_i = \{P_i^1, ..., P_i^K\}$, such that $P_i^k \cap P_i^{k'} = \varnothing$ for all k, k', and $\bigcup_{k=1}^{K} P_i^k = \Omega$.

The interpretation is that, when the state is ω, player i observes which element of her information partition contains ω.

Definition 45.1: A **correlated equilibrium** of a strategic game $G = \langle N, (A_i), (u_i) \rangle$ consists of

1. A finite probability space (Ω, π)

2. For each player, $i \in N$, a **partition** $\wp_i$ of Ω.

3. For each player, $i \in N$, a function, $\sigma_i : \Omega \to A_i$, where $\sigma_i(\omega) = \sigma_i(\omega')$ whenever $\omega \in P_i^k$ and $\omega' \in P_i^k$ for some $P_i^k \in \wp_i$ (the **measurability** condition)

such that, for all $i \in N$ and every $\wp_i$-measurable function $\tau_i : \Omega \to A_i$, we have

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)). \tag{2}$$

Comments:

1. The probability space and information partitions are endogenous, part of the equilibrium. Sometimes a correlated equilibrium is defined for a given information structure (see Aumann's original definition).

2. For a given information structure, a player's information structure is his/her type, so a correlated equilibrium is a Bayesian Nash equilibrium.

3. For a given information structure, we can define a "correlated extension" G^*, in which an action is a measurable function σ_i. A Nash equilibrium of G^* is a correlated equilibrium of G.

4. Equivalent to inequality (2) about ex ante expected utility, we could have imposed an inequality conditional on every element in player i's information partition.

5. For any pure strategy NE, there is a corresponding correlated equilibrium yielding the same outcome. For any mixed strategy NE, there is a corresponding correlated equilibrium yielding the same distribution of outcomes. However, the information structure must allow for the required independent randomizations.

Proposition 46.2: Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game. Any convex combination of correlated equilibrium payoff profiles of G is a correlated equilibrium payoff profile of G.

Claim: It is possible for a correlated equilibrium payoff profile to be outside the convex hull of the set of pure and mixed strategy Nash equilibrium payoff profiles.

Example 46.3: (Chicken)

		player 2 L (passive)	R (aggessive)
player 1	T (passive)	$6, 6$	$2, 7$
	B (aggressive)	$7, 2$	$0, 0$

NE payoff profiles are $(7, 2)$ and $(2, 7)$, and there is a mixed strategy NE payoff profile, $(4\frac{2}{3}, 4\frac{2}{3})$.

With perfectly correlated signals (public events), the set of correlated equilibrium payoff profiles is the convex hull of these three points.

With signals that are imperfectly correlated, new payoffs are possible. Let $\Omega = \{a, b, c\}$, $\pi(a) = \pi(b) = \pi(c) = \frac{1}{3}$.

$\wp_1 = \{\{a\}, \{b, c\}\}$, and $\wp_2 = \{\{a, b\}, \{c\}\}$.

$\sigma_1(a) = B$, $\sigma_1(b) = \sigma_1(c) = T$.

$\sigma_2(a) = \sigma_2(b) = L, \quad \sigma_2(c) = R$.

To check that σ is a correlated equilibrium, suppose player 1 observes $\{a\}$. Since he stands to receive a payoff of 7, obviously no profitable deviation exists. Suppose he observes $\{b, c\}$. Conditional expected utility from playing T is $\frac{1}{2}(6) + \frac{1}{2}(2) = 4$, while deviating to B yields conditional expected utility of $\frac{1}{2}(7) + \frac{1}{2}(0) = 3\frac{1}{2}$. Symmetric argument for player 2.

Notice that the desire to follow your signal is self enforcing, and not a knife edge situation.

Ex ante expected utility is $\frac{1}{3}(7) + \frac{1}{3}(6) + \frac{1}{3}(2) = 5$, and $(5, 5)$ is outside the convex hull of pure and mixed strategy NE payoffs. More complicated examples could be constructed with correlated equilibrium payoff profiles that Pareto dominate pure and mixed strategy NE payoffs.