Misbehavior in Common-Value Auctions*

Dan Levin†
The Ohio State University

James Peck‡
The Ohio State University

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Abstract

We study misbehavior by bidding rings or an auctioneer in the ascending English and the sealed-bid Sophi auctions with common values. These auctions are outcome equivalent in equilibrium, without misbehavior. We characterize the optimal misbehavior of the ring or the auctioneer and then compare/rank those formats. The ring is stable, since ring members have an incentive to join and reveal their signals to each other. Under a separability assumption, the ring can do no better in the dynamic English format than in the static Sophi format, and behavior does not change if ordinary bidders are informed about the ring's existence.

1. Introduction

Misbehavior in auctions by the selling side (auctioneer), or by a subset of bidders (rings), have long been known to exist. Cassady (1967) documents the auctioneer's practice of spiking the price by the use of phantom bids, or "bids from

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†Department of Economics, The Ohio State University, 1945 North High Street, Columbus, OH 43210, e-mail: levin.32@osu.edu.

‡Department of Economics, The Ohio State University, 1945 North High Street, Columbus, OH 43210, email: peck.33@osu.edu.

the chandelier," and by cooperating with a confederate, sometimes called "shill bidding" or "trotting." He also documents misbehavior by a group of bidders or "ring," who cooperate (collude) to acquire the item at a lower price. Examples of auctions with bidder rings, sometimes small rings of two or three bidders, include auctions for antiques, fish, timber, and wool.[1]

Most of the theoretical literature on misbehavior in auctions was done within the independent private-values model. In this paper, we study misbehavior in the common-value environment, where bidders have affiliated private signals or estimates of the value at the time bidding takes place, but where the ex post value of the object is the same for all. There is a common-value component of many auctions, such as mineral rights or oil and gas leases (see Wilson (1979), Milgrom (1981), or Hendricks et al (2008)). The common-value environment alters the optimal (mis)behavior of the auctioneer/shill team, since the team can manipulate the beliefs of bidders as well as the auction price. The common-value environment affects the optimal misbehavior of a bidding ring, for similar reasons. In section 2, we discuss how our approach differs from the small literature on misbehavior in common value auctions. In our analysis, the ring consists of a subset of bidders (assumed for simplicity to be just two) who share the payoff when they win the item. The characterization of optimal misbehavior by the ring in the original auction is of central importance, because we demonstrate that the same characterization obtains when potential ring members can choose whether or not to form a ring after observing their private signals. We are thinking about an environment in which most individuals are not cheaters, so that a given auction is unlikely to have a bidding ring or a shill bidder. Therefore, ordinary bidders act as the standard theory specifies. However, misbehavior does occur, and we are interested in understanding the form such misbehavior takes.[2] Furthermore, we show that, under a separability assumption, equilibrium outcomes do not change when ordinary bidders are aware of the ring.

[1] According to Cassady (1967), "The term 'ring' apparently derives from the fact that in a settlement sale following the auction, members of the collusive arrangement form a circle or ring to facilitate observation of their trading behavior by the ring leader."

[2] To simplify the analysis, we assume that the misbehavior comes as a surprise to the ordinary bidders. Under common knowledge of a small probability that the auctioneer had the technology to collude with a shill, or that two bidders could collude, the outcome would not be much different. We would not dismiss applied theory that ignored the fact that collusion is possible, if unlikely, in order to study the behavior of ordinary bidders who do not suspect collusion. For the same reason, we should not dismiss an analysis of collusion, about which ordinary bidders are unaware, if the collusion is unlikely.

A second contribution of this paper is that, in addition to the English auction, we present a static mechanism, called the Sophi auction, which is outcome equivalent to the English auction in equilibrium, assuming that there is no misbehavior (see Levin and Reiss (2018)). The Sophi auction has some desirable behavioral properties. It internalizes, in its pricing rule, the correction for the winner's curse; in equilibrium, bidders simply bid or report their estimate of the value. The Sophi auction is not currently used in practice, but maybe it ought to be used. An Average Bid Mechanism is used in practice as an attempt to reduce the winner's curse and eliminate defaults, but there is no symmetric and strictly monotone equilibrium (see Chang et al (2014)). For this reason, Sophi is superior to the Average Bid Mechanism. The Sophi auction eliminates the bidders' need to update dynamically their beliefs after every dropout, based on knowledge of the joint distribution of signals, as would be required in the (outcome equivalent) English auction. Furthermore, under the assumption that the common value is the average of all the signals, which we call Assumption Z, the pricing rule is intuitive, simple, and is immune to the Wilson critique.

Here are our main results for the optimal misbehavior by the auctioneer. For the English auction, the auctioneer/shill delays exiting the auction until only one other bidder remains. If the auctioneer's signal exceeds, or is near enough to, the inferred second order statistic of the ordinary bidders, then the auctioneer remains in the auction to further push up the price, taking the risk of retaining the item at an expected loss. The Sophi auction is no longer outcome equivalent to the English auction with misbehavior. The English is strictly better for the auctioneer than Sophi.

Here are our main results for the optimal misbehavior by the bidding ring. For the English auction, one of the ring members will exit at the first possible opportunity. Under a regularity condition, we characterize the dropout point of the other ring member, based on an indifference condition. For signal realizations below a threshold, there is an interior solution for how long to remain in the auction. For signal realizations above the threshold, the ring remains in the auction and guarantees securing the object. The English auction yields the ring a weakly higher payoff than the Sophi auction, and a strictly higher payoff whenever the optimal misbehavior in the English auction must use the information revealed by dropouts. Under a separability assumption (Assumption S, much weaker than Assumption Z), it turns out that the English auction and the Sophi auction yield the same outcomes. Amazingly, under Assumption S, the ordinary bidders are choosing the same strategies that they would be employing with full knowledge of the existence of the ring and its equilibrium strategy!

For most of our analysis, we assume that the ring forms and agrees an equal share of the profit, before bidders observe their signals. Then, in Section 5, we consider a game in which potential ring members cannot commit to participating in the ring until after they observe their signals. We show that the ring is stable, since (i) every signal type would choose to join the ring in equilibrium, (ii) ring members have an incentive to truthfully reveal their signals to each other, and (iii) ring members have an incentive to bid as directed in the auction. We describe how to extend the analysis straightforwardly to potential rings with more than two members.

There is another value to our investigation. For lab experiments of private value and common value auctions, there is robust evidence showing that dynamic auction formats perform much better (i.e., closer to Nash equilibrium) than in their strategic equivalent static cousins.[3] Our analysis opens an intriguing new element, that the dynamic versions allow for more profitable misbehavior, at least with regard to the English auction vs. the Sophi auction, so that the static version might be better for ordinary bidders. Thus, perhaps the static version *should* be used, but the auctioneer (whether misbehaving or not) might prefer the dynamic version.[4]

1.1. Literature Review

There is a substantial literature on bidding rings in models with independent private values. Graham and Marshall (1987) consider a second-price auction with a reserve price and an English auction with a reserve price, where a "ring center" recruits all of the bidders to join the ring and designs a pre-auction mechanism with a fixed payment to incentivize truth telling. Behavior in the auction itself is trivial; since the ring member with the highest value bids his value, other ring members have no incentive to compete. See also Mailath and Zemsky (1991) for the case of heterogeneous bidders. McAfee and McMillan (1992) consider a first-price auction, mostly with an all-inclusive cartel. For the usual case of a declining hazard rate, the optimal tradeoff, between efficiency and a low auction

[3] More recently, Li (2017) formalizes the concept of "obviously dominant strategy," which holds for the dynamic IPV English auction, but not for its static cousin, the IPV second price auction.

[4] For the case of the ring, this conclusion rests on an alternative equilibrium selection, in which the ring remains in the auction until there is only one other bidder remaining. See Remark 1.

price, is to have each bidder (whose value exceeds the reserve price) make a bid equal to the reserve price. If side payments are allowed, there is an efficient mechanism with payments from the highest valuation to the others in order to achieve incentive compatibility. The actual auction price is the reserve price. The bidders are assumed to be able to commit to their behavior in the auction, perhaps due to threat of harm or supergame considerations. Marshall and Marx (2007) study both first-price and second-price auctions with heterogeneous bidders and a ring that is not all-inclusive. Bidders inside and outside the ring observe the mechanism (which is important for the first-price format). In equilibrium, the ring is able to suppress ring competition with the second-price format. With the first-price format, ring competition can be suppressed if the mechanism has the ability to submit all bids, but if the mechanism can only recommend bids, more than one bidder is instructed to bid, which is inefficient.

