Department of Economics
The Ohio State University
Econ 817 Game Theory

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Prof. James Peck
Prof. David Schmeidler

Homework #2 Answers

1. O-R, exercise 146.1.

Answer: The minmax payoffs are given by $v_1 = v_2 = 1$, which implies that player 1 must receive a payoff of at least 1 in any subgame perfect equilibrium of the repeated game. Player 2's payoff exceeds player 1's payoff by at least 1 at any action profile of the stage game, so player 2 must receive a payoff of at least 2 in any subgame perfect equilibrium of the repeated game. Suppose $((A, A), (A, A), ...)$ is the outcome path of a subgame perfect equilibrium. Player 2's payoff is 3. By deviating to D in the first period, player 2 receives a payoff of 5 in period 1 and a continuation payoff of at least 2, because the contintuation strategies after the deviation must form a subgame perfect equilibrium. Therefore, the deviation yields a payoff of at least

$$(1-\delta)(5+\sum_{t=1}^{\infty}\delta^t 2)=\frac{7}{2},$$

which is greater than 3.

2. Consider the following simplified version of Blackjack. A shuffled deck contains only two types of cards, 1's and 2's, with an equal number of each type. The object is to have the sum of your cards come as close to 2 as possible, without going over. Player 1 looks at the card he is dealt, and decides whether to "hit" or "stick." If he decides to hit, then another card is dealt and his total is the sum of his two cards. After observing the number of cards taken by player 1 but not the face values of the cards, player 2 observes her own card. She then decides whether to hit or stick. If she decides to hit, then another card is dealt and her total is the sum of her two cards.

If player 1's total is greater than 2, then he loses, and payoffs for the two players are $(-1, 1)$. Player 1 also loses if his total is 1 and player 2's total is 2.

If player 1's total is 2 or less and player 2's total is more than 2, then player 1 wins, and payoffs are $(1, -1)$. Player 1 also wins if his total is 2 and player 2's total is 1.

If both totals are 1 or both totals are 2, then the game is a tie, and payoffs are $(0,0)$.

Find a sequential equilibrium of this game, remembering to specify beliefs as well as strategies. Be neat, be clear, and explain your notation.

Answer: Since player 2 wins whenever player 1's total is greater than 2, we will consider the game to be over at that point. Clearly, in any sequential equilibrium, a player must stick with a total of 2. Also, when player 1 draws a second card and the game is not over, his total is 2, so player 2 must draw a second card if her first card is 1. Thus, we must specify player 1's choice when his first card is 1, and we must specify player 2's choice when her first card is 1 and player 1 sticks after one card. It turns out that both players should hit with probability one. Here is the sequential equilibrium behavior at each information set, where for example the histories $(1,S,1)$ and $(2,S,1)$ are in the same information set for player 2, who observes that player 1 sticks but does not observe player 1's card:

$$\begin{aligned} \beta_1(1)(H) &= 1, \quad \beta_1(2)(H) = 0, \\ \beta_2(2,S,1)(H) &= 1, \quad \beta_2(2,S,2)(H) = 0, \\ \beta_2(1,H,1,1)(H) &= 1, \quad \beta_2(1,H,1,2)(H) = 0. \end{aligned}$$

There are only two non-singleton information sets, with beliefs given by:

$$\begin{aligned} \mu(2,S,1) &= 1 \text{ and } \mu(1,S,1) = 0, \\ \mu(2,S,2) &= 1 \text{ and } \mu(1,S,2) = 0. \end{aligned}$$

To verify that this is a sequential equilibrium, start with player 2 in the information set containing the history $(2,S,1)$. Clearly player 2 must hit, given player 1's strategy. Now consider player 1 whose first card is 1. If he hits, his expected payoff is

$$\frac{1}{2}(-1) + \frac{1}{2}(\frac{1}{4}) = -\frac{3}{8}. \tag{1}$$

The first term on the left side of (1) reflects drawing a 2 and going over, and the second term reflects drawing a 1, in which case player 1 wins if player 2 receives a 1, hits, and receives a 2 (otherwise it is a tie). If player 1 sticks, his expected payoff is

$$\frac{1}{2}(-1) + \frac{1}{2}(\frac{1}{2}[-1] + \frac{1}{2}[1]) = -\frac{1}{2}.$$

Thus, hitting is sequentially rational. To verify consistency, simply assume that players choose the opposite action with probability ε. As $\varepsilon \to 0$, the strategy profile converges to β and beliefs from Bayes' rule converge to μ. For example,

$$\begin{aligned} \mu^\varepsilon(2,S,1) &= \frac{1-\varepsilon}{(1-\varepsilon)+\varepsilon} \to 1 \\ \mu^\varepsilon(1,S,1) &= \frac{\varepsilon}{(1-\varepsilon)+\varepsilon} \to 0, \end{aligned}$$

and similarly if player 2's first card is 2.