Pearce, "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica* 1984

Rationalizability is a weaker (broader) solution concept than Nash equilibrium. It looks at the implications of common knowledge of rationality, without imposing consistency requirements on strategy profiles.

Suppose we are interested in narrowing the set of "reasonable" actions we predict a player might choose, with no communication or observation of past play.

(A1) Each player forms a subjective probability distribution over the other players' actions, and the distribution cannot contradict his/her knowledge.

(A2) Each player chooses an action that maximizes expected utility, based on the subjective beliefs about others' actions.

(A3) The structure of the game, and the rationality of players based on (A1) and (A2), is common knowledge.

Let $N = \{1, ..., n\}$ and let $\overline{H}$ denote the convex hull of the set H.

The procedure to remove mixed strategies that are not rationalizable is the following.

Start with $H_i \subseteq \triangle(A_i)$, for $i \in N$.

Let $H_i(0) = H_i$.

Inductively define $H_i(t)$ by

$$H_i(t) = \left\{ \begin{array}{c} \alpha_i \in H_i(t-1) : \exists \gamma \in \times_{j \in N} \overline{H}_j(t-1) \\ \text{such that } \alpha_i \text{ is a best response} \\ \text{in } H_i(t-1) \text{ to } \gamma. \end{array} \right\}$$

Thus, retain α_i if it is a best response to some conjecture over α_{-i}, that have not been removed at an earlier stage.

Define

$$R_i(H_1, ..., H_n) = \bigcap_{t=1}^{\infty} H_i(t).$$

Denote $M_i = \triangle(A_i)$ for $i \in N$, and $M = (M_1, ..., M_n)$.

Definition: Given the strategic game $\langle N, (A_i), (u_i) \rangle$, the set of **rationalizable** strategies for player i is $R_i(M)$. The profile $\alpha = (\alpha_1, ..., \alpha_n)$ is rationalizable if $\alpha_i \in R_i(M)$ for all $i \in N$.

Is every strategy consistent with (A1)-(A3) rationalizable? Yes, because if everyone is rational, then we must have $\alpha_i \in H_i(1)$. If everyone knows that everyone is rational, they must be best responding to conjectures restricted to elements of $\times_{j \in N} \overline{H}_j(1)$, which requires $\alpha_i \in H_i(2)$, and so on.

Is every rationalizable strategy consistent with (A1)-(A3)? Yes, because any choice $\alpha_i \in R_i(M)$ can be justified as a best response to others' strategies, $\alpha_i \in H_i(1)$, so everyone is rational. We have $\alpha_i \in H_i(2)$, so players rationally best respond based on the knowledge that others are rational, and so on.

Definition: $H_i \subseteq M_i$ has the **pure strategy property** if $\alpha_i \in H_i$ implies every pure strategy given positive weight by α_i is also in H_i.

Proposition 1 (Pearce): If $H_i \subseteq M_i$, and if H_i is closed, nonempty, and satisfies the pure strategy property for all $i \in N$, then

(a) $H_i(t)$ is closed, nonempty, and satisfies the pure strategy property for all $i \in N$, $t = 1, 2, ...$

(b) for some integer k, $H_i(t) = H_i(k)$ for all $t \geq k$ and all $i \in N$.

Corollary: For all $i \in N$, the set of rationalizable strategies $R_i(M)$ is nonempty and contains at least one pure strategy.

The following definitions will allow us to say more about $R_i(M)$.

Definition: For sets $H_i \subseteq M_i$, $i \in N$, we say $(H_i)_{i \in N}$ has the **best response property** if $\alpha_i \in H_i$ implies $\exists \gamma \in \times_{j \in N} \overline{H}_j$, such that α_i is a best response to γ.

Definition: $E_i = \{x_i \in M_i$: $\exists (X_j)_{j \in N}$ with the best response property, and $x_i \in X_i\}$.

Proposition 2 (Pearce): $E_i = R_i(M)$ for all $i \in N$.

Corollary: Any pure strategy that is played with positive probability in any Nash equilibrium (pure or mixed) is rationalizable.

Rationalizability and Iterated Elimination of Strictly Dominated Strategies

Definition: A mixed strategy, $\alpha_i \in M_i$, is **strictly dominated** if there exists $\gamma_i \in M_i$ such that for all $\alpha_{-i} \in M_{-i}$, we have

$$U_i(\alpha_{-i}, \gamma_i) > U_i(\alpha_{-i}, \alpha_i).$$

For 2-person games, a strategy is strictly dominated if and only if there is no conjecture for which the strategy is a best response. Therefore, the set of rationalizable strategies is the set of strategies remaining after iterated elimination of strictly dominated strategies (the order of elimination does not matter).

For 3-person games, the set of rationalizable strategies may be strictly smaller than the set resulting from iterated elimination of strictly dominated strategies.

Figure 58.1 in O-R Player 3 picks the matrix, and all players receive the same payoff.

		player 2 L	player 2 R	
player 1	U	8	0	M_1
	D	0	0	

		player 2 L	player 2 R	
player 1	U	0	0	M_2
	D	0	8	

		player 2 L	player 2 R	
player 1	U	4	0	M_3
	D	0	4	

		player 2 L	player 2 R	
player 1	U	3	3	M_4
	D	3	3	

In this game, M_3 is **not** strictly dominated. For example, the mixed strategy $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ yields the same payoff.

However, M_3 is never a best response to any conjectures about the other players. It is easy to check that for M_3 to be weakly preferred to both M_1 and M_2, we must have $\alpha_1(U) + \alpha_2(L) = 1$. But if the probabilities sum to one, then M_4 is strictly preferred. Thus, M_3 is **not** rationalizable.

However, if conjectures can allow for other players' strategies to be correlated, then strict domination is equivalent to never a weak best response (see O-R Lemma 60.1).

Then the new definition of rationalizability is equivalent to iterated elimination of strictly dominated strategies.

In the example, M_3 would be rationalizable, based on the conjecture that UL and DR occur with probability $\frac{1}{2}$.

Cautious Rationalizability: Pearce considers a form of "perfection" to eliminate strategies that risk lower payoffs with nothing to gain.

		player 2 L	R
player 1	U	$1, 1$	$0, 0$
	D	$0, 0$	$0, 0$

Unlike the usual interpretation of players making mistakes with small probability, here we go through the usual process of eliminating strategies, but we require conjectures to put (at least) small probability on all remaining actions. Actions that have been eliminated are not considered, so there are no mistakes involved.