Hendricks et al (2008) model an all-inclusive ring in a first-price auction model with common values. It is assumed that the bidders can commit to abide by the mechanism, including any required side payments. In general, a mechanism that achieves ex post efficiency and budget balance may not be possible, because a bidder with a very high signal may be better off going alone rather than making the required payments to the other bidders. With independent signals, however, efficiency and budget balance can be achieved. Of course, the issue is whether bidders can be incentivized to join the ring, since the object is always acquired at the reserve price in the actual auction.

Our starting point is that the mechanism designer employed by the ring may face constaints other than participation, incentive compatibility, and budget balance. It is costly to approach all the other bidders with a proposal to form a ring when there are potential criminal penalties. Due to the complication of properly modeling the costs of being discovered, and the likelihood of being discovered (which might depend on the mechanism), we take a different approach that focuses on the auction itself. We assume that the set of potential ring members is exogenous and not all-inclusive (for simplicity, containing 2 bidders). The literature typically assumes that ring members can commit to their behavior in the auction. However, this assumption is just for convenience, since we show that the ring is stable and that the analysis does not change significantly with larger rings. The literature assumes that non-ring members either do not exist, exist but know the makeup of the ring, or employ a dominant strategy in the auction. We assume that the non-ring members believe that there is no ring, thereby capturing the idea that the ring can manipulate the beliefs of the non-ring members. Cassady

(1967, p. 189) claims that “collusive buying in auctions is not very common,” so the assumption that the non-ring members do not suspect a ring can be taken as an accurate approximation for many auctions. However, it is worth repeating that, under Assumption S, ordinary bidders would not change their bids, even if they knew of the existence of the ring!

There is a small literature on shill bidding in common value auctions. Vincent (1995) provides an explanation of why an auctioneer, in a second-price auction with interdependent values, might want to keep the reserve price secret, where the bidders know that there is a secret reserve price. Although keeping the reserve price might seem to violate the linkage principle of Milgrom and Weber (1982), it does not, because keeping the reserve price secret will induce more participation in the auction. Lamy (2009) studies first-price and second-price auctions, with interdependent values, in which the auctioneer does not receive a signal and cannot commit not to use a shill. Equilibrium in the first-price auction is immune to shill bidding, but equilibrium in the second-price auction is very complicated (even with no auctioneer signal), with the auctioneer employing a mixed strategy. The auctioneer would be strictly better off with the ability to commit not to use a shill bid. Our model considers a case nearly opposite to that of Lamy (2009), where the probability that the auctioneer can use a shill is small enough that the ordinary bidders believe that there is no shill. As a result, the auctioneer is able to manipulate the bidders’ beliefs and generate “excitement” by having the shill remain in the auction. Also, we study an English auction format (and the related Sophi format), and we model the auctioneer/shill as having a signal. Chakraborty and Kosmopoulou (2004) consider a model in which it is common knowledge that the auctioneer can employ a shill with a certain probability. The reason that they are able to solve an otherwise intractable model is that bidders receive one of only two possible signals.

Akbarpour and Li (2019) develop the notion of a credible mechanism, in which the mechanism designer or auctioneer has no beneficial “safe” deviations from following the mechanism’s rules. According to this notion, the Sophi auction (with no shill bidder) is credible if the reports are simultaneously and publicly announced by the bidders. However, if reports are privately communicated, the auctioneer can lie about the reports other than the highest report, thereby raising the price with no risk of the deviation being detected. We consider a different sort of auctioneer misbehavior. In our setting, the auctioneer commits to follow the auction rules, in the sense that the mapping from bidder actions to outcomes is carried out honestly. The misbehavior consists of conspiring with one of the bidders.

2. The Common-Value Model

Consider a general common-value (CV) auction framework with n bidders, where the common value, V, and the signals, $X = (X_1, ..., X_n)$ are random variables drawn from a joint distribution function, $F(V, X)$, with support $[\underline{v}, \overline{v}] \times [\underline{x}, \overline{x}]^n$. We assume that bidders are risk neutral and that F is symmetric across bidders. The interpretation is that each bidder i privately observes a signal (estimate), $X_i = x_i$. Denote the random variable for the j-th order statistic of the signals by Y^j, and the realization as y^j. From $F(V, X)$, we can derive various marginal and conditional distributions, which we indicate by a subscript. For example, we denote the probability that the order statistics, $(Y^1, Y^2, ..., Y^n)$, are less than or equal to $y = (y^1, y^2, y^3, ..., y^n)$ by $F_{Y^1,Y^2,...,Y^n}(y^1, y^2, y^3, ..., y^n)$. All joint signal vectors are assumed to occur with positive density. We assume that there exists a finite expected value of V, conditional on $x = (x_1, ..., x_n)$, denoted by the continuously differentiable function, $\mu(x) \equiv E[V|x]$. The function, μ, determines a continuously differentiable function, which we denote by $h(y) \equiv E[V|y]$, which is assumed to satisfy $\partial h(y)/\partial y^j \geq 0$ for all j and $\partial h(y)/\partial y^1 > 0$.[5]

The following, Assumption S (for separability), greatly simplifies the analysis and yields sharp and surprising results. We also use a special case of Assumption S, which we call Assumption Z, mainly for illustrative purposes.

Assumption S (Separability): $\mu(x)$ is separable if for all $i \neq j$, for all $x^a_{-(i,j)}$ and $x^b_{-(i,j)}$, and for all (x_i, x_j) and $(x_i, x_j)'$, we have

$$\mu((x_i, x_j), x^a_{-(i,j)}) \gtreqless \mu((x_i, x_j)', x^a_{-(i,j)})$$

implies

$$\mu((x_i, x_j), x^b_{-(i,j)}) \gtreqless \mu((x_i, x_j)', x^b_{-(i,j)}).$$

Assumption Z: $\mu(x) = \frac{1}{n}(\sum_{j=1}^{n} x_j)$.

[5] This assumption is weaker than the commonly used assumption that the $(n+1)$ random variables are positively affiliated. [With the extra assumption that $\partial h(y)/\partial y^1$ is strictly positive.]

In words, separability means that, if two vectors of signals differ only in the i and j components, and the vector with (x_i, x_j) yields a higher (respectively, lower) expectation of V than the vector with $(x_i, x_j)'$ does, then this relationship will continue to hold if we change the non-(i, j) components of the vectors. Although Assumption S is restrictive, it holds for a wide class of environments, including

$$V = \sum_{j=1}^{n} \phi(x_j) + \varepsilon,$$

where ϕ is an arbitrary increasing function and ε is white noise.

The English auction: The English auction is an indirect mechanism, in which there is a clock that rises continuously, starting at the expected value associated with the worst possible realization of signals, $h(\underline{x}, ..., \underline{x})$. Bidders decide whether to remain in the auction or drop out. As soon as only one bidder remains, the clock stops and that bidder wins the auction, and the payment is the clock price at that point. All bidders observe the history of dropouts and associated clock prices. See Milgrom and Weber (1982).

The Sophi (sophisticated) Auction: Sophi is a direct mechanism where each bidder i reports a signal $\widetilde{x}_i$, and the bidder who reports the highest signal (say bidder i) wins and pays, $P^S(\widetilde{x}) = E(V|Y^1 = \widetilde{y}^2, Y^2 = \widetilde{y}^2, Y^3 = \widetilde{y}^3, ..., Y^n = \widetilde{y}^n)$. That is, the payment is the expectation of V, conditional on the signals of the other bidders equaling $\widetilde{x}_{-i}$, and the signal of bidder i equaling the highest report of the other bidders. All other bidders pay nothing. In equilibrium, we have $P^S(\widetilde{x}) = E(V|y^2, y^2, y^3, ..., y^n)$. Levin and Reiss (2018) proved that, if a direct mechanism satisfies the properties that (1) the bidder with the highest signal receives the object, (2) a bidder who does not receive the object pays nothing, and (3) truthful reporting is an ex-post (no regret) equilibrium, then the bidder who wins the object pays according to the Sophi rule.

