

## Knowledge and Common Knowledge

Game Theory requires us to be interested in knowledge of “parameters” like costs, valuations, and demand, but also knowledge about what other players know.

Consider a finite probability space  $(\Omega, p)$  and an information partition for each player  $i$ .

That is,  $\wp_i = \{P_i^1, \dots, P_i^K\}$ , such that  $P_i^k \cap P_i^{k'} = \emptyset$  for all  $k, k'$ , and  $\bigcup_{k=1}^K P_i^k = \Omega$ .

Let  $P_i(\omega)$  denote that element of player  $i$ 's information partition containing  $\omega$ .

An **event** is a subset of  $\Omega$ , with the interpretation that event  $E$  occurs when the state is  $\omega$  if  $\omega \in E$ .

Player  $i$  **knows**  $E$  in state  $\omega$  if  $P_i(\omega) \subseteq E$ .

Definition: Player  $i$ 's **knowledge function** is given by

$$K_i(E) = \{\omega \in \Omega : P_i(\omega) \subseteq E\}.$$

$K_i(E)$  is the set of states in which player  $i$  knows  $E$ .

Some Facts about knowledge generated by partitions:

1.  $K_i(\Omega) = \Omega$ .
2.  $E \subseteq F$  implies  $K_i(E) \subseteq K_i(F)$ . If  $F$  occurs whenever  $E$  occurs, then whenever you know  $E$  you also know  $F$ .
3.  $K_i(E) \cap K_i(F) = K_i(E \cap F)$ . If you know  $E$  and know  $F$ , then you know that both  $E$  and  $F$  occur.
4.  $K_i(E) \subseteq E$ . Whenever you know that  $E$  occurs, then  $E$  in fact occurs.

5.  $K_i(E)$  is the union of all elements of  $i$ 's partition that are contained in  $E$ :

$$K_i(E) = \bigcup_{j: P_i^j \subseteq E} P_i^j$$

6.  $K_i(E) = K_i(K_i(E))$ . If you know  $E$ , then you know that you know  $E$ .

Example:  $\Omega = \{1, 2, \dots, 16\}$  and  $E = \{1, 2, 3, 6, 7\}$ .

$$\wp_i = \{P_i^1, \dots, P_i^8\} =$$

$$\left\{ \begin{array}{l} \{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \\ \{9, 13\}, \{10, 14\}, \{11, 15\}, \{12, 16\} \end{array} \right\}.$$

Then we have  $K_i(E) = \{2, 3, 6, 7\}$ .

Definition: An event,  $E$ , is **common knowledge** at  $\omega$  if 1 knows  $E$ , 2 knows  $E$ , 1 knows 2 knows  $E$ , 2 knows 1 knows  $E$ , and so on.

$$\omega \in K_1(E)$$

$$\omega \in K_2(E)$$

$$\omega \in K_1(K_2(E))$$

$$\omega \in K_2(K_1(E)), \text{ and so on.}$$

Definition:  $\wp_1$  is a **coarsening** of (or coarser than)  $\wp_2$  if for all  $P_1^j \in \wp_1$ , there exists  $(P_2^\kappa)_{\kappa=1}^k$  such that

$$P_2^\kappa \in \wp_2 \text{ for } \kappa = 1, \dots, k$$
$$\bigcup_{\kappa=1}^k P_2^\kappa = P_1^j.$$

If  $\wp_1$  is a **coarsening** of  $\wp_2$ , then  $\wp_2$  is a **refinement** of (or finer than)  $\wp_1$ .

Definition: The **meet** of  $\wp_1$  and  $\wp_2$  (written  $\wp_1 \wedge \wp_2$ ) is the finest common coarsening of  $\wp_1$  and  $\wp_2$ . The **join** of  $\wp_1$  and  $\wp_2$  (written  $\wp_1 \vee \wp_2$ ) is the coarsest common refinement of  $\wp_1$  and  $\wp_2$ .

Example:  $\Omega = \{1, 2, \dots, 16\}$  and  $E = \{1, 2, 3, 6, 7\}$ .

$$\wp_1 = \{P_1^1, \dots, P_1^8\} =$$

$$\left\{ \begin{array}{l} \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \\ \{9, 10\}, \{11, 12\}, \{13, 14\}, \{15, 16\} \end{array} \right\}.$$

$$\wp_2 = \{P_2^1, \dots, P_2^8\} =$$

$$\left\{ \begin{array}{l} \{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \\ \{9, 13\}, \{10, 14\}, \{11, 15\}, \{12, 16\} \end{array} \right\}.$$

$$\wp_1 \wedge \wp_2 = \left\{ \begin{array}{l} \{1, 2, 5, 6\}, \{3, 4, 7, 8\} \\ \{9, 10, 13, 14\}, \{11, 12, 15, 16\} \end{array} \right\}$$

$$\wp_1 \vee \wp_2 = \{\{1\}, \{2\}, \dots, \{15\}, \{16\}\}$$

$$K_1(E) = \{1, 2\}$$

$$K_2(E) = \{2, 3, 6, 7\}$$

$$K_1(K_2(E)) = \emptyset$$

$$K_2(K_1(E)) = \emptyset$$

Therefore,  $E$  is not common knowledge in any state.

