

A Battle of Price Manipulators and the Market Game Foundations for Rational Expectations Equilibrium*

James Peck
The Ohio State University

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Abstract

Manipulation of prices and convergence to rational expectations equilibrium is studied in a two-period Shapley-Shubik market game without noise traders. Informed players with initially long and short positions (bulls and bears) seek to manipulate consumer expectations in opposite directions. In equilibrium, bears and uninformed consumers sell up to their short-sale limits in period 1. Bulls buy in period 1 but receive arbitrage losses, as the period-1 price exceeds the period-2 price. When the number of bulls and bears approaches infinity, the equilibrium converges to the REE. Without short-sale constraints we have an unstable situation, with bulls and bears each seeking to out-leverage the other, with no revealing equilibrium possible.

1. Introduction

Manipulation of prices by agents with superior information is often thought to be both profitable to the individual and damaging to society. The classic example of price manipulation occurs when an agent ultimately wants to sell a commodity, but first buys in an attempt to influence the price at which she can sell in the

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future. However, few models are available to study such manipulation. Noise trader models have been used effectively, but we are interested here in exploring the implications of fully rational behavior. Auction models can be difficult to work with if buyers at one stage become sellers at another stage. Demand-curve submission games hold promise, but so far the tractable dynamic models make undesirable simplifying assumptions.¹ The present paper adopts a two-period Shapley-Shubik strategic market game. We reach the surprising conclusion that an informed trader may be unable to successfully manipulate prices when all agents are rational. Indeed, an informed trader may be at a disadvantage relative to uninformed traders. As a by-product of our analysis, we show that as the number of informed traders approaches infinity, the equilibrium allocation converges to what would obtain in a fully-revealing Rational Expectations Equilibrium (REE).

In the market game model, the same two goods are traded on two consecutive spot markets, where agents only care about their final holdings. Good y is the numeraire good, which can be thought of as money, and good x is the good whose price is being manipulated, which can be thought of as shares of stock in some firm. There is a continuum of consumers or small traders who are modeled as players in the game, and there is a finite number of large traders who value only the numeraire good. Some large traders are "bulls," who are endowed with a positive position in good x , and benefit from being able to sell their endowment at a high price. Other large traders are "bears," who are endowed with a negative position in good x , and benefit from a low price. Large traders observe the state of nature θ , drawn from a continuous distribution, but ordinary consumers are uninformed. We impose short-sale constraints and show that in equilibrium, the unconstrained traders (bulls) are at a disadvantage. Bulls purchase good x in period 1 in an attempt to manipulate the price and thereby manipulate the inferences of ordinary consumers. Still, the period 1 price in state θ , $p^1(\theta)$ reveals the state and the period 2 price in state θ , $p^2(\theta)$ is what would prevail in a fully revealing REE, so consumers are not fooled in equilibrium. However, $p^1(\theta) > p^2(\theta)$ always holds, so the bulls lose money on the round trip transaction of buying in period 1 and then selling in period 2. The bears and ordinary consumers sell up to their short-sale limits in period 1, and make arbitrage profits as a result.

Here is the intuition for why the unconstrained side of the market is at a disadvantage relative to the constrained side. We show that, in a type-symmetric

¹Vives (2011) offers an interesting and sophisticated analysis of a supply-function submission game. The model is static, and therefore not suited to the study of the type of manipulation in which an agent manipulates the price at one stage, and then unwinds her trade at a later stage.

equilibrium, $p^1(\theta)$ is a strictly increasing function, revealing the state, and that $p^2(\theta)$ is what would prevail in a fully revealing REE.² Suppose that $p^1(\theta) = p^2(\theta)$ holds for all θ , so that arbitrage profits are zero for everyone. If a bull were to increase slightly her bid in period 1, this would cause an increase in p^1 . It would also cause an increase in p^2 , because consumers would not be aware of the deviation and would be fooled into expecting a higher θ , causing them to increase their bids in period 2. The dominant effect would be an increase in profits from liquidating her long position in period 2. To prevent this type of deviation, equilibrium requires $p^1(\theta) > p^2(\theta)$ for all θ . This price wedge balances a tradeoff in which bidding more in period 1 (i) raises consumers' expectations and p^2 , thereby raising the liquidated value of the bull's long position, and (ii) increases the quantity of good x that is bought and then sold at a loss. Notice the similarity to the signaling in the Milgrom and Roberts (1982) model of predatory pricing. In that model, the equilibrium exhibits a wedge between the established firm's pre-entry price and the monopoly price corresponding to its cost type. The pre-entry price is decreasing in the cost type, thereby revealing the cost. Nonetheless, the pre-entry price is below the monopoly price, balancing a tradeoff between the effects of deviating to a lower pre-entry price: (i) fooling the entrant into thinking that the cost type is lower, thereby reducing the level of entry, and (ii) sacrificing profits in the pre-entry period by charging too low a price.

With $p^1(\theta) > p^2(\theta)$ for all θ , the bears are selling up to their short-sale constraints in period 1 (increasing their negative position), then buying back enough of good x in period 2 to unwind their positions at a profit. Ordinary consumers are also selling their entire endowments of good x in period 1, then buying back enough of good x in period 2 to reach their desired consumption level. Bears and consumers would like to sell more in period 1 because they are "selling high and buying low," but short-selling constraints prevent them from doing so.

What would happen if bears were not subject to short-sale constraints? Then just as bulls have an incentive to buy more in period 1 unless the incentive to manipulate consumer beliefs upwards is balanced by arbitrage losses, bears would have an incentive to sell more in period 1 unless the incentive to manipulate consumer beliefs downwards is balanced by arbitrage losses. The resulting battle between bulls and bears attempting to manipulate the market in opposite directions would lead each side to try to outleverage the other, with no revealing equilibrium possible. Thus, without trading constraints on at least one side of the market in this model, we would have an unstable and highly volatile situation.

²We adopt the convention that the REE price is strictly increasing in θ .

This is a market game model whose equilibrium prices and allocation converge to the competitive REE price and allocation. While this has been accomplished in certain auction market examples in which many participants do not trade, we accomplish the task in a framework allowing for all agents to trade in the equilibrium. The price in period 1 reveals the state, and the price in period 2 is the REE price. If the number of bulls and bears is large, or if the initial positions of bulls and bears are small relative to the endowments of the consumer sector, then the price wedge is small, so the price in period 1 is also close to the REE price.

In Section 2, we discuss the relevant literatures. Section 3 contains the model with short-sale restrictions, with the equilibrium characterization and result on convergence to REE. Section 4 considers the model without short-sale restrictions. Section 5 offers some concluding remarks.

2. Literature Review

There is a large literature on manipulation in noise-trader models, which will not be reviewed in detail.³ One could argue that noise traders are no different from rational traders who are constrained in some way, such as the present paper's restriction on short-selling. However, being explicit about the constraints enables us to understand how particular restrictions affect equilibrium behavior. For example, in the present paper, we show that taking away short-selling constraints, rather than eliminating the incentive to manipulate prices, leads instead to an unstable situation as manipulators on each side of the market attempt to outmanipulate each other. While the present paper has constrained traders, the magnitude of the noise embodied in their behavior is zero. Interestingly, in equilibrium these traders make arbitrage profits at the expense of the unconstrained traders.