In equilibrium, *Sophi* "asks" bidders to simply bid their signals, because the price rule corrects for the adverse selection. In practice, this may help less sophisticated bidders, who typically ignore the adverse selection and overbid, to avoid or mitigate the winner's curse (WC).

Comparisons between English and Sophi

The simple equilibrium bidding rule for the Sophi auction sharply contrasts with the complicated equilibrium strategies in CV auctions such as in first-price and English auctions, where bidders do not bid, or drop at, their signal. In the symmetric equilibrium of a first-price auction, each bidder ought to bid as if holding the highest signal, and then use a complicated Bayesian calculation to correct her estimation of the CV and also decide on the proper (optimal) shading. In English auctions, each bidder typically (with the lowest signal holder as a possible exception) drops at a lower clock price than her own signal, while remaining bidders continuously need to update after each drop-out.[6] Levin and Reiss (2018) show that, in both the Sophi and English auctions, the highest signal holder wins the object and pays $E(V|y^2, y^2, y^3, ..., y^n)$. Thus, the Sophi auction and the English auction are allocation and price equivalent.

Suppose that Assumption Z is known to hold. Assumption Z amounts to assuming that the average of the n private signals is a sufficient statistic for $E[V]$, but it assumes nothing about the process generating signals. No informational requirements are placed on the auctioneer in the English auction, since the English auction is immune to the Wilson critique. However, heavy informational requirements are placed on bidders. Even under Assumption Z, a bidder in the English auction must know the joint distribution, $F(V, X)$, and perform a complicated Bayes' rule computation to infer the other bidders' types after each dropout. Now consider the Sophi auction. The equilibrium bidding rule under Sophi is as simple as can be, to bid one's signal, so no informational requirements are placed on the bidders. Under Assumption Z, the payment rule, when bidder 1 makes the largest bid and bidder 2 makes the second largest bid, is:

$$P^S(\widetilde{x}) = \frac{1}{n}(\widetilde{x}_2 + \widetilde{x}_2 + \widetilde{x}_3 + ... + \widetilde{x}_n) = \frac{1}{n}[(\sum_{i=1}^{n} \widetilde{x}_i) - (\widetilde{x}_1 - \widetilde{x}_2)]. \qquad (2.1)$$

This payment rule is simple, and imposes no informational requirements on the auctioneer. In particular, the auctioneer does not need to know anything about the joint distribution, $F(V, X)$, in order to implement this mechanism, so Sophi is immune to the Wilson critique if Assumption Z is known to hold, imposing no informational requirements on the auctioneer or the bidders. If Assumption Z

[6] Sophi has an additional normative appeal. Instructing bidders to "just bid your estimate" does not only sound simple, but it is also optimal (in equilibrium). It is much harder to see what bidding advice one could instruct bidders in other commonly used auctions.

does not hold, there is a tradeoff between the informational requirements placed on the bidders in the English auction and the informational requirements placed on the auctioneer in the Sophi auction. In actual auction environments, especially when Assumption Z is a reasonable approximation, a mechanism using pricing rule (2.1) "à la Sophi" might yield outcomes closer to the equilibrium predictions of the English auction than the English auction mechanism itself.

3. Misbehavior by the Auctioneer.

Here we consider a model with n ordinary bidders and one shill bidder, who observes a signal, but is secretly in league with the auctioneer. We assume that the ordinary bidders treat the auctions as having $n+1$ ordinary bidders, which can be considered a limiting case of a small probability that one of the bidders is a shill. Let us use the previous notations for the bidders' signals and let the signal of the auctioneer/shill be given by the random variable $S \equiv X_{n+1}$, with realization s. As a slight change in notation for the shill model, for $j = 1, ..., n$, we denote the j-th order statistic of the *ordinary* bidders as Y^j, with realization y^j.[7]

3.1. Optimal misbehavior by the auctioneer in the English auction

The analysis begins with the following observations.

Observation 1: Any bidder who holds signal y^i and remains active, in equilibrium infers the vector of signals of the ordinary bidders who have dropped out, $(y^{i+1}, ..., y^n)$, from the drop-out prices.

Observation 2: Since the auctioneer can stay active in the auction and drop out at the same clock-price as the holder of the second highest signal, y^2, she would never drop earlier. Dropping at a lower price is dominated by dropping at the same price as the holder of y^2, since it raises the payment without any risk of not selling. Thus, after the holder of the second highest signal, y^2, drops, both the auctioneer and the holder of y^1 learn the vector $(y^2, y^3, ...y^n)$.

Observation 3: Since the auctioneer never drops before the holder of y^2, that bidder does not observe the signals, s or y^1.

[7] Here the joint distribution of the value and signals is given by $F(V, X) = F(V, X_1, ..., X_{n+1})$.

Observation 4: The holder of y^2 drops at $d_2 = E[V|(Y^1 = y^2, S = y^2, Y^3 = y^3, ...Y^n = y^n)$. (see Milgrom and Weber (1982)).

Suppressing the dependence on $(y^2, y^3, ..., y^n)$, define $V(y^1)$ by

$$V(y^1) = E[V|Y^1 = y^1, S = y^1, Y^2 = y^2, Y^3 = y^3, ..., Y^n = y^n]. \tag{3.1}$$

Since both the auctioneer and the holder of y^1 can infer the vector $(y^2, y^3, ...y^n)$, and since the holder of y^1 does not know that the holder of s is a shill bidder, $V(y^1)$ is the maximum willingness to pay by the holder of y^1 at the time when the holder of y^2 drops.[8] Note that the holder of y^1 treats the shill as an ordinary bidder, so (3.1) is simply the expectation of V, conditional on his own signal being y^1, the signals of the drop-out bidders being $(y^2, y^3, ...y^n)$, and the signal of the remaining bidder (who happens to be the shill) being y^1.

Let $\widehat{V}(s^m, s)$ be defined by

$$\widehat{V}(s^m, s) = E[V|Y^1 = s^m, S = s, Y^2 = y^2, Y^3 = y^3, ..., Y^n = y^n],$$

where $s^m \equiv \max\{s, y^2\}$.[9] We can interpret $\widehat{V}(s^m, s)$ as the maximum willingness to pay of a **regular** bidder who holds signal s, and "happens" to still be active at the price d_2, where the holder of y^2 drops. Note that $s \lesseqgtr y^2$ holds as $\widehat{V}(s^m, s) \lesseqgtr d_2$ holds.

We stated in Observation 2 that the auctioneer never drops before d_2, but would she ever drop strictly after y^2, (at a strictly higher price than d_2), **even** when $s \leq y^2$ holds? Consider an auctioneer who contemplates mimicking the dropping price of a regular bidder who is type $t > y^2$. Then, given the behavior of the bidder holding y^1, the auctioneer wins the auction and retains the good if $y^1 < t$ holds, and the auctioneer succeeds in selling the good at a higher price if $y^1 > t$ holds. In the latter case, her revenue is equal to the dropping price of a regular bidder who is type t, given by

$$d_t = E[V|Y^1 = t, S = t, Y^2 = y^2, Y^3 = y^3, ..., Y^n = y^n].$$

[8] This is well known from the literature. See Matthews (1977) and Milgrom and Weber (1982).

[9] This notation is used to cover the possibility that $s < y^2$ holds, although the auctioneer never drops before d_2.

Suppressing the dependence on $(s, y^2, y^3, ...y^n)$, define $G(t)$, and associated density $g(t)$, by

$$G(t) = F_{Y^1|S,Y^2...Y^n}(t) = \Pr[Y^1 \leq t \mid S = s, Y^2 = y^2, Y^3 = y^3, ..., Y^n = y^n],$$

which is the probability that the first order statistic is not higher than t, given the realizations of $(S = s, Y^2 = y^2, Y^3 = y^3, ..., Y^n = y^n)$.

