Knowledge and Common Knowledge

Game Theory requires us to be interested in knowledge of "parameters" like costs, valuations, and demand, but also knowledge about what other players know.

Consider a finite probability space (Ω, p) and an information partition for each player i.

That is, $\wp_i = \{P_i^1, ..., P_i^K\}$, such that $P_i^k \cap P_i^{k'} = \varnothing$ for all k, k', and $\bigcup_{k=1}^{K} P_i^k = \Omega$.

Let $P_i(\omega)$ denote that element of player i's information partition containing ω.

An **event** is a subset of Ω, with the interpretation that event E occurs when the state is ω if $\omega \in E$.

Player i **knows** E in state ω if $P_i(\omega) \subseteq E$.

Definition: Player i's **knowledge function** is given by

$$K_i(E) = \{\omega \in \Omega : P_i(\omega) \subseteq E\}.$$

$K_i(E)$ is the set of states in which player i knows E.

Some Facts about knowledge generated by partitions:

1. $K_i(\Omega) = \Omega$.

2. $E \subseteq F$ implies $K_i(E) \subseteq K_i(F)$. If F occurs whenever E occurs, then whenever you know E you also know F.

3. $K_i(E) \cap K_i(F) = K_i(E \cap F)$. If you know E and know F, then you know that both E and F occur.

4. $K_i(E) \subseteq E$. Whenever you know that E occurs, then E in fact occurs.

5. $K_i(E)$ is the union of all elements of i's partition that are contained in E:

$$K_i(E) = \bigcup_{j : P_i^j \subseteq E} P_i^j$$

6. $K_i(E) = K_i(K_i(E))$. If you know E, then you know that you know E.

Example: $\Omega = \{1, 2, ..., 16\}$ and $E = \{1, 2, 3, 6, 7\}$.

$\wp_i = \{P_i^1, ..., P_i^8\} =$

$$\left\{ \begin{array}{c} \{1,5\}, \{2,6\}, \{3,7\}, \{4,8\}, \\ \{9,13\}, \{10,14\}, \{11,15\}, \{12,16\} \end{array} \right\}.$$

Then we have $K_i(E) = \{2, 3, 6, 7\}$.

Definition: An event, E, is **common knowledge** at ω if 1 knows E, 2 knows E, 1 knows 2 knows E, 2 knows 1 knows E, and so on.

$\omega \in K_1(E)$

$\omega \in K_2(E)$

$\omega \in K_1(K_2(E))$

$\omega \in K_2(K_1(E))$, and so on.

Definition: $\wp_1$ is a **coarsening** of (or coarser than) $\wp_2$ if for all $P_1^j \in \wp_1$, there exists $(P_2^\kappa)_{\kappa=1}^k$ such that

$$\begin{aligned} P_2^\kappa &\in \wp_2 \text{ for } \kappa = 1, ..., k \\ \bigcup_{\kappa=1}^{k} P_2^\kappa &= P_1^j. \end{aligned}$$

If $\wp_1$ is a **coarsening** of $\wp_2$, then $\wp_2$ is a **refinement** of (or finer than) $\wp_1$.

Definition: The **meet** of $\wp_1$and $\wp_2$ (written $\wp_1 \wedge \wp_2$) is the finest common coarsening of $\wp_1$and $\wp_2$. The **join** of $\wp_1$and $\wp_2$ (written $\wp_1 \vee \wp_2$) is the coarsest common refinement of $\wp_1$and $\wp_2$.

Example: $\Omega = \{1, 2, ..., 16\}$ and $E = \{1, 2, 3, 6, 7\}$.

