Knowledge and Common Knowledge

Game Theory requires us to be interested in knowledge of “parameters” like costs, valuations, and demand, but also knowledge about what other players know. 

Consider a finite probability space \((\Omega, p)\) and an information partition for each player \(i\).

That is, \(\varnothing_i = \{P_i^1, \ldots, P_i^K\}\), such that \(P_i^k \cap P_i^{k'} = \varnothing\) for all \(k, k'\), and \(\bigcup_{k=1}^{K} P_i^k = \Omega\).

Let \(P_i(\omega)\) denote that element of player \(i\)’s information partition containing \(\omega\).

An **event** is a subset of \(\Omega\), with the interpretation that event \(E\) occurs when the state is \(\omega\) if \(\omega \in E\).
Player $i$ **knows** $E$ in state $\omega$ if $P_i(\omega) \subseteq E$.

**Definition:** Player $i$’s **knowledge function** is given by

$$K_i(E) = \{ \omega \in \Omega : P_i(\omega) \subseteq E \}.$$

$K_i(E)$ is the set of states in which player $i$ knows $E$.

Some Facts about knowledge generated by partitions:

1. $K_i(\Omega) = \Omega$.

2. $E \subseteq F$ implies $K_i(E) \subseteq K_i(F)$. If F occurs whenever E occurs, then whenever you know E you also know F.

3. $K_i(E) \cap K_i(F) = K_i(E \cap F)$. If you know E and know F, then you know that both E and F occur.

4. $K_i(E) \subseteq E$. Whenever you know that E occurs, then E in fact occurs.
5. $K_i(E)$ is the union of all elements of $i$’s partition that are contained in $E$:

$$K_i(E) = \bigcup_{j: P_i^j \subseteq E} P_i^j$$

6. $K_i(E) = K_i(K_i(E))$. If you know $E$, then you know that you know $E$.

Example: $\Omega = \{1, 2, \ldots, 16\}$ and $E = \{1, 2, 3, 6, 7\}$.

$\wp_i = \{P_i^1, \ldots, P_i^8\} =$

$$\left\{ \begin{array}{l}
\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \\
\{9, 13\}, \{10, 14\}, \{11, 15\}, \{12, 16\}
\end{array} \right\}.$$ 

Then we have $K_i(E') = \{2, 3, 6, 7\}$. 
Definition: An event, $E$, is **common knowledge** at $\omega$ if 1 knows $E$, 2 knows $E$, 1 knows 2 knows $E$, 2 knows 1 knows $E$, and so on.

$$\omega \in K_1(E)$$

$$\omega \in K_2(E)$$

$$\omega \in K_1(K_2(E))$$

$$\omega \in K_2(K_1(E))$$, and so on.

Definition: $\wp_1$ is a **coarsening** of (or coarser than) $\wp_2$ if for all $P_1^j \in \wp_1$, there exists $(P_2^\kappa)_{\kappa=1}^k$ such that

$$P_2^\kappa \in \wp_2 \text{ for } \kappa = 1, \ldots, k$$

$$\bigcup_{\kappa=1}^k P_2^\kappa = P_1^j.$$

If $\wp_1$ is a coarsening of $\wp_2$, then $\wp_2$ is a **refinement** of (or finer than) $\wp_1$. 
Definition: The **meet** of \( \wp_1 \) and \( \wp_2 \) (written \( \wp_1 \wedge \wp_2 \)) is the finest common coarsening of \( \wp_1 \) and \( \wp_2 \). The **join** of \( \wp_1 \) and \( \wp_2 \) (written \( \wp_1 \vee \wp_2 \)) is the coarsest common refinement of \( \wp_1 \) and \( \wp_2 \).

Example: \( \Omega = \{1, 2, \ldots, 16\} \) and \( E = \{1, 2, 3, 6, 7\} \).

\[ \wp_1 = \{P_1^1, \ldots, P_1^8\} = \]
\[
\left\{ \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \\
\{9, 10\}, \{11, 12\}, \{13, 14\}, \{15, 16\} \right\}.
\]

\[ \wp_2 = \{P_2^1, \ldots, P_2^8\} = \]
\[
\left\{ \{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \\
\{9, 13\}, \{10, 14\}, \{11, 15\}, \{12, 16\} \right\}.
\]

\[ \wp_1 \wedge \wp_2 = \left\{ \{1, 2, 5, 6\}, \{3, 4, 7, 8\}, \\
\{9, 10, 13, 14\}, \{11, 12, 15, 16\} \right\} \]

\[ \wp_1 \vee \wp_2 = \{\{1\}, \{2\}, \ldots, \{15\}, \{16\}\} \]
\[ K_1(E) = \{1, 2\} \]

\[ K_2(E) = \{2, 3, 6, 7\} \]

\[ K_1(K_2(E)) = \emptyset \]

\[ K_2(K_1(E)) = \emptyset \]

Therefore, \( E \) is not common knowledge in any state.