We can now write the maximization problem of the auctioneer as

$$\max_{t \geq s^m} \int_{y^2}^{t} \widehat{V}(\tau, s) dG(\tau) + [1 - G(t)]V(t). \tag{3.2}$$

In (3.2), the first term represents the event in which the auctioneer wins the item (by mimicking type t) and retains its value, which is $\widehat{V}(Y^1, s)$. In the second term, $[1 - G(t)]$ is the probability that $Y^1 \geq t$ holds, resulting in the auctioneer losing the item and receiving the price, $V(t)$. Recall, $V(t)$ is the price that the auctioneer receives by mimicking an ordinary bidder of type t, which is not her own value at that point. Also note, the auctioneer is mimicking a type, $t \geq s^m$, while the integral runs from y^2 to t. The reason is that, if $s > y^2$ holds, the auctioneer will never mimic a type less than s, while y^1 can be anything above y^2.

Let t^* denote the optimal type to mimic, solving (3.2). The first order condition for an interior solution is

$$0 = \widehat{V}(t, s) - V(t) + [\frac{1 - G(t)}{g(t)}]\frac{\partial V(t)}{\partial t}. \tag{3.3}$$

The following proposition shows that an interior solution occurs if s is greater than, or even smaller than, but close enough to, y^2. In that case, the auctioneer chooses $t^* > s^m$, and risks retaining the item at an expected loss. On the other hand, if s is very low relative to y^2, the auctioneer might drop immediately after the bidder with y^2 drops.

Proposition 1: *For all values of s satisfying $s \geq y^2$, and for some values of s strictly less than y^2, the optimal auctioneer misbehavior in the English auction solves (3.3), and $t^* > s^m$ holds, so the auctioneer risks retaining the item at an expected loss.*

Proof. The right side of (3.3) is the derivative of the auctioneer's payoff with respect to t. Since $[\frac{1-G(y^2)}{g(y^2)}]\frac{\partial V(y^2)}{\partial t}$ is strictly positive, and $\widehat{V}(t, s) \geq V(t)$ holds for

$t \leq s$, it follows that the right side of (3.3) is strictly positive for $t \leq s$, so we must have $t^* > s$. Therefore, $s \geq y^2$ implies $t^* > \max\{s, y^2\}$. By continuity, there is an $\varepsilon > 0$, such that for $s \geq y^2 - \varepsilon$, the right side of (3.3) is strictly positive for all $t \leq y^2$. ■

Under natural regularity conditions, the right side of (3.3) is strictly decreasing in t. This would be the case, for example, when $\widehat{V}(t,s) - V(t)$ is decreasing in t, $V(t)$ is concave, and we have the standard condition that $[\frac{1-G(y^2)}{g(y^2)}]$ is strictly decreasing in t. If so, then the optimal auctioneer misbehavior involves a cutoff s, which depends on y^2, below which the auctioneer drops immediately after the bidder with y^2 drops, and above which the auctioneer drops strictly after the bidder with y^2 drops.

Proposition 2: *Assume that $[\frac{1-G(t)}{g(t)}]$ is decreasing in t, and that Assumption Z holds. Then the optimal auctioneer misbehavior in the English auction is characterized as follows. The auctioneer drops immediately after the bidder with y^2 drops if we have*

$$s \leq y^2 - \frac{2(1-G(y^2))}{g(y^2)}. \tag{3.4}$$

Otherwise, the auctioneer drops strictly after the bidder with y^2 drops, and t^* is the unique solution to

$$s = t - \frac{2(1-G(t))}{g(t)}. \tag{3.5}$$

Proof. Define

$$K \equiv y^2 + y^3 + ... + y^n.$$

Then, under Assumption Z, $\widehat{V}(t,s)$ and $V(t)$ simplify to

$$\begin{aligned} \widehat{V}(t,s) &= \frac{t+s+K}{n+1} \quad \text{and} \\ V(t) &= \frac{2t+K}{n+1}. \end{aligned}$$

The right side of (3.3) then equals

$$\frac{1}{n+1}[s - t + \frac{2(1-G(t))}{g(t)}]. \tag{3.6}$$

Since $\frac{(1-G(t))}{g(t)}$ is decreasing in t, it follows that (3.6) is strictly decreasing. If (3.4) holds, then (3.6) is negative when evaluated at $t = y^2$, and therefore negative for $t > y^2$, which implies the auctioneer payoff is higher by dropping when the bidder with y^2 drops, rather than waiting. If (3.4) does not hold, then (3.6) is positive when evaluated at $t = y^2$. Also, (3.6) is negative when evaluated at $t = \overline{x}$, so there must be an interior solution to the auctioneer's problem, which is the unique solution to (3.5). ■

3.2. Optimal misbehavior by the auctioneer in the Sophi auction

Without misbehavior, the English and Sophi auctions lead to the same equilibrium outcomes for any realization of signals. However, the auctioneer's ability to manipulate the auction secretly, with a shill bidder, differs across the two formats. We show in this subsection that the auctioneer always receives a higher expected payoff, conditional on her signal, in the English auction than in the Sophi auction, as long as her signal is less than $\overline{x}$.

To characterize the optimal auctioneer misbehavior in the Sophi auction, first notice that for *given* $(y^2, y^3, ...y^n)$, the auctioneer's payoff as a function of the reported type t is given by

$$\int_{y^2}^{t} \widehat{V}(\tau, s)dG(\tau) + [1 - G(t)]V(t),$$

which follows from the rules for the Sophi payments and the truthful reporting of the ordinary bidders. The overall payoff of an auctioneer, with signal s but reporting t, is given by

$$\int_{y^2,y^3,...,y^n} \left[\int_{y^2}^{t} \widehat{V}(\tau, s)dG(\tau) + [1 - G(t)]V(t) \right] dF_{Y^2,...,Y^n|S}(y^2, y^3, ..., y^n|s). \quad (3.7)$$

The optimal auctioneer misbehavior, given signal s, is to report the value of $t > s$ that maximizes (3.7).

3.3. Comparisons between English and Sophi with a misbehaving auctioneer

Proposition 3: *For all $s < \overline{x}$, a misbehaving auctioneer strictly prefers the English auction over the Sophi auction.*

Proof. Comparing (3.2) and (3.7), it is clear that the auctioneer in the English auction can achieve the Sophi payoff by setting t^*, for each realization of $(y^2, y^3, ..., y^n)$, equal to the auctioneer's Sophi report, which we denote as t^{*Sophi}. To show that the English is strictly preferred, it suffices to show that there are some realizations of $(y^2, y^3, ..., y^n)$, occurring with positive probability, such that t^{*Sophi} is not a solution to (3.2).

First we show that $t^{*Sophi} < \overline{x}$ holds. Differentiating (3.7) with respect to t, and evaluating at $t = \overline{x}$, yields

$$\int_{y^2,y^3,...,y^n} \left[\widehat{V}(\overline{x}, s) - V(\overline{x})\right] dF_{Y^2,...,Y^n|S}(y^2, y^3, ..., y^n|s). \tag{3.8}$$

Since, for $s < \overline{x}$, the term in brackets in (3.8) is negative for each $(y^2, y^3, ..., y^n)$, the auctioneer strictly increases profits by reducing t below $\overline{x}$, so $t^{*Sophi} < \overline{x}$ holds.

Thus, there must be a positive probability that $y^2 > t^{*Sophi}$ holds. In all such cases, however, t^{*Sophi} is not a solution to (3.2), because $t^* \geq y^2$ must hold. Intuitively, the auctioneer should not drop before the bidder with y^2 drops. ■

The idea behind the proof of Proposition 3 is that there is always a chance that y^2 is very high. In the Sophi auction, the auctioneer does not observe y^2, and ends up reporting $t^{*Sophi} < y^2$. In the English auction, this would correspond to the auctioneer dropping before the bidder with y^2, which cannot be optimal. More generally, we would expect that t^* should depend non-trivially on $(y^2, y^3, ..., y^n)$, in which case the English auction yields a higher payoff.