An early paper on manipulation involving fully rational agents is Allen and Gale (1992). A large trader called a manipulator makes profits in equilibrium, because the market cannot distinguish between trade by a manipulator and trade by an informed trader. The informed trader knows the content of an announcement that will take place about the stock value (either good or bad). Although

³Some of these papers build on Kyle (1985). Some papers on manipulation in noise-trader models include Vila (1989), Kumar and Seppi (1992), Bagnoli and Lipman (1996), Chakraborty and Yilmaz (2004), and Goldstein and Guembel (2008).

path-breaking, the model restricts the trading options of the large traders, and price formation is not explicitly modeled. Allen, Litov, and Mei (2006) present a model of price manipulation with small arbitrageurs and a large manipulator who threatens to corner the market. The settlement price in the event of a corner is exogenously specified, and there is a random amount of supply that emerges at date 1 that might prevent the corner. The present paper looks at more routine environments in which cornering the market is not a possibility, and models competing (potential) manipulators who seek to push the price in opposite directions.

The Shapley-Shubik market game model⁴ employed in this paper has the advantage that prices are explicitly modeled. Offers to sell a commodity and bids of money to purchase the commodity are placed on a "trading post," with the price set to clear the market. This structure closely corresponds to a market in which all traders place market orders. It has a desirable feature for a model of price manipulation, that the more a trader buys (respectively, sells), the higher (respectively, lower) the price. The paper by Dubey, Geanakoplos, and Shubik (1987, henceforth DGS), like the present paper, considers a two-period market game model with agents who are asymmetrically informed about the state of nature. DGS forcefully makes the point that being too small to affect the price is different from being a price taker. Although prices reveal the state of nature, this revelation occurs after bids and offers have been submitted, so informed agents are able to benefit from their information and receive higher utility than otherwise-identical uninformed agents. Unlike the present paper, in DGS, consumption occurs in (and utility is derived from) each period. Thus, agents are unable to make trades in order to manipulate the price in the first period, and then unwind those trades in the second period. In the present paper on the other hand, the same goods (in terms of providing utility) are traded in both periods, so uninformed agents have the option to delay their trade until the state is revealed. The two models have very different implications for the value of information and convergence to REE.

Jackson and Peck (1999) employ a static market game model, and show that prices reveal the state of nature, but uninformed traders are at a disadvantage. It is also shown that the asset is mispriced in equilibrium, with prices possibly being more volatile than dividends. In contrast, the present paper shows the surprising result that uninformed traders can actually take advantage of the unsuccessful attempt by the informed bulls to manipulate prices. Equilibrium mispricing of the asset can occur in both papers. Peck and Shell (1990) consider a static market

⁴For a partial list of papers in this literature, see Shapley and Shubik (1977), Dubey and Shubik (1978), Postlewaite and Schmeidler (1978), and Peck, Shell, and Spear (1992).

game model without uncertainty, where agents are able to make arbitrarily large short sales with a punishment for being unable to deliver on their promises. Short selling enhances market thickness, allowing for Nash equilibria that are arbitrarily close to the competitive equilibrium. Similarly, Ghosal and Morelli (2004) considers retrading over a sequence of markets, but without uncertainty. It is shown that retrading allows for thicker markets and more efficient equilibria. In the present paper, the revealing equilibrium becomes closer to the REE as the short-sale limit is increased, but allowing for arbitrarily large short sales is problematic, eliminating all type-symmetric equilibria with the state revealed in period 1. McLean, Peck, and Postlewaite (2005) consider a static market game model with two states of nature and privately informed producers. With few producers, fully revealing equilibrium is impossible, because of the incentive to manipulate consumer beliefs; however, with many producers, each producer is informationally small, and fully revealing equilibrium exists.

There are well-known paradoxes associated with REE, stemming from the fact that price-formation is not explicitly modeled.⁵ Reny and Perry (2006) is the state of the art in providing the game theoretic foundations of REE, yet agents desire to consume at most one unit. The contribution of the present paper to the REE literature is to provide a game theoretic foundation of REE in a setting that allows consumption and purchases of multiple units. Indeed, the clearing-house feature of the Shapley-Shubik market game allows goods to be divisible and allows each agent in the economy to trade.

Fostel and Geanakoplos (2008) present a model that shares with the current paper a focus on leverage and an equilibrium in which assets are mispriced. However, leverage is endogenous, depending on collateral requirements that are imposed in order to avoid the possibility of bankruptcy. In the present paper, leverage is limited by short-sale constraints, and bankruptcy issues do not arise. Angeletos, Lorenzoni, and Pavan (2010) shares with the current paper the feature of information spillovers from informed to uninformed agents, and equilibria in which assets are mispriced. However, both of these papers have no large traders and do not address questions of manipulation.

⁵For an early REE paper, see Radner (1979). Also see the discussions and references in Jordan and Radner (1982), Milgrom (1981), and DGS (1987).

3. The Model

There are two goods, x and y , and a continuum of states of nature, θ , where the state is drawn from a continuous and strictly increasing c.d.f., $G(\theta)$, with support $[\underline{\theta}, \bar{\theta}]$. The economy is comprised of two types of potential manipulators who observe θ , labeled bulls and bears, and a continuum of uninformed consumers. The set of consumers is denoted by C , where consumer $h \in C$ has the endowment vector, (ω_h^x, ω_h^y) , and is a von Neumann-Morgenstern expected utility maximizer with the concave and quasi-linear Bernoulli utility function, $u_h(x_h, \theta) + y_h$.⁶

In addition to consumers, there are n bulls and n bears, who care only about consumption of good y . That is, utility is linear in consumption of good y and does not depend on consumption of good x as long as it is nonnegative; negative consumption of good x yields utility of negative infinity. Denote the set of bulls as A_+ and the set of bears as A_- . We assume that each bull or bear has an endowment of good y that is large enough to make all desired purchases. Bulls begin the game with a positive endowment or long position of good x , $\omega \geq 0$, and bears begin the game with a negative endowment or short position of good x , $-\omega$. For the special case of $\omega = 0$, bulls and bears are identical except for their labels.

We assume that the competitive economy corresponding to the model has a unique fully revealing REE, where the price of good x in terms of good y is given by $f(\theta)$. Without loss of generality, assume that $f(\theta)$ is strictly increasing. We also assume that $f(\underline{\theta}) = 0$ and that $f(\theta)$ is continuously differentiable.⁷ Denote the REE consumption for consumer h in state θ as $(\hat{x}_h(\theta), \hat{y}_h(\theta))$. To interpret the model as bulls and bears manipulating an asset market, think of good x as the asset and good y as money or general wealth, and think of $(\hat{x}_h(\theta), \hat{y}_h(\theta))$ as the desired portfolio of an ordinary consumer. For another interpretation that better connects the model with the REE literature, think of good x as the non-numeraire commodity and good y as the numeraire commodity, where consumer utility depends on the consumption bundle and the realized state. Then the bulls and bears are informed consumers whose aggregate net demands are exactly zero in every state in the REE.