Claim: An event  $E$  is common knowledge at  $\omega$  if  $E$  contains that element of  $\wp_1 \wedge \wp_2$  (the meet) containing  $\omega$ .

The following definition allows us to prove this claim:  $\omega'$  is **reachable** from  $\omega$  if there is a sequence,  $P^1, P^2, \dots, P^T$  such that  $\omega \in P^1$ ,  $\omega' \in P^T$ , and consecutive  $P^j$  intersect and belong alternatively to  $\wp_1$  and  $\wp_2$ .

1 knows  $E$  means that  $E$  contains  $P_1(\omega)$ .

1 knows 2 knows  $E$  means that  $E$  contains all elements of  $\wp_2$  that intersect  $P_1(\omega)$ .

1 knows 2 knows 1 knows  $E$  means that  $E$  contains all  $P^3$  in  $\wp_1$  that intersect a  $P^2$  in  $\wp_2$  that intersect  $P_1(\omega)$ .

Therefore,  $E$  is common knowledge at  $\omega$  if and only if  $E$  contains all  $\omega'$  reachable from  $\omega$ .

Proof of Claim: We must show that  $M \equiv \{\omega' \text{ reachable from } \omega\}$  is the element of  $\wp_1 \wedge \wp_2$  containing  $\omega$ .

$M$  is the union of one or more elements of  $\wp_1$  and of  $\wp_2$ . Therefore,  $M$  must be the union of one or more elements of  $\wp_1 \wedge \wp_2$ , including the element containing  $\omega$ .

If  $M$  consisted of more than one element of  $\wp_1 \wedge \wp_2$ , and if  $\omega'$  is not in the element containing  $\omega$ , then  $\omega'$  is not reachable.

Let  $A$  be an event. Then  $q_i(\omega)$  is the posterior probability of  $A$ , given  $i$ 's information

$$q_i(\omega) = \frac{\text{pr}(A \cap P_i(\omega))}{\text{pr}(P_i(\omega))}$$

Theorem (Aumann): Let  $q_1$  and  $q_2$  be numbers. Assume common priors. If  $E = \{\omega : q_1(\omega) = q_1 \text{ and } q_2(\omega) = q_2\}$  is common knowledge at  $\omega$ , then  $q_1 = q_2$ .

Note:  $A$  is the event on which players form posteriors, and is not necessarily known. The event that is common knowledge is that the posteriors are  $q_1$  and  $q_2$ .

Proof: Let  $M$  be the element of  $\wp_1 \wedge \wp_2$  containing  $\omega$ . Then  $M = \bigcup_j P_1^j$ , where we have the union of disjoint elements of  $\wp_1$ .

Because  $E$  is common knowledge at  $\omega$ , we must have  $q_1(\omega') = q_1$  for all  $\omega' \in M$ .

Therefore, for all  $P_1^j \subseteq M$ ,

$$\frac{pr(A \cap P_1^j)}{pr(P_1^j)} = q_1.$$

Cross multiplying,  $pr(A \cap P_1^j) = q_1 pr(P_1^j)$ .

Summing over (disjoint)  $P_1^j \subseteq M$ , we have  $pr(A \cap M) = q_1 pr(M)$ .

Similarly,  $pr(A \cap M) = q_2 pr(M)$ .

Therefore,  $q_1 = q_2$ .

## Discussion:

1. The assumption that the information partitions themselves are common knowledge is without loss of generality. If player 1 does not know  $\wp_2$  then we should expand  $\Omega$  to allow  $\wp_2$  [the partition over the original state space] to be different in different states.
2. Common knowledge of posteriors is a very strong assumption. How is it achieved?
3. The puzzle of the hats:  $n$  people are seated around a table wearing either a white or a black hat. The people cannot see their own hat, but they see the remaining  $n - 1$  hats. It is common knowledge that all people are rational. An observer says, "At least one of the hats is white. I will count slowly, and after each number, you can raise your hand if you know the color of your hat." If there are  $k$  people with white hats, then they all raise their hand when the observer counts to  $k$ .
4. Extra points and field goals in football.