4. Misbehavior by a Ring of Bidders

Here we consider a model with $n-1$ ordinary bidders and a "ring" of two bidders, each of whom observes a signal, but are secretly in league with each other. We do not model the legal or reputational risks of forming a ring, so this section assumes that the ring members have committed to share equally the surplus. In the next section, we show that the ring is stable, by solving the augmented game in which potential ring members, after observing their signal, decide whether or not to form a ring, and whether or not to truthfully reveal their signal. In the augmented game, no matter what private signals each potential member has, the ring forms and ring members behave as analyzed in this section. Extension to a larger ring is simple and does not add anything to the analysis. We assume that the ordinary bidders treat the auctions as having $n + 1$ ordinary bidders, which can be considered a limiting case of a small probability that there is a ring. However, we show in Proposition 7 that, under Assumption S, the ordinary bidders are choosing the same strategies that they would choose in the game in which they know that there is a ring. Let us use the previous notations for the ordinary bidders' signals and let the signals of the ring be given by the random variables $S_1 \equiv X_1$ and $S_2 \equiv X_2$ with realizations s_1 and s_2. The ring members are bidders 1 and 2, but as a reminder, we will refer to them as bidders R1 and R2 (where R stands for ring). For $j = 1, ..., n-1$, we denote the j-th order statistic of the *ordinary* bidders as Y^j, with realization y^j.

4.1. Optimal misbehavior by the ring in the English auction

The analysis begins with the following observations.

Observation 1: In order to keep the payment to the auctioneer for winning as low as possible, the ring will have one of its members, say bidder R2, drop immediately, so the ordinary bidders incorrectly infer $S_2 = \underline{x}$.

Observation 2: Any bidder who holds signal y^i and remains active, in equilibrium infers the vector $(y^{i+1}, ..., y^{n-1})$ from the drop-out prices.

Observation 3: If the remaining member of the ring, bidder R1, is still active when the holder of the second highest signal, y^2, drops then both bidder R1 and the holder of y^1 learn the vector $(y^2, y^3, ..., y^{n-1})$. Bidder R1 knows s_2, but the holder of y^1 incorrectly infers $S_2 = \underline{x}$. In this case, the holder of y^2 drops at the price $E[V|(Y^1 = y^2, S_1 = y^2, S_2 = \underline{x}, Y^3 = y^3, ...Y^{n-1} = y^{n-1})$. See Milgrom and Weber (1982).

Suppressing the dependence on $(y^2, y^3, ..., y^{n-1})$, define $V^R(y^1)$ by

$$V^R(y^1) = E[V|Y^1 = y^1, S_1 = y^1, S_2 = \underline{x}, Y^2 = y^2, ..., Y^{n-1} = y^{n-1}].$$

The superscript, R, is used to distinguish the notation for the model with a ring from the similar notation in the previous section (for the model with the shill bidder). $V^R(y^1)$ is the maximum willingness to pay by the holder of y^1 at the time when the holder of y^2 drops. This is simply the expectation of V, conditional on his own signal being y^1, the signals of the ordinary drop-out bidders being $(y^2, y^3, ...y^{n-1})$, the signal of bidder R2 being $\underline{x}$, and the signal of bidder R1 being y^1.

Let $\widehat{V}^R(s_1, s_2, y^1)$ be defined by

$$\widehat{V}^R(s_1, s_2, y^1) = E[V|Y^1 = y^1, S_1 = s_1, S_2 = s_2, Y^2 = y^2, ..., Y^{n-1} = y^{n-1}]. \quad (4.1)$$

We can interpret $\widehat{V}^R(s_1, s_2, y^1)$ as the expectation of V for the ring, conditional on $(s_1, s_2, y^1, y^2, ..., y^{n-1})$.

We can think of the ring's problem, after bidder R2 of the ring drops at $\underline{x}$, as bidder R1 "pretending" to be an ordinary bidder with signal t. That is, the remaining ring bidder drops out when the holder of y^1 is revealed to have signal $y^1 \geq t$. Notice that, once bidder R1 has determined that winning the object cannot be profitable, the dropping point does not matter, as long as it occurs before the holder of y^2 drops. The equilibrium we select assumes that bidder R1 drops at the first instant that continuing cannot be profitable. See Remark 1 for a discussion of alternative equilibrium selections. To be precise, suppose that the history of drops reveals $(y^j, ..., y^{n-1})$ and the current clock price reveals that remaining ordinary bidders have signals of at least t, $y^{j-1} \geq t$. Also suppose that we have, for all $(y^1, ..., y^{j-1})$, $\widehat{V}^R(s_1, s_2, y^1) \leq V^R(y^1)$. Then an active bidder R1 will drop at that point.

If bidder R1 has not yet dropped by the time the holder of y^2 drops, the ring's payoff, given $(y^2, ..., y^{n-1}, s_1, s_2)$, is equal to the following expression.

$$\int_{y^2}^{t} \left[\widehat{V}^R(s_1, s_2, y^1) - V^R(y^1)\right] dF_{Y^1|Y^2,...,Y^{n-1},S_1,S_2}(y^1|y^2, ..., y^{n-1}, s_1, s_2). \quad (4.2)$$

Thus, the optimal misbehavior, given $(y^2, ..., y^{n-1}, s_1, s_2)$, is found by choosing t^* to maximize (4.2) for t.[10] Some of our results depend on the following regularity condition.

Assumption Reg (Regularity): $\widehat{V}^R(s_1, s_2, y^1) - V^R(y^1)$ is strictly decreasing in y^1 for all (s_1, s_2, y^1).

Assumption Reg, although restrictive, is quite reasonable. Note that we condition on only one signal being y^1 in the expectation of V in $\widehat{V}^R(s_1, s_2, y^1)$, but we condition on two signals being y^1 in the expectation of V in $V^R(y^1)$.

Proposition 4: *Under Assumption Reg, the optimal misbehavior of the ring in the English auction, with signals (s_1, s_2), is characterized as follows. If we have*

$$\widehat{V}^R(s_1, s_2, y^2) - V^R(y^2) \leq 0, \quad (4.3)$$

[10] For some signal realizations, it is optimal for the ring member R1 to remain in the auction beyond the point at which an ordinary bidder with signal $\overline{x}$ would drop. In this case, R1 "pretends" to be type $t^* = \overline{x}$, but off the equilibrium path, R1 would remain in the auction until price $\widehat{V}^R(s_1, s_2, \overline{x})$, believing that the deviating bidder(s) is type $\overline{x}$.

then the remaining ring bidder (if still active) drops immediately when the holder of y^2 drops. If we have

$$\widehat{V}^R(s_1, s_2, \overline{x}) - V^R(\overline{x}) \geq 0, \tag{4.4}$$

then (on the equilibrium path) the remaining ring bidder remains in the auction until she wins. Otherwise, the remaining ring bidder drops at the unique t^* solving

$$\widehat{V}^R(s_1, s_2, t) - V^R(t) = 0 \tag{4.5}$$

for t. That is, she drops out at the clock price when the remaining bidder is revealed to have signal $y^1 \geq t^*$.

Proof. The left side of (4.5) is the expected payoff of the ring, conditional on winning the object at price $V^R(y^1)$. Since $\widehat{V}^R(s_1, s_2, y^1) - V^R(y^1)$ is strictly decreasing in y^1, the expected payoff is strictly decreasing in y^1. If (4.3) holds, then the payoff from winning is non-positive when the holder of y^2 drops out, and the payoff decreases as the clock price rises, so the optimal misbehavior, if still active at that point, is to drop immediately. If (4.4) holds, then the payoff from winning is non-negative, even if the ring outwaits a remaining bidder with signal $\overline{x}$. Since the payoff at a lower clock is always strictly positive, the optimal misbehavior is to remain in the auction until the clock price reaches, $\widehat{V}^R(s_1, s_2, \overline{x})$, which is weakly greater than $V^R(\overline{x})$, thereby winning the auction. Finally, if neither (4.3) nor (4.4) hold, then there must be a solution to (4.5), t^*, strictly between $V^R(y^2)$ and $V^R(\overline{x})$. For $y^1 < t^*$, winning the object is profitable, and for $y^1 > t^*$, winning the object is unprofitable, so the optimal misbehavior of the ring is to remain in the auction until the other bidder is revealed to be at least type t^*. ■

Corollary to Proposition 4: *Define $K^j \equiv y^j + y^{j+1} + ... + y^{n-1}$. Then under Assumption Z, the optimal misbehavior by the ring in the English auction is characterized by the remaining ring member pretending to be type $t^* = \min[s_1 + s_2 - \underline{x}, \overline{x}]$. When ordinary bidders with signals $y^j, y^{j+1}, ..., y^{n-1}$ have dropped, revealing their signals, the remaining ring member will be next to drop if the clock price reaches b^*, given by*

$$\begin{aligned} b^* &= \frac{j(s_1 + s_2 - \underline{x}) + \underline{x} + K^j}{n+1} \quad \text{if we have } s_1 + s_2 < \overline{x}, \\ b^* &= \frac{(j-1)\overline{x} + s_1 + s_2 + K^j}{n+1} \quad \text{if we have } \overline{x} \leq s_1 + s_2. \end{aligned}$$

In the first instance, the remaining ring member drops out when the other active bidders are revealed to be at least type $s_1 + s_2 - \underline{x}$, and in the second instance, the remaining ring member waits until everyone else drops (on the equilibrium path).