To model the price formation process, and to allow for the possibility of prices

⁶The quasi-linearity assumption plays no role in the analysis of price manipulation, but matters for the comparison of the equilibrium allocation to the REE. In the concluding remarks, we discuss the extension to a general utility function.

⁷The assumption that $f(\underline{\theta}) = 0$ is not needed for any of the results, and is only made to avoid an additional case following period 1 prices that are below the lowest equilibrium value.

revealing information that can later be used by consumers, we consider a dynamic market structure with two periods. All agents make bids of good y and offers of good x in each period, with prices determined according to a Shapley-Shubik clearing house. For simplicity, we assume that agents cannot both bid and offer in the same period. All agents observe the price in period 1 before making their choices for period 2. However, we think of this trading process as occurring within a narrow window of real time, with consumption occurring afterwards, so that utility depends only on the net position achieved at the end of period 2.

Consumers can offer and bid up to their endowment, but no more. Thus, consumer h 's action set in period 1 is given by

$$\{(b_h^1, q_h^1) \in \mathfrak{R}_+^2 : q_h^1 \leq \omega_h^x, b_h^1 \leq \omega_h^y, q_h^1 b_h^1 = 0\}.$$

Consumers observe the period 1 price but not the state or the bids or offers themselves. Thus, consumer h 's bids and offers in period 2 can depend on the observed period 1 price, and cannot exceed her current holdings.

Bulls and bears cannot carry a short position of good x greater than $\bar{\omega}$ into period 2, where we have $\bar{\omega} > \omega$. Because endowments of good y are assumed to be large, we do not limit the size of bids. Thus, bull i in state θ has an action set in period 1 given by

$$\{(b_i^1(\theta), q_i^1(\theta)) \in \mathfrak{R}_+^2 : q_i^1 b_i^1 = 0, q_i^1(\theta) \leq \bar{\omega} + \omega\},$$

and bear j in state θ has an action set in period 1 is given by

$$\{(b_j^1(\theta), q_j^1(\theta)) \in \mathfrak{R}_+^2 : q_j^1 b_j^1 = 0, q_j^1(\theta) \leq \bar{\omega} - \omega\}.$$

Denoting a strategy profile for the entire game as σ , the price of good x in period 1 when the state is θ is given by

$$p^1(\theta, \sigma) = \frac{\sum_{i \in A_+} b_i^1(\theta) + \sum_{j \in A_-} b_j^1(\theta) + \int_{h \in C} b_h^1 dh}{\sum_{i \in A_+} q_i^1(\theta) + \sum_{j \in A_-} q_j^1(\theta) + \int_{h \in C} q_h^1 dh}. \quad (3.1)$$

In period 2, a consumer's bid and offer can depend on the period 1 price and the chosen period 1 action, and a bull's or a bear's bid and offer can depend on the period 1 price, the observed state, and the chosen period 1 action. For simplicity, we require that an agent cannot offer a positive bid and a positive offer in period 2, and we require that bids and offers do not exceed an agent's current holdings. We denote period 2 actions as $(b_h^2(p^1, b_h^1, q_h^1), q_h^2(p^1, b_h^1, q_h^1))$ for consumer

h , $(b_i^2(\theta, p^1, b_i^1, q_i^1), q_i^2(\theta, p^1, b_i^1, q_i^1))$ for bull i , and $(b_j^2(\theta, p^1, b_j^1, q_j^1), q_j^2(\theta, p^1, b_j^1, q_j^1))$ for bear j .

Since we must evaluate deviations from σ , we will need notation for the price in period 2 in state θ under strategy profile σ , following an arbitrary action profile in period 1, s^1 , which determines a period 1 price, p^1 . This price is the sum of the bids divided by the sum of the offers, given by

$$p^2(\theta, \sigma; s^1) = \frac{\sum_{i \in A_+} b_i^2(\theta, p^1, b_i^1, q_i^1) + \sum_{j \in A_-} b_j^2(\theta, p^1, b_j^1, q_j^1) + \int_{h \in C} b_h^2(p^1, b_h^1, q_h^1) dh}{\sum_{i \in A_+} q_i^2(\theta, p^1, b_i^1, q_i^1) + \sum_{j \in A_-} q_j^2(\theta, p^1, b_j^1, q_j^1) + \int_{h \in C} q_h^2(p^1, b_h^1, q_h^1) dh} \quad (3.2)$$

The final allocation is determined according to the standard Shapley-Shubik market clearing rules. Offers sent to the market are subtracted from a player's holdings of good x , and bids sent to the market are subtracted from a player's holdings of good y . However, offers are then sold, returning an amount of good y equal to the offer multiplied by the price; bids are used to purchase a quantity of good x equal to the amount of the bid divided by the price.⁸ For consumer h , final consumption in state θ under strategy profile σ is given by⁹

$$x_h(\theta, \sigma) = \omega_h^x + \frac{b_h^1}{p^1(\theta, \sigma)} - q_h^1 + \frac{b_h^2(p^1, b_h^1, q_h^1)}{p^2(\theta, \sigma)} - q_h^2(p^1, b_h^1, q_h^1) \quad (3.3)$$

$$y_h(\theta, \sigma) = \omega_h^y - b_h^1 + q_h^1 p^1(\theta, \sigma) - b_h^2(p^1, b_h^1, q_h^1) + q_h^2(p^1, b_h^1, q_h^1) p^2(\theta, \sigma). \quad (3.4)$$

For bull i and bear j , the final allocation, net of the endowment of good y , is given

⁸This specification presumes that prices are nonzero and that the final consumptions are nonnegative. If all bids are zero, leading to a price of zero, then we adopt the convention that $0/0=0$ (i.e., all offers are lost). If some agent receives negative consumption of some good, we cancel all trades, wipe out the initial positions of the bulls and bears, and confiscate all endowments of any agent with negative consumption. This harsh penalty ensures that no one is unable to meet their obligations in equilibrium.