Claim: An event \( E \) is common knowledge at \( \omega \) if \( E \) contains that element of \( \wp_1 \land \wp_2 \) (the meet) containing \( \omega \).

The following definition allows us to prove this claim: \( \omega' \) is reachable from \( \omega \) if there is a sequence, \( P^1, P^2, \ldots, P^T \) such that \( \omega \in P^1, \omega' \in P^T \), and consecutive \( P^j \) intersect and belong alternatively to \( \wp_1 \) and \( \wp_2 \).
1 knows $E$ means that $E$ contains $P_1(\omega)$.

1 knows 2 knows $E$ means that $E$ contains all elements of $\wp_2$ that intersect $P_1(\omega)$.

1 knows 2 knows 1 knows $E$ means that $E$ contains all $P^3$ in $\wp_1$ that intersect a $P^2$ in $\wp_2$ that intersect $P_1(\omega)$.

Therefore, $E$ is common knowledge at $\omega$ if and only if $E$ contains all $\omega'$ reachable from $\omega$.

Proof of Claim: We must show that $M \equiv \{\omega' \text{ reachable from } \omega\}$ is the element of $\wp_1 \wedge \wp_2$ containing $\omega$.

$M$ is the union of one or more elements of $\wp_1$ and of $\wp_2$. Therefore, $M$ must be the union of one or more elements of $\wp_1 \wedge \wp_2$, including the element containing $\omega$.

If $M$ consisted of more than one element of $\wp_1 \wedge \wp_2$, and if $\omega'$ is not in the element containing $\omega$, then $\omega'$ is not reachable.
Let $A$ be an event. Then $q_i(\omega)$ is the posterior probability of $A$, given $i$’s information

$$q_i(\omega) = \frac{pr(A \cap P_i(\omega))}{pr(P_i(\omega))}$$

Theorem (Aumann): Let $q_1$ and $q_2$ be numbers. Assume common priors. If $E = \{\omega : q_1(\omega) = q_1$ and $q_2(\omega) = q_2\}$ is common knowledge at $\omega$, then $q_1 = q_2$.

Note: $A$ is the event on which players form posteriors, and is not necessarily known. The event that is common knowledge is that the posteriors are $q_1$ and $q_2$. 
Proof: Let $M$ be the element of $\wp_1 \land \wp_2$ containing $\omega$. Then $M = \bigcup P^j_1$, where we have the union of disjoint elements of $\wp_1$.

Because $E$ is common knowledge at $\omega$, we must have $q_1(\omega') = q_1$ for all $\omega' \in M$.

Therefore, for all $P^j_1 \subseteq M$,

$$\frac{pr(A \cap P^j_1)}{pr(P^j_1)} = q_1.$$ 

Cross multiplying, $pr(A \cap P^j_1) = q_1 pr(P^j_1)$.

Summing over (disjoint) $P^j_1 \subseteq M$, we have $pr(A \cap M) = q_1 pr(M)$.

Similarly, $pr(A \cap M) = q_2 pr(M)$.

Therefore, $q_1 = q_2$. 

Discussion:

1. The assumption that the information partitions themselves are common knowledge is without loss of generality. If player 1 does not know $\varnothing_2$ then we should expand $\Omega$ to allow $\varnothing_2$ [the partition over the original state space] to be different in different states.

2. Common knowledge of posteriors is a very strong assumption. How is it achieved?

3. The puzzle of the hats: $n$ people are seated around a table wearing either a white or a black hat. The people cannot see their own hat, but they see the remaining $n-1$ hats. It is common knowledge that all people are rational. An observer says, “At least one of the hats is white. I will count slowly, and after each number, you can raise your hand if you know the color of your hat.” If there are $k$ people with white hats, then they all raise their hand when the observer counts to $k$.

4. Extra points and field goals in football.