Proof of Corollary. Under Assumption Z, if all of the remaining active bidders have signal $t^* = s_1 + s_2 - \underline{x}$, the value to the ring is

$$\frac{(j-1)t^* + s_1 + s_2 + K^j}{n+1},$$

and the current price is

$$\frac{(j-1)t^* + t^* + \underline{x} + K^j}{n+1}.$$

Since these two expressions are equal, the ring's expected profits are zero at this point. Were the ring to wait longer, and win the object, the ring's profits are given by

$$\widehat{V}^R(s_1, s_2, y^1) - V^R(y^1) = \frac{s_1 + s_2 + K^1}{n+1} - \frac{y^1 + \underline{x} + K^1}{n+1} = \frac{s_1 + s_2 - \underline{x} - y^1}{n+1}. \quad (4.6)$$

Since $y^1 > t^*$ holds, the resulting profits from winning must be negative. Similarly, it is easy to see that the profits from winning are positive if the ring pretends to be type t^* and wins the auction, so pretending to be type t^* is optimal for any history.

When $s_1 + s_2 < \overline{x}$ holds, the clock price at which the remaining ring member drops is therefore

$$\frac{j(s_1 + s_2 - \underline{x}) + \underline{x} + K^j}{n+1}.$$

When $\overline{x} \le s_1 + s_2$ holds, then on the equilibrium path, it is profitable to win the object no matter what signals the ordinary bidders have. If the price rises above

$$\frac{j\overline{x} + K^j}{n+1}$$

with $j-1$ ordinary bidders remaining, which is off the equilibrium path, the ring is assumed to believe that each of the remaining ordinary bidders have the signal $\overline{x}$. It is optimal for the remaining ring member to remain in the auction until the price reaches

$$\frac{(j-1)\overline{x} + s_1 + s_2 + K^j}{n+1}.$$

■

4.2. Optimal misbehavior by the ring in the Sophi auction

To characterize the optimal ring misbehavior in the Sophi auction, first notice that one of the ring members (say, bidder R2) will report type $\underline{x}$, in order to keep the price as low as possible. We can write the ring's payoff, when bidder R1 reports type t, as

$$\int_{y^2,y^3,...,y^{n-1}} \left[\int_{y^2}^{t} \Delta(s_1, s_2, y^1) dF_{Y^1|Y^2,...,Y^{n-1},S_1,S_2}(y^1|y^2, ..., y^{n-1}, s_1, s_2) \right] \times dF_{Y^2,...,Y^{n-1}|S_1,S_2}(y^2, y^3, ..., y^{n-1}|s_1, s_2), \quad (4.7)$$

where $\Delta(s_1, s_2, y^1)$ is defined by

$$\Delta(s_1, s_2, y^1) = \widehat{V}^R(s_1, s_2, y^1) - V^R(y^1).$$

Notice that, when $t > y^1$ holds and the ring wins the auction, $\widehat{V}^R(s_1, s_2, y^1)$ is the expectation of V and $V^R(y^1)$ is the price the ring pays, according to the Sophi pricing rule and the ordinary bidders' truthful reporting.

The optimal misbehavior for the ring in the Sophi auction is to choose t to maximize (4.7).

4.3. Comparisons between English and Sophi with a ring

We show below, in Proposition 5, that the ring is weakly better off under the English auction than under the Sophi auction, and typically strictly better off. However, Proposition 6 shows that, when Assumption S holds, then the two auction formats yield the same outcomes for the ring. Proposition 6 also shows that, under Assumption S, the ring's strategy is without regret. That is, assuming ordinary bidders play their equilibrium strategies (bidding as if all $n+1$ bidders are ordinary bidders), the ring has no incentive to change its strategy, even if they could be informed of the signals of all the ordinary bidders. We conclude this subsection with a remark about comparisons between English and Sophi, when the ring chooses one of its other best responses under the English auction.

Proposition 5: *The optimal misbehavior of the ring in the English auction, with signals (s_1, s_2), yields the ring a payoff strictly higher than the optimal misbehavior in the Sophi auction, except when t^* depends on $(y^2, ..., y^{n-1})$ in a trivial way* *[where trivial means that the optimal misbehavior can be achieved with a constant t^*].*

Proof. For *given* $(y^2, ..., y^{n-1})$, the payoff in the Sophi auction, for the ring with signals (s_1, s_2), is given by

$$\int_{y^2}^{t} \left[\widehat{V}^R(s_1, s_2, y^1) - V^R(y^1)\right] dF_{Y^1|Y^2,...,Y^{n-1},S_1,S_2}(y^1|y^2, ..., y^{n-1}, s_1, s_2).$$

This is exactly the payoff received in the English auction, given $(y^2, ..., y^{n-1}, s_1, s_2)$, when the ring drops out when the holder of y^1 is revealed to have signal $y^1 \geq t$. Thus, if the optimal misbehavior for the ring in the Sophi auction, with signals (s_1, s_2), is to report $t^{*SophiR}$, the same payoff can be achieved in the English auction by dropping out when the holder of y^1 is revealed to have signal $y^1 \geq t^{*SophiR}$, for each realization of $(y^2, ..., y^{n-1})$. It follows that the ring receives a weakly higher payoff in the English auction than in the Sophi auction.

If, for the ring with signals (s_1, s_2), the optimal misbehavior can be achieved with a constant t^* that does not depend on $(y^2, ..., y^{n-1})$, then reporting t^* in the Sophi auction allows the ring to receive the same payoff as in the English auction for each realization of $(y^2, ..., y^{n-1})$. On the other hand, if t^* depends nontrivially on $(y^2, ..., y^{n-1})$, then requiring the ring in the English auction to choose a constant t, independent of $(y^2, ..., y^{n-1})$, leads to some suboptimal choices and a strict reduction in payoff. ■

It will be convenient to let $h^R(s_1, s_2, y^1, y^2, ..., y^{n-1})$ denote the expected value of the object, conditional on the signals of the ring and the order statistics of the ordinary bidders.

Proposition 6: *Suppose that Assumption S holds. Then the optimal misbehavior of the ring, with signals (s_1, s_2), yields the same outcome in the English auction and in the Sophi auction. Furthermore, the optimal misbehavior is without regret.*

Proof.

Case 1. First, suppose that, in the ring's optimal strategy in the English auction, after some history, $\widetilde{y}^2, ..., \widetilde{y}^{n-1}$, the ring is active and pretends to be type $t^* < \overline{x}$, dropping at the clock price $h^R(t^*, \underline{x}, t^*, \widetilde{y}^2, ..., \widetilde{y}^{n-1})$.

Claim 1: Given this history, t^* must satisfy

$$h^R(s_1, s_2, t^*, \widetilde{y}^2, ..., \widetilde{y}^{n-1}) - h^R(t^*, \underline{x}, t^*, \widetilde{y}^2, ..., \widetilde{y}^{n-1}) = 0. \tag{4.8}$$

Proof of Claim 1: Suppose the left side of (4.8) is strictly positive. Then by continuity, for an interval of signal y^1 sufficiently close to, but greater than, t^*, we have

$$h^R(s_1, s_2, y^1, \widetilde{y}^2, ..., \widetilde{y}^{n-1}) - h^R(y^1, \underline{x}, y^1, \widetilde{y}^2, ..., \widetilde{y}^{n-1}) > 0.$$

Thus, waiting a little longer will allow the ring to obtain positive profit when the remaining ordinary bidder has a signal near t^*, and this event occurs with positive probability, contradicting the optimality of t^*. Similarly, if the left side of (4.8) is strictly negative, then for an interval of signal y^1 sufficiently close to, but less than, t^*, we have

$$h^R(s_1, s_2, y^1, \widetilde{y}^2, ..., \widetilde{y}^{n-1}) - h^R(y^1, \underline{x}, y^1, \widetilde{y}^2, ..., \widetilde{y}^{n-1}) < 0.$$

Dropping a little earlier will allow the ring to avoid negative profit when the remaining ordinary bidder has a signal near t^*, and this event occurs with positive probability, contradicting the optimality of t^*. This establishes Claim 1.