⁹To consider consumption that would arise following a deviation, we will sometimes use the notation $x_h(\theta, \sigma; s^1)$ or $y_h(\theta, \sigma; s^1)$ to denote consumption in state θ under strategy profile σ , following the period 1 action profile s^1 .

by

$$\begin{aligned}
x_i(\theta, \sigma) &= \omega + \frac{b_i^1(\theta)}{p^1(\theta, \sigma)} - q_i^1(\theta) + \frac{b_i^2(\theta, p^1, b_i^1, q_i^1)}{p^2(\theta, \sigma)} - q_i^2(\theta, p^1, b_i^1, q_i^1) \\
y_i(\theta, \sigma) &= -b_i^1(\theta) + q_i^1(\theta)p^1(\theta, \sigma) - b_i^2(\theta, p^1, b_i^1, q_i^1) + q_i^2(\theta, p^1, b_i^1, q_i^1)p^2(\theta, \sigma) \\
x_j(\theta, \sigma) &= -\omega + \frac{b_j^1(\theta)}{p^1(\theta, \sigma)} - q_j^1(\theta) + \frac{b_j^2(\theta, p^1, b_j^1, q_j^1)}{p^2(\theta, \sigma)} - q_j^2(\theta, p^1, b_j^1, q_j^1) \\
y_j(\theta, \sigma) &= -b_j^1(\theta) + q_j^1(\theta)p^1(\theta, \sigma) - b_j^2(\theta, p^1, b_j^1, q_j^1) + q_j^2(\theta, p^1, b_j^1, q_j^1)p^2(\theta, \sigma).
\end{aligned}$$

Our solution concept is weak sequential equilibrium (WSE), in which the only consistency requirement is that beliefs satisfy Bayes' rule wherever possible.¹⁰ The relevant beliefs are the beliefs of consumers about the state conditional on their information in period 2.¹¹ In the WSE constructed below, beliefs assign probability one to a single state, denoted by $\theta^e(p^1)$. We construct a type-symmetric equilibrium in which bulls offer zero and make positive bids in period 1, while consumers and bears bid zero and offer up to their short-sale constraints in period 1. The maximum possible offer by consumers and bears in period 1, denoted by \bar{x}^1 , is given by

$$\bar{x}^1 = n(\bar{\omega} - \omega) + \int_{h \in C} \omega_h^x dh.$$

Proposition 1: *The following strategy profile¹² and beliefs constitute a symmetric*

¹⁰Mas-Colell et al (1995) refer to this concept as weak perfect Bayesian equilibrium.

¹¹Given the strategy profile, beliefs about the state determine beliefs about the period 1 actions of each player.

¹²See Remark 1 below.

WSE:

$$\text{bull } i : \quad b_i^1(\theta) = \frac{f(\theta)(\bar{x}^1 + \omega)}{n}, \quad q_i^1(\theta) = 0, \quad (3.5)$$

$$b_i^2(\theta, p^1, b_i^1, q_i^1) = 0, \quad q_i^2(\theta, p^1, b_i^1, q_i^1) = \omega + \frac{b_i^1}{p^1} - q_i^1, \quad (3.6)$$

$$\text{bear } j : \quad b_j^1(\theta) = 0, \quad q_j^1(\theta) = \bar{\omega} - \omega, \quad (3.7)$$

$$b_j^2(\theta, p^1, b_j^1, q_j^1) = (q_j^1 + \omega - \frac{b_j^1}{p^1})f(\theta^e(p^1)), \quad q_j^2(\theta, p^1, b_j^1, q_j^1) = 0, \quad (3.8)$$

$$\text{consumer } h : \quad b_h^1 = 0, \quad q_h^1 = \omega_h^x, \quad (3.9)$$

$$b_h^2(p^1, b_h^1, q_h^1) = \left[\hat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \right] f(\theta^e(p^1)), \quad (3.10)$$

$$q_h^2(p^1, b_h^1, q_h^1) = 0.$$

$$\theta^e(p^1) = f^{-1}\left(\frac{p^1 \bar{x}^1}{\bar{x}^1 + \omega}\right) \quad \text{if } p^1 \leq \frac{\bar{x}^1 + \omega}{\bar{x}^1} f(\bar{\theta}) \quad (3.11)$$

$$\theta^e(p^1) = \bar{\theta} \quad \text{if } p^1 > \frac{\bar{x}^1 + \omega}{\bar{x}^1} f(\bar{\theta}) \quad (3.12)$$

Along the equilibrium path, prices are given by

$$p^1(\theta) = \left(\frac{\bar{x}^1 + \omega}{\bar{x}^1} \right) f(\theta) \quad \text{and} \quad (3.13)$$

$$p^2(\theta) = f(\theta). \quad (3.14)$$

Remark 1. The specified actions in period 2 presume that agents did not deviate by so much in period 1 to put them on the opposite side of the market in period 2. For completeness, if for bull i we have

$$\omega + \frac{b_i^1}{p^1} - q_i^1 \leq 0,$$

then her action in period 2 is

$$b_i^2(\theta, p^1, b_i^1, q_i^1) = (q_i^1 - \omega)f(\theta^e(p^1)), \quad q_i^2(\theta, p^1, b_i^1, q_i^1) = 0.$$

If, for bear j we have

$$q_j^1 + \omega - \frac{b_j^1}{p^1} \leq 0,$$

then his action in period 2 is

$$b_j^2(\theta, p^1, b_j^1, q_j^1) = 0, \quad q_j^2(\theta, p^1, b_j^1, q_j^1) = \frac{b_j^1}{p^1} - q_j^1 - \omega.$$

If for consumer h we have

$$\widehat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \leq 0,$$

then consumer h 's action in period 2 is

$$b_h^2(p^1, b_h^1, q_h^1) = 0, \quad q_h^2(p^1, b_h^1, q_h^1) = - \left[\widehat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \right].$$

Extending the proof of Proposition 1 to establish sequential rationality for these deviations in period 1 straightforwardly mirrors the proof presented, so we omit the details.

Proof of Proposition 1. Equation (3.13) follows immediately from substituting (3.5), (3.7), and (3.9) into (3.1). To derive equation (3.14), we see from (3.11) and (3.1) that consumers' beliefs assign probability one to the correct state, $\theta^e(p^1(\theta)) = \theta$ for all θ . Therefore, along the equilibrium path, in period 2 and state θ , each bear bids $\bar{\omega}f(\theta)$ and for each h , consumer h bids $\widehat{x}_h(\theta)f(\theta)$. Each bull offers

$$\omega + \frac{f(\theta)(\bar{x}^1 + \omega)}{np^1(\theta)} = \omega + \frac{f(\theta)(\bar{x}^1 + \omega)}{n \left(\frac{\bar{x}^1 + \omega}{\bar{x}^1} \right) f(\theta)} = \omega + \frac{\bar{x}^1}{n}.$$

Therefore, along the equilibrium path, we have

$$p^2(\theta) = \frac{n\bar{\omega}f(\theta) + f(\theta) \int_{h \in C} \widehat{x}_h(\theta) dh}{n\omega + \bar{x}^1} = f(\theta) \left[\frac{n\bar{\omega} + \int_{h \in C} \widehat{x}_h(\theta) dh}{n\bar{\omega} + \int_{h \in C} \omega_h^x dh} \right]. \quad (3.15)$$

Because bulls and bears do not demand any of good x in the REE, market clearing requires

$$\int_{h \in C} \widehat{x}_h(\theta) dh = n\omega + n(-\omega) + \int_{h \in C} \omega_h^x dh,$$

so (3.15) simplifies to $p^2(\theta) = f(\theta)$.

Now let us verify that (3.5)-(3.12) is a WSE.