$\wp_1 = \{P_1^1, ..., P_1^8\} =$

$$\left\{ \begin{array}{c} \{1,2\},\{3,4\},\{5,6\},\{7,8\}, \\ \{9,10\},\{11,12\},\{13,14\},\{15,16\} \end{array} \right\}.$$

$\wp_2 = \{P_2^1, ..., P_2^8\} =$

$$\left\{ \begin{array}{c} \{1,5\},\{2,6\},\{3,7\},\{4,8\}, \\ \{9,13\},\{10,14\},\{11,15\},\{12,16\} \end{array} \right\}.$$

$$\wp_1 \wedge \wp_2 = \left\{ \begin{array}{c} \{1,2,5,6\},\{3,4,7,8\} \\ \{9,10,13,14\},\{11,12,15,16\} \end{array} \right\}$$

$$\wp_1 \vee \wp_2 = \{\{1\},\{2\},...,\{15\},\{16\}\}$$

$K_1(E) = \{1, 2\}$

$K_2(E) = \{2, 3, 6, 7\}$

$K_1(K_2(E)) = \varnothing$

$K_2(K_1(E)) = \varnothing$

Therefore, E is not common knowledge in any state.

Claim: An event E is common knowledge at ω if E contains that element of $\wp_1 \wedge \wp_2$ (the meet) containing ω.

The following definition allows us to prove this claim: ω' is **reachable** from ω if there is a sequence, $P^1, P^2, \ldots, P^T$ such that $\omega \in P^1$, $\omega' \in P^T$, and consecutive P^j intersect and belong alternatively to $\wp_1$ and $\wp_2$.

1 knows E means that E contains $P_1(\omega)$.

1 knows 2 knows E means that E contains all elements of $\wp_2$ that intersect $P_1(\omega)$.

1 knows 2 knows 1 knows E means that E contains all P^3 in $\wp_1$ that intersect a P^2 in $\wp_2$ that intersect $P_1(\omega)$.

Therefore, E is common knowledge at ω if and only if E contains all ω' reachable from ω.

Proof of Claim: We must show that $M \equiv \{\omega'$ reachable from $\omega\}$ is the element of $\wp_1 \wedge \wp_2$ containing ω.

M is the union of one or more elements of $\wp_1$ and of $\wp_2$. Therefore, M must be the union of one or more elements of $\wp_1 \wedge \wp_2$, including the element containing ω.

If M consisted of more than one element of $\wp_1 \wedge \wp_2$, and if ω' is not in the element containing ω, then ω' is not reachable.

Let A be an event. Then $q_i(\omega)$ is the posterior probability of A, given i's information

$$q_i(\omega) = \frac{pr(A \cap P_i(\omega))}{pr(P_i(\omega))}$$

Theorem (Aumann): Let q_1 and q_2 be numbers. Assume common priors. If $E = \{\omega : q_1(\omega) = q_1 \text{ and } q_2(\omega) = q_2\}$ is common knowledge at ω, then $q_1 = q_2$.

Note: A is the event on which players form posteriors, and is not necessarily known. The event that is common knowledge is that the posteriors are q_1 and q_2.

Proof: Let M be the element of $\wp_1 \wedge \wp_2$ containing ω. Then $M = \bigcup_j P_1^j$, where we have the union of disjoint elements of $\wp_1$.

Because E is common knowledge at ω, we must have $q_1(\omega') = q_1$ for all $\omega' \in M$.

Therefore, for all $P_1^j \subseteq M$,

$$\frac{pr(A \cap P_1^j)}{pr(P_1^j)} = q_1.$$

Cross multiplying, $pr(A \cap P_1^j) = q_1 pr(P_1^j)$.

Summing over (disjoint) $P_1^j \subseteq M$, we have $pr(A \cap M) = q_1 pr(M)$.

Similarly, $pr(A \cap M) = q_2 pr(M)$.

Therefore, $q_1 = q_2$.

Discussion:

1. The assumption that the information partitions themselves are common knowledge is without loss of generality. If player 1 does not know $\wp_2$ then we should expand Ω to allow $\wp_2$ [the partition over the original state space] to be different in different states.

2. Common knowledge of posteriors is a very strong assumption. How is it achieved?

3. The puzzle of the hats: n people are seated around a table wearing either a white or a black hat. The people cannot see their own hat, but they see the remaining $n-1$ hats. It is common knowledge that all people are rational. An observer says, "At least one of the hats is white. I will count slowly, and after each number, you can raise your hand if you know the color of your hat." If there are k people with white hats, then they all raise their hand when the observer counts to k.

4. Extra points and field goals in football.