Since t^* satisfies (4.8), then it follows immediately from Assumption S that, for all $y^2, ..., y^{n-1}$, the same t^* satisfies

$$h^R(s_1, s_2, t^*, y^2, ..., y^{n-1}) - h^R(t^*, \underline{x}, t^*, y^2, ..., y^{n-1}) = 0. \tag{4.9}$$

Claim 2: For all histories $y^2, ..., y^{n-1}$, pretending to be t^* is the unique optimal continuation.

Proof of Claim 2: First, we show that $t^* \geq \max[s_1, s_2]$ must hold. Otherwise, there is a positive probability of signal realizations satisfying $t^* < y^1 < \max[s_1, s_2]$. For these realizations, the profit from winning,

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(y^1, \underline{x}, y^1, y^2, ..., y^{n-1}). \tag{4.10}$$

would be positive, so remaining in the auction yields strictly higher profits.

From (4.9) and Assumption S (substituting y^1 for t^*), we have

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(t^*, \underline{x}, y^1, y^2, ..., y^{n-1}) = 0. \tag{4.11}$$

Suppose $y^1 > t^*$ holds. Then if the ring were to bid enough to acquire the object, by substituting (4.11) into (4.10), the ring's profits would be

$$h^R(t^*, \underline{x}, y^1, y^2, ..., y^{n-1}) - h^R(y^1, \underline{x}, y^1, y^2, ..., y^{n-1}). \tag{4.12}$$

Since the expectation of V is strictly increasing in the highest signal,[11] and since $y^1 > t^*$ holds, this profit expression must be negative. Therefore, the ring is strictly worse off pretending to be a type greater than t^*.

Suppose $t^* \geq y^1$ holds. If, in addition, we have $\max[s_1, s_2] \leq y^1$, then, since the expectation of V is strictly increasing in the highest signal, (4.12) must be strictly positive. If, we have $y^1 < \max[s_1, s_2]$, then, since the expectation of V is strictly increasing in the highest signal, the profits from obtaining the object, (4.10), are strictly greater than

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(\max[s_1, s_2], \underline{x}, y^1, y^2, ..., y^{n-1}).$$

Since $t^* \geq \max[s_1, s_2]$ holds, the previous expression is weakly greater than

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(t^*, \underline{x}, y^1, y^2, ..., y^{n-1}),$$

which is zero, by (4.11). Therefore, the ring is strictly worse off if it pretends to be a type below t^*. This establishes Claim 2.

Claim 3: Suppose we have a history, $y^j, ..., y^{n-1}$, satisfying $y^j \leq t^*$, for some $j > 2$. Then pretending to be t^* is the unique optimal continuation.

Proof of Claim 3: Pretending to be t^* means dropping at the price

$$h^R(t^*, \underline{x}, t^*, ..., t^*, y^j, ..., y^{n-1}).$$

In the English auction, when the price reaches $h^R(t^*, \underline{x}, t^*, ..., t^*, y^j, ..., y^{n-1})$, the ring would be receiving zero profits if all of the remaining ordinary bidders drop simultaneously at that price. If the remaining ring member remains past that price, and wins the object, the ring's profits are given by (4.12), which is negative, since we have $y^1 > t^*$. Thus, pretending to be a type greater than t^* yields strictly lower expected profits.

If the remaining ring member pretends to be a type below t^*, then there is a positive probability that all of the ordinary bidders have signals below t^*, in which case the foregone profits of winning are (4.12). Since $y^1 < t^*$ holds under this scenario, the foregone profits are positive, as established in the proof of Claim 2. This establishes Claim 3.

[11] Our maintained assumption, $\partial h(y)/\partial y^1 > 0$, for the model with the ring, translates into the assumption that h^R is strictly increasing in the highest of all signals, including the ring's signals.

Case 2. Suppose that, in the ring's optimal strategy in the English auction, after some history, $\widetilde{y}^2, ..., \widetilde{y}^{n-1}$, the ring is active and pretends to be type $t^* = \overline{x}$. Then the profits from winning the object must be non-negative when we have $y^1 = \overline{x}$,

$$h^R(s_1, s_2, \overline{x}, \widetilde{y}^2, ..., \widetilde{y}^{n-1}) - h^R(\overline{x}, \underline{x}, \overline{x}, \widetilde{y}^2, ..., \widetilde{y}^{n-1}) \geq 0.$$

(If negative, then pretending to be below $\overline{x}$ will strictly increase profits.) By Assumption S, we must have

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(\overline{x}, \underline{x}, y^1, y^2, ..., y^{n-1}) \geq 0$$

for all $y^1, y^2, ..., y^{n-1}$. Actual profits, when the ordinary bidders' signals are $y^1, y^2, ..., y^{n-1}$, are given by

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(y^1, \underline{x}, y^1, y^2, ..., y^{n-1}),$$

which is strictly positive for all $y^1 < \overline{x}$. Therefore, pretending to be type $t^* = \overline{x}$ is the unique optimal strategy in this case.

Thus, for both Case 1 and Case 2, in the English auction, the remaining ring member pretends to be type t^*, independent of the history. Since t^* is unique and independent of the history, we must have $t^* = t^{*Sophi}$. It follows that the two formats yield the same outcome.

The optimal misbehavior is without regret: Suppose $y^1 > t^*$ holds, so the ring's profits are zero. If the ring were to bid enough to acquire the object, the ring's profits would be

$$h^R(t^*, \underline{x}, y^1, y^2, ..., y^{n-1}) - h^R(y^1, \underline{x}, y^1, y^2, ..., y^{n-1}),$$

as derived above using Assumption S. Since $y^1 > t^*$ holds, then this profit expression is non-positive. Suppose, instead that $y^1 \leq t^*$ holds, so the ring obtains the object. Then the above profit expression is non-negative, which follows from $y^1 \leq t^*$.

■

Remark 1: Proposition 6 establishes that, under Assumption S, the outcomes are the same under both auction formats. However, the ring in the English auction has a continuum of optimal strategies, because once the ring determines that it can never profitably win the object, the remaining ring member can delay dropping up until the ordinary bidder with signal y^2 drops. Our selection, of the strategy in which the remaining ring member exits as soon as it knows that it can never profitably win the object, is consistent with what we assume about ordinary bidders.[12] However, the ring is special in some sense, and it is interesting to consider other optimal choices. Suppose that Assumption S holds, but the remaining ring member in the English auction pretends to be the maximum of t^* and y^2. Then one can prove that the ring receives the same payoff under the two formats, the auctioneer is better off under the English, and the ordinary bidders are better off under Sophi.[13] If Assumption S does not hold, the situation is more complicated, because some realizations of $(s_1, s_2, y^1, y^2, ..., y^{n-1})$ might induce the second ring member to pretend to be a lower type in the English auction than in the Sophi auction, so there are some realizations for which the auctioneer is better off with the Sophi auction. Nonetheless, the possiblity of the ring delaying exit in the English auction, without taking the risk of obtaining the object, is a feature that benefits the auctioneer. Thus, the ring can manipulate the auction in two ways. Having one ring member drop immediately hurts the auctioneer (while benefitting itself); having the remaining ring member remain, at least until y^2 drops, benefits the auctioneer (without hurting itself). This leads to the interesting possibility that the auctioneer is better off ex ante by the existence of the ring in the English auction.

4.4. Common Knowledge of the Ring

We show in Proposition 7 below that, under Assumption S, the strategies of the ring and the ordinary bidders are equilibrium strategies of the model in which the existence of the ring is common knowledge. This result may seem highly counterintuitive, especially for the English auction. With a secret ring, the immediate dropout by one of the ring members causes ordinary bidders to infer a signal of $\underline{x}$, but when the existence of the ring is common knowledge, the immediate drop by one of the ring members provides no inference to ordinary bidders about that ring member's signal. The correct intuition is based on the fact that the ring is pre-

[12] In a model without misbehavior, imagine how awkward it would be to study equilibria in which bidders arbitrarily delay their exit until 2 or 3 bidders remain.