Beliefs are consistent. To see this, notice that all values of p^1 between 0 and $\left(\frac{\bar{x}^1 + \omega}{\bar{x}^1}\right) f(\bar{\theta})$ are on the equilibrium path, and since $f'(\theta) > 0$ we can invert $p^1(\theta)$ to infer the correct state. Bayes' rule yields (3.11). For $p^1 > \left(\frac{\bar{x}^1 + \omega}{\bar{x}^1}\right) f(\bar{\theta})$, the observed price is too high to be inconsistent with the strategy profile. Consumers believe that the state is $\bar{\theta}$ as specified in (3.12); since Bayes' rule does not apply, these beliefs are consistent.

Sequential rationality is satisfied in period 2 for each agent and each information set. For consumer h observing (p^1, b_h^1, q_h^1) and with beliefs $\theta^e(p^1)$, whose actions have a negligible effect on the period 2 price, $b_h^2(p^1, b_h^1, q_h^1)$ and $q_h^2(p^1, b_h^1, q_h^1)$ must satisfy

$$\begin{aligned} & \max_{b_h^2, q_h^2} u_h(x_h, \theta^e) + y_h \\ & \text{subject to} \\ x_h &= \omega_h^x + \frac{b_h^1}{p^1} - q_h^1 + \frac{b_h^2}{f(\theta^e)} - q_h^2, \\ y_h &= \omega_h^y - b_h^1 + q_h^1 p^1 - b_h^2 + q_h^2 f(\theta^e), \\ 0 &\leq q_h^2 \leq \omega_h^x + \frac{b_h^1}{p^1} - q_h^1, \\ 0 &\leq b_h^2 \leq \omega_h^y - b_h^1 + q_h^1 p^1, \\ b_h^2 q_h^2 &= 0 \end{aligned} \tag{3.16}$$

Due to the quasi-linearity of the utility function, a necessary and sufficient condition for feasible bids and offers to solve (3.16) is that consumption of good x satisfy

$$\frac{\partial u_h(x_h, \theta^e)}{\partial x_h} = f(\theta^e). \tag{3.17}$$

However, condition (3.17) is also the necessary and sufficient condition characterizing utility maximization in the REE in state θ^e , with solution $x_h = \hat{x}_h(\theta^e)$. Since (3.10) gives rise to $x_h = \hat{x}_h(\theta^e)$ in (3.16), period 2 consumer behavior is sequentially rational.

Clearly, it is sequentially rational for a bull to bid zero and offer her entire holdings of good x in period 2, as specified in (3.6). To show that (3.8) is sequentially rational, the following lemma will be useful.

Lemma 1: *Assume that agents did not deviate by so much in period 1 to put them on the opposite side of the market in period 2. That is, for each bull i , bear j , and consumer h , we have*

$$\begin{aligned}\omega + \frac{b_i^1}{p^1} - q_i^1 &> 0, \\ q_j^1 + \omega - \frac{b_j^1}{p^1} &> 0, \\ \widehat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} &> 0.\end{aligned}$$

Then for arbitrary period 1 actions s^1 satisfying the above inequalities, the period 2 actions given by (3.6), (3.8), (3.10), and (3.11) yield $p^2(\theta, \sigma; s^1) = f(\theta^e(p^1))$.

Proof of Lemma 1. Given period 1 actions, p^1 , (3.6), (3.8), (3.10), and (3.11), the price in period 2 is given by

$$p^2(\theta, \sigma; s^1) = \frac{\sum_{j \in A_-} (q_j^1 - \frac{b_j^1}{p^1}) + n\omega + \int_{h \in C} [\widehat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1}] dh}{\sum_{i \in A_+} (\frac{b_i^1}{p^1} - q_i^1) + n\omega} f(\theta^e(p^1)). \quad (3.18)$$

Dividing both sides of (3.18) by $f(\theta^e(p^1))$, multiplying numerator and denominator on the right side by p^1 , and using the market clearing property of REE, that $\int_{h \in C} [\widehat{x}_h(\theta^e(p^1)) - \omega_h^x] = 0$ holds, we have

$$\frac{p^2(\theta, \sigma; s^1)}{f(\theta^e(p^1))} = \frac{p^1 \left[\sum_{j \in A_-} q_j^1 + n\omega + \int_{h \in C} q_h^1 dh \right] - \sum_{j \in A_-} b_j^1 - \int_{h \in C} b_h^1 dh}{p^1 \sum_{i \in A_+} (n\omega - q_i^1) + \sum_{i \in A_+} b_i^1}. \quad (3.19)$$

Denoting the left side of (3.19) as K (suppressing the dependence on the state and the period 1 actions), cross multiplying, and manipulating the expression, we have

$$\begin{aligned}& (K - 1) \sum_{i \in A_+} b_i^1 + \left[\sum_{i \in A_+} b_i^1 + \sum_{j \in A_-} b_j^1 + \int_{h \in C} b_h^1 dh \right] \\ &= p^1 \left[\sum_{j \in A_-} q_j^1 + \sum_{i \in A_+} q_i^1 + \int_{h \in C} q_h^1 dh \right] + (1 - K) p^1 (n\omega - \sum_{i \in A_+} q_i^1).\end{aligned}$$

From the definition of p^1 , the second term on the left side of the above equation cancels the first term on the right side, leaving

$$(K - 1) \sum_{i \in A_+} b_i^1 = (1 - K)p^1(n\omega - \sum_{i \in A_+} q_i^1). \quad (3.20)$$

For each bull i , we have $\omega + \frac{b_i^1}{p^1} - q_i^1 > 0$, so summing over $i \in A_+$ yields the conclusion from (3.20) that $K = 1$ must hold. ■

From Lemma 1, by bidding according to (3.8) in period 2, bear j induces the price $p^2(\theta, \sigma; s^1) = f(\theta^e(p^1))$. His net position of good x entering period 2 is $-(q_j^1 + \omega - \frac{b_j^1}{p^1})$, and his net purchases in period 2 are

$$\frac{(q_j^1 + \omega - \frac{b_j^1}{p^1})f(\theta^e(p^1))}{p^2(\theta, \sigma; s^1)},$$

which equals $(q_j^1 + \omega - \frac{b_j^1}{p^1})$. Thus, bear j exactly closes out his position of good x , so sequential rationality is satisfied at all information sets in period 2.

Sequential rationality is satisfied in period 1. Sequential rationality for each consumer follows from the fact that consumers have a negligible influence on prices, and from the fact that $p^1(\theta) \geq p^2(\theta)$ holds for each θ . Consumption of good x in state θ is $\hat{x}_h(\theta)$ independent of consumer h 's action in period 1, but consumption of good y is highest when the consumer adopts (3.9).

To verify sequential rationality by bulls in period 1, express the profit of bull i , given her continuation strategy and the equilibrium strategies of other agents, as a function of θ and arbitrary period 1 actions (b_i^1, q_i^1) , as

$$\pi_i^{bull}(\theta, b_i^1, q_i^1) = -b_i^1 + q_i^1 p^{1,bull}(\theta, b_i^1, q_i^1) + \left[\omega + \frac{b_i^1}{p^{1,bull}(\theta, b_i^1, q_i^1)} - q_i^1 \right] f(\theta^e(p^{1,bull}(\theta, b_i^1, q_i^1))). \quad (3.21)$$

Equation (3.21) uses Lemma 1, capturing the effect that bull i has on p^1 , and the resulting effect on consumer behavior in period 2. The term $p^{1,bull}(\theta, b_i^1, q_i^1)$ is the price in period 1 when the state is θ and all agents other than bull i are choosing the strategies specified in (3.5)-(3.12), given by

$$p^{1,bull}(\theta, b_i^1, q_i^1) = \frac{\frac{(n-1)}{n}f(\theta)(\bar{x}^1 + \omega) + b_i^1}{\bar{x}^1 + q_i^1}. \quad (3.22)$$

Substituting (3.11) and (3.22) into (3.21) and simplifying, we have

$$\pi_i^{bull}(\theta, b_i^1, q_i^1) = \frac{n-1}{n}\omega f(\theta). \quad (3.23)$$

It follows that bull i receives profit that is independent of (b_i^1, q_i^1) , so the specified strategy satisfies sequential rationality.

To verify sequential rationality by bears in period 1, the objective function for bear j , given his continuation strategy and the equilibrium strategies of other agents, as a function of θ and arbitrary period 1 actions (b_j^1, q_j^1) , is given by:

$$\pi_j^{bear}(\theta, b_j^1, q_j^1) = -b_j^1 + q_j^1 p^{1,bear}(\theta, b_j^1, q_j^1) + \left[-\omega + \frac{b_j^1}{p^{1,bear}(\theta, b_j^1, q_j^1)} - q_j^1 \right] f(\theta^e(p^{1,bear}(\theta, b_j^1, q_j^1))), \quad (3.24)$$

where $p^{1,bear}(\theta, b_j^1, q_j^1)$ is the price in period 1 when the state is θ and all agents other than bear j are choosing the strategies specified in (3.5)-(3.12), given by

$$p^{1,bear}(\theta, b_j^1, q_j^1) = \frac{f(\theta)(\bar{x}^1 + \omega) + b_j^1}{\bar{x}^1 + q_j^1 - \bar{\omega}}.$$

Since (3.11) implies

$$f(\theta^e(p^{1,bear}(\theta, b_j^1, q_j^1))) = \frac{p^{1,bear}(\theta, b_j^1, q_j^1)\bar{x}^1}{\bar{x}^1 + \omega},$$

expression (3.24) can be simplified to

$$\pi_j^{bear}(\theta, b_j^1, q_j^1) = \left[\frac{\omega}{\bar{x}^1 + \omega} \right] [-b_j^1 + (q_j^1 - \bar{x}^1)p^{1,bear}(\theta, b_j^1, q_j^1)]. \quad (3.25)$$

It is easy to see from (3.25) that $\pi_j^{bear}(\theta, b_j^1, q_j^1)$ is strictly decreasing in b_j^1 and strictly increasing in q_j^1 . Therefore, it is sequentially rational to bid zero and offer up to one's short-sale limit. ■

One might think that bulls can take advantage of their market power and their ability to manipulate the beliefs of consumers, but this is not the case. Bulls increase their long position in period 1 in an effort to manipulate the price and convince consumers that good x is more valuable, but consumers understand this incentive and correct their beliefs accordingly. Bulls actually bid the price in

period 1 above the price in period 2, and lose money as a result as bears and consumers take the other side of the transactions. Still, if the bulls do not bid up the price, consumers would mistakenly think that the state is lower than it actually is, reminiscent of Milgrom and Roberts (1982). Notice that the side of the market that is not constrained in its ability to manipulate prices is actually worse off than they would be under full information. Bulls receive profits according to (3.23), but under full information they would receive $\omega f(\theta)$. As the number of bulls approaches infinity, this difference approaches zero. The following proposition shows that the BNE prices and allocation converge to the REE price and allocation, as we replicate the economy.

Proposition 2: *Consider an r -fold replication of the economy. If $\omega = 0$ holds, then for the equilibrium characterized in Proposition 1, the prices and allocation coincide with the REE for all r . If $\omega > 0$ holds, then the equilibrium converges to the REE as $r \rightarrow \infty$, in the following sense. For all θ , $p^1(\theta)$ converges to the REE price, $p^2(\theta)$ is exactly the REE price, and the allocation (uniformly across consumers and arbitrageurs) converges to the REE allocation.*

Proof. We have an equilibrium for the replicated economy by substituting rn for n and $r\bar{x}^1$ for \bar{x}^1 in the construction given in Proposition 1.¹³ If $\omega = 0$ holds, then we have $p^1(\theta) = p^2(\theta) = f(\theta)$ for all θ , and the result follows immediately. If $\omega > 0$ holds, we have

$$\lim_{r \rightarrow \infty} p^1(\theta; r) = \lim_{r \rightarrow \infty} \left(\frac{r\bar{x}^1 + \omega}{r\bar{x}^1} \right) f(\theta) = f(\theta).$$

Consumer i receives consumption of good x equal to her REE level, $\hat{x}_h(\theta)$, based on the argument given in the proof of Proposition 1. Because the augmentation of her income, by selling her endowment in period 1 and buying it back in period 2, is converging to zero, her consumption of good y converges to the REE level, $\hat{y}_h(\theta)$. It is easy to see that the bulls and bears receive the consumption that they receive under the REE. ■

It is interesting to note that the convergence result in Proposition 2 applies to the most paradoxical environment discussed in the REE literature. In the competitive economy, each bull is a net seller of ω units of good x , and each bear is a net buyer of ω units of good x , independent of θ . Thus, the net trades of all

¹³Consumers are assumed to be replicated as well, but this is not essential for the result.

informed agents do not depend on the state of nature, so the question might arise as to how that information can become embedded in the price. The resolution of that paradox is that we model the price formation process with a sequence of markets. It turns out that the overall net trades of good x by informed agents (across the two markets) do not depend on the state, but the net trades of good x on the first market are state dependent and revealing.

4. The Model without Short-Sale Restrictions

From Proposition 1, we see that relaxing short-sale restrictions with a higher \bar{x}^1 leads to a thicker market and smaller price distortions.¹⁴ What would happen without any short-sale restrictions at all? It turns out that the resulting economy is unstable, with no revealing equilibrium. Bulls and bears want to manipulate consumer beliefs by trading more than the other side, with no solution with finite bids and offers. To see this, suppose that there is a fully revealing type-symmetric equilibrium in which bulls choose a bid function $b^1(\theta)$ and bears choose an offer function $q^1(\theta)$ in period 1. To help simplify the analysis, we will assume that bulls and bears observe all period 1 bids and offers. We also assume that consumers are not allowed to participate on the period 1 market, although they observe the period 1 price. The equilibrium period 1 price in state θ is then given by

$$p^1(\theta) = \frac{b^1(\theta)}{q^1(\theta)}.$$

The period 1 price in state θ , given that the other players are employing their equilibrium strategies and bull i chooses the period 1 bid b_i^1 , and the period 1 price in state θ , given that the other players are employing their equilibrium strategies and bear j chooses the period 1 offer q_j^1 , are given by

$$\begin{aligned} p^1(b_i^1, \theta) &= \frac{b_i^1 + (n-1)b^1(\theta)}{nq^1(\theta)} \\ p^1(q_j^1, \theta) &= \frac{nb^1(\theta)}{q_j^1 + (n-1)q^1(\theta)} \end{aligned}$$

The following lemma is similar to Lemma 1.

¹⁴See Peck and Shell (1990).

Lemma 2. *In the model without short-sale restrictions, suppose we have an open-market,¹⁵ fully revealing, type-symmetric WSE with bulls choosing the bid function $b^1(\theta)$ and bears choosing the offer function $q^1(\theta)$ in period 1. Then for all p^1 such that $p^1 = p^1(\theta)$ holds for some θ , we have:*

(i) *consumer beliefs assign probability one to a single state, denoted by $\theta^e(p^1)$, which is the unique solution to $p^1 = p^1(\theta)$, and*

(ii) *for all period-1 action profiles s^1 leading to the price p^1 , the final holdings of good x for each bull and bear is zero, and the price in period 2 is $p^2(\theta, \sigma; s^1) = f(\theta^e(p^1))$.*

Proof of Lemma 2. Since the WSE is fully revealing, consistency requires consumers observing p^1 to have beliefs assigning probability one to the state solving $p^1 = p^1(\theta)$. Furthermore, consumers assign probability one to the continuation strategies of the other players in state θ following the price $p^1(\theta)$, and thus the period 2 price $p^2(\theta^e(p^1))$. Because consumers are negligible, sequential rationality for consumer h requires that (b_h^2, q_h^2) solve the following optimization problem

$$\begin{aligned} & \max_{b_h^2, q_h^2} u_h(x_h, \theta^e(p^1)) + y_h \\ & \text{subject to} \end{aligned} \tag{4.1}$$

$$\begin{aligned} x_h &= \omega_h^x + \frac{b_h^2}{p^2(\theta^e(p^1))} - q_h^2, \\ y_h &= \omega_h^y - b_h^2 + q_h^2 p^2(\theta^e(p^1)), \\ 0 &\leq q_h^2 \leq \omega_h^x, \\ 0 &\leq b_h^2 \leq \omega_h^y, \\ b_h^2 q_h^2 &= 0 \end{aligned}$$

Due to the quasi-linearity of the utility function, a necessary and sufficient condition for feasible bids and offers to solve (4.1) is that anticipated consumption of good x satisfy

$$\frac{\partial u_h(x_h, \theta^e(p^1))}{\partial x_h} = p^2(\theta^e(p^1)). \tag{4.2}$$

It is straightforward to see that consumer h 's anticipated consumption of good x satisfying (4.2), and therefore consumer h 's bids and offers solving (4.1), are uniquely determined.

¹⁵Open-market means that bids and offers are positive.

Let A_+^1 denote the set of bulls or bears with a nonnegative net holding of good x after period 1, and let A_-^1 denote the set of bulls or bears with a negative net holding of good x after period 1. Denote the endowment of good x of player i as ω_i , so we have $\omega_i = \omega$ for a bull and $\omega_i = -\omega$ for a bear.

Any bull or bear with a positive net holding of good x will offer exactly that amount in period 2, by sequential rationality, so we have

$$q_i^2 = \frac{b_i^1}{p^1} - q_i^1 + \omega_i \quad \text{for } i \in A_+^1.$$

For $i \in A_-^1$, sequential rationality for player i requires a bid in period 2, such that final holdings of good x are zero (given the profile of actions in period 2). Thus, we have

$$x_i = \frac{b_i^1}{p^1} - q_i^1 + \omega_i + \frac{b_i^2}{p^2(\theta, \sigma; s^1)} = 0. \quad (4.3)$$

The period 2 price is then given by

$$p^2(\theta, \sigma; s^1) = \frac{\sum_{i \in A_-^1} b_i^2 + \int_{h \in C} b_h^2 dh}{\sum_{i \in A_+^1} (\frac{b_i^1}{p^1} - q_i^1 + \omega_i) + \int_{h \in C} q_h^2 dh}. \quad (4.4)$$

From (4.3), we have

$$0 = \sum_{i \in A_-^1} x_i = \sum_{i \in A_-^1} (\frac{b_i^1}{p^1} - q_i^1 + \omega_i) + \frac{\sum_{i \in A_-^1} b_i^2}{p^2(\theta, \sigma; s^1)}. \quad (4.5)$$

We also have

$$\begin{aligned} & \sum_{i \in A_-^1} (\frac{b_i^1}{p^1} - q_i^1 + \omega_i) + \sum_{i \in A_+^1} (\frac{b_i^1}{p^1} - q_i^1 + \omega_i) \\ &= \frac{\sum_i b_i^1}{p^1} - \sum_i q_i^1 = 0. \end{aligned}$$

Therefore, (4.5) becomes

$$\begin{aligned} 0 &= - \sum_{i \in A_+^1} (\frac{b_i^1}{p^1} - q_i^1 + \omega_i) + \frac{\sum_{i \in A_-^1} b_i^2}{p^2(\theta, \sigma; s^1)}, \quad \text{or} \\ \sum_{i \in A_+^1} (\frac{b_i^1}{p^1} - q_i^1 + \omega_i) &= \frac{\sum_{i \in A_-^1} b_i^2}{p^2(\theta, \sigma; s^1)}. \end{aligned} \quad (4.6)$$

Substituting (4.6) into (4.4), we have

$$p^2(\theta, \sigma; s^1) = \frac{\sum_{i \in A^1} b_i^2 + \int_{h \in C} b_h^2 dh}{\frac{\sum_{i \in A^1} b_i^2}{p^2(\theta, \sigma; s^1)} + \int_{h \in C} q_h^2 dh}.$$

Cross-multiplying and simplifying, we have

$$p^2(\theta, \sigma; s^1) = \frac{\int_{h \in C} b_h^2 dh}{\int_{h \in C} q_h^2 dh}. \quad (4.7)$$

From (4.7) and the allocation rule, (3.3) and (3.4), we have

$$\begin{aligned} \int_{h \in C} x_h(\theta, \sigma; s^1) dh &= \int_{h \in C} \omega_h^x dh \quad \text{and} \\ \int_{h \in C} y_h(\theta, \sigma; s^1) dh &= \int_{h \in C} \omega_h^y dh. \end{aligned} \quad (4.8)$$

Since consumers' bids and offers in the actual state following the period 1 action profile s^1 are the same as in state $\theta^e(p^1)$ on the equilibrium path, and since in both cases the period 2 price is the ratio of consumer bids to consumer offers as a result of (4.7), it follows that $p^2(\theta, \sigma; s^1) = p^2(\theta^e(p^1))$ holds. From (4.2), we have

$$\frac{\partial u_h(x_h(\theta, \sigma; s^1), \theta^e(p^1))}{\partial x_h} = p^2(\theta, \sigma; s^1). \quad (4.9)$$

Combining (4.8) and (4.9), we see that $p^2(\theta, \sigma; s^1)$ is a market-clearing price for the competitive economy when consumers believe the state is $\theta^e(p^1)$, implying $p^2(\theta, \sigma; s^1) = f(\theta^e(p^1))$. ■

From Lemma 2, in any fully revealing type-symmetric WSE, the equilibrium price in period 2, following any period 1 action profile yielding the price p^1 , is $f(\theta^e(p^1))$. Sequential rationality requires bull i to choose b_i^1 to maximize the objective function given by

$$\pi_i(b_i^1, \theta) = -b_i^1 + \left(\omega + \frac{b_i^1}{p^1(b_i^1, \theta)}\right) f(\theta^e(p^1(b_i^1, \theta))).$$

Differentiating with respect to b_i^1 , we have the first order condition,

$$0 = -1 + \left(\omega + \frac{b_i^1}{p^1(b_i^1, \theta)}\right) f'(\theta^e(p^1(b_i^1, \theta))) \frac{\partial \theta^e(p^1(b_i^1, \theta))}{\partial p^1} \frac{\partial p^1(b_i^1, \theta)}{\partial b_i^1} \quad (4.10)$$

$$+ f(\theta^e(p^1(b_i^1, \theta))) \frac{\partial}{\partial b_i^1} \left(\frac{b_i^1}{p^1(b_i^1, \theta)}\right).$$

Evaluating at $b_i^1 = b^1(\theta)$ and $\theta^e(p^1(b_i^1, \theta)) = \theta$, (4.10) becomes

$$0 = -1 + (\omega + q^1(\theta)) f'(\theta) \frac{\partial \theta^e(p^1(\theta))}{\partial p^1} \frac{1}{nq^1(\theta)}$$

$$+ f(\theta) \frac{q^1(\theta)(n-1)}{nb^1(\theta)},$$

which simplifies to

$$f'(\theta) \frac{\partial \theta^e(p^1(\theta))}{\partial p^1} \left(\frac{\omega}{nq^1(\theta)} + \frac{1}{n}\right) = 1 - \frac{f(\theta)(n-1)}{np^1(\theta)}. \quad (4.11)$$

Similarly, the objective function for bear j is given by

$$\pi_j(q_j^1, \theta) = q_j^1 p^1(q_j^1, \theta) - (q_j^1 + \omega) f(\theta^e(p^1(q_j^1, \theta))).$$

The first order condition is

$$0 = p^1(q_j^1, \theta) + q_j^1 \frac{\partial p^1(q_j^1, \theta)}{\partial q_j^1} - f(\theta^e(p^1(q_j^1, \theta))) \quad (4.12)$$

$$- (q_j^1 + \omega) f'(\theta^e(p^1(q_j^1, \theta))) \frac{\partial \theta^e(p^1(q_j^1, \theta))}{\partial p^1} \frac{\partial p^1(q_j^1, \theta)}{\partial q_j^1}.$$

Evaluating at $q_j^1 = q^1(\theta)$ and $\theta^e(p^1(q_j^1, \theta)) = \theta$, (4.12) becomes

$$0 = \frac{b^1(\theta)}{q^1(\theta)} + \left[-\frac{b^1(\theta)}{nq^1(\theta)}\right] - f(\theta) - (q^1(\theta) + \omega) f'(\theta) \frac{\partial \theta^e(p^1(q_j^1, \theta))}{\partial p^1} \left[-\frac{b^1(\theta)}{n(q^1(\theta))^2}\right],$$

which simplifies to

$$0 = \frac{(n-1)b^1(\theta)}{nq^1(\theta)} - f(\theta) + (q^1(\theta) + \omega) f'(\theta) \frac{\partial \theta^e(p^1(q_j^1, \theta))}{\partial p^1} \left[\frac{b^1(\theta)}{n(q^1(\theta))^2}\right]. \quad (4.13)$$

Multiplying (4.13) by $q^1(\theta)/b^1(\theta)$ and simplifying, we have

$$0 = \frac{(n-1)}{n} - \frac{q^1(\theta)f(\theta)}{b^1(\theta)} + f'(\theta) \frac{\partial \theta^e(p^1(\theta))}{\partial p^1} \left(\frac{\omega}{nq^1(\theta)} + \frac{1}{n} \right). \quad (4.14)$$

Since the last term in (4.14) is the left side of (4.11), and both conditions are necessary, we have the necessary condition

$$0 = \frac{(n-1)}{n} - \frac{q^1(\theta)f(\theta)}{b^1(\theta)} + 1 - \frac{f(\theta)(n-1)}{np^1(\theta)},$$

which simplifies to

$$f(\theta) = p^1(\theta). \quad (4.15)$$

Since (4.15) holds for all θ , we have

$$f'(\theta) \frac{\partial \theta^e(p^1(\theta))}{\partial p^1} = \frac{f'(\theta)}{(p^1)'(\theta)} = 1,$$

which, when substituted into (4.11), yields after simplifying,

$$\frac{\omega}{nq^1(\theta)} = 0.$$

Whenever $\omega > 0$ holds, we have a contradiction to the supposition that there can be a WSE of this form.

The argument given above also allows us to conclude that there cannot be an open-market, type-symmetric WSE in which the bears are choosing the bid function $b^1(\theta)$ and bulls are choosing the offer function $q^1(\theta)$ in period 1. Simply change ω to $-\omega$ in the expressions for $\pi_i(b_i^1, \theta)$ (which now applies to bears) and $\pi_j(q_j^1, \theta)$ (which now applies to bulls), and in subsequent calculations. We have proved the following result.

Proposition 3. *In the model without short-sale restrictions, there does not exist an open-market, fully revealing, type-symmetric WSE.*

Remark 2. Throughout the analysis, we have maintained the assumption that bulls and bears have a large enough endowment of good y to make all desired purchases. In the context of Proposition 3, we should think of the model without short-sale restrictions as also imposing no restrictions regarding bidding above

one's endowment of good y . The model without short-sale restrictions but with the restriction that bids cannot exceed holdings of good y has not yet been solved, but a reasonable conjecture would be that there is a fully revealing WSE in which bulls bid up to their limit in period 1, and bears submit offers such that the price in period 1 is below $f(\theta)$.

5. Concluding Remarks

For the model of Section 3, a bull is indifferent as to how much to bid in period 1. If information acquisition is costly, one must complicate the model in order to generate an incentive to gather information about θ . If bulls and bears have initial holdings of good x that depend on θ , these players will have to monitor the state in order to determine their optimal action in period 2. Another possibility is to change the information structure so that privately informed players are not informationally small (see McLean and Postlewaite (2002)), perhaps with the introduction of risk aversion.

If consumer utility functions were not quasi-linear, then activity in period 1 would generate income effects, and the price in period 2 would no longer equal the REE price for the original economy, $f(\theta)$. However, I would conjecture that, under suitable regularity conditions, (i) the result about the unconstrained manipulators being at a disadvantage persists, and (ii) income effects become small as the economy is replicated, so that the equilibrium converges to the REE.

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