[13] If the ring chooses any optimal strategy of waiting until at least t^*, but sometimes waiting longer, it is still the case that the auctioneer is better off with the English auction, and the ordinary bidders are better off under Sophi.

pared to compete up until the point of zero expected profits. If the ordinary bidder with signal y^1 wins the object, the price is $h^R(t^*, \underline{x}, t^*, y^2, ..., y^{n-1})$, and the value to this bidder is $h^R(s_1, s_2, y^1, y^2, ..., y^{n-1})$. However, since t^* is such that the value to the ring equals the clock price at the point of dropping, $h^R(s_1, s_2, t^*, y^2, ..., y^{n-1})$ equals $h^R(t^*, \underline{x}, t^*, y^2, ..., y^{n-1})$. Therefore, we can write the profits of the ordinary bidder with signal y^1 as

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(s_1, s_2, t^*, y^2, ..., y^{n-1}).$$

From this expression, we see that this ordinary bidder is best responding by treating the ring's behavior as if they were ordinary bidders.

There is a technical issue that must be resolved. For the English auction, for some signal realizations, bidder R1 is willing to remain in the auction strictly beyond the price at which an ordinary bidder of type $\overline{x}$ would drop. The analysis of previous subsections defined $t^* = \overline{x}$, but in fact, R1 is sometimes willing to bid more than the price at which type $\overline{x}$ would drop, $h^R(\overline{x}, \underline{x}, \overline{x}, y^2, ..., y^{n-1})$. This is important when the ordinary bidders know about the ring, because if R1 is known to drop at or below the price $h^R(\overline{x}, \underline{x}, \overline{x}, y^2, ..., y^{n-1})$, an ordinary bidder with signal near $\overline{x}$ would want to win the auction at that price. However, if R1 believes that an ordinary bidder who remains beyond the drop price of type $\overline{x}$ is type $\overline{x}$, and drops at the zero profit price, $h^R(s_1, s_2, \overline{x}, y^2, ..., y^{n-1})$, then as we show, ordinary bidders are best responding with their sincere strategies. In essence, the zero profit point for R1 corresponds to a type "greater than" $\overline{x}$.

For the Sophi auction, we need to modify the game to allow bidders to report types greater than $\overline{x}$.[14] Otherwise, if the ordinary bidders know about the ring, an ordinary bidder with signal near $\overline{x}$ would want to report $\overline{x}$. We do not change the distribution of signals but the pricing rule must be extended to cover the case of multiple reports greater than $\overline{x}$ (where the expectation of V is not well defined under the standard pricing rule). Here is the extension of the pricing rule: if t is the highest report and there are other reports greater than $\overline{x}$, all reports higher than $\overline{x}$ are truncated to $\overline{x}$, and the *lowest* report is increased by $(t - \overline{x})$ or increased to $\overline{x}$, whichever is less. Thus, when R1 submits the highest report, t^*, and the ordinary bidder with signal y^1 is the only ordinary bidder reporting more than $\overline{x}$, R1 wins and pays the price $h^R(\overline{x}, \underline{x} - \overline{x} + t^*, \overline{x}, y^2, ..., y^{n-1})$.[15]

[14] We note that this issue does not arise when $\overline{x} = \infty$ holds.

[15] If we have $\underline{x} - \overline{x} + t^* > \overline{x}$, then the price would be $h^R(\overline{x}, \overline{x}, \overline{x}, y^2, ..., y^{n-1})$, but R1 would never want to submit a report that high.

Proposition 7: *Suppose that Assumption S holds. Consider (i) the optimal strategy of the ring, as characterized by bidder R2 pretending to be type* $\underline{x}$ *and bidder R1 pretending to be type* t^**, and (ii) the "sincere" strategies of the ordinary bidders as characterized earlier. For the English auction, this strategy profile is an equilibrium profile of the game in which the existence of the ring is common knowledge. For the extended Sophi auction in which the existence of the ring is common knowledge, this strategy profile is an equilibrium profile, except that* t^* *may be greater than* $\overline{x}$*; the outcomes are as characterized earlier. Furthermore, sincere bidding by the ordinary bidders is without regret.*

Proof. Suppose the the existence of the ring is common knowledge. We know that, if the ordinary bidders are choosing the same strategy that they choose with the secret ring, the ring is best responding. To show that we have an equilibrium that is without regret, we will now show that the ordinary bidders do not have a profitable deviation for any realization of the signals.

(English) All signal realizations fall into one of the following cases.

Case 1: $y^1 > t^* \geq y^2$.

The ordinary bidder with signal y^1 wants to win the object. To see this, the profits, expected value minus the price, are given by

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(t^*, \underline{x}, t^*, y^2, ..., y^{n-1}).$$

From the proof of Proposition 6, (4.9) holds, so we have

$$h^R(s_1, s_2, t^*, y^2, ..., y^{n-1}) = h^R(t^*, \underline{x}, t^*, y^2, ..., y^{n-1}).$$

Thus, profits can be written as

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(s_1, s_2, t^*, y^2, ..., y^{n-1}). \tag{4.13}$$

Since $y^1 > t^*$ holds, then by our maintained assumptions on h, (4.13) is positive.

An ordinary bidder with signal y^j (for $j > 1$) does not want to obtain the object. To see this, the profits from winning would be

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(t^*, \underline{x}, y^1, y^2, ..., y^{j-1}, y^1, y^{j+1}, ..., y^{n-1}),$$

which, by (4.9) and Assumption S, is equal to

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(s_1, s_2, y^1, y^2, ..., y^{j-1}, y^1, y^{j+1}, ..., y^{n-1}). \quad (4.14)$$

Since $y^1 > y^j$ holds, then by our maintained assumptions on h, (4.14) is non-positive.

Case 2: $y^1 \geq y^2 > t^*$.

The ordinary bidder with signal y^1 wants to win the object. To see this, the profits, expected value minus the price, are given by

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(t^*, \underline{x}, y^2, y^2, ..., y^{n-1}).$$

From (4.9) and Assumption S, so we have

$$h^R(s_1, s_2, y^2, y^2, ..., y^{n-1}) = h^R(t^*, \underline{x}, y^2, y^2, ..., y^{n-1}).$$

Thus, profits can be written as

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(s_1, s_2, y^2, y^2, ..., y^{n-1}). \quad (4.15)$$

Since $y^1 \geq y^2$ holds, then by our maintained assumptions on h, (4.15) is non-negative.

An ordinary bidder with signal y^j (for $j > 1$) does not want to obtain the object. To see this, the profits from winning would be

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(t^*, \underline{x}, y^1, y^2, ..., y^{j-1}, y^1, y^{j+1}, ..., y^{n-1}),$$

which, by (4.9) and Assumption S, is equal to (4.14). Since $y^1 \geq y^j$ holds, then by our maintained assumptions on h, (4.14) is non-positive.

Case 3: $\overline{x} > t^* \geq y^1$.

An ordinary bidder with signal y^j (for $j \geq 1$) does not want to obtain the object. To see this, the profits from winning would be

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(t^*, \underline{x}, y^1, y^2, ..., y^{j-1}, t^*, y^{j+1}, ..., y^{n-1}),$$

which, by (4.9) and Assumption S, is equal to

$$h^R(t^*, \underline{x}, y^1, y^2, ..., y^{j-1}, y^j, y^{j+1}, ..., y^{n-1}) - h^R(t^*, \underline{x}, y^1, y^2, ..., y^{j-1}, t^*, y^{j+1}, ..., y^{n-1}). \quad (4.16)$$

Since $t^* \geq y^j$ holds, (4.16) is non-positive.

Case 4: $t^* = \overline{x}$.

Bidder R1 believes that an ordinary bidder who remains beyond the drop price of type $\overline{x}$ is type $\overline{x}$, and therefore is willing to bid up to the zero profit price, $h^R(s_1, s_2, \overline{x}, y^2, ..., y^{n-1})$. An ordinary bidder with signal y^j (for $j \geq 1$) does not want to obtain the object. To see this, the price the bidder would have to pay is $h^R(s_1, s_2, \overline{x}, y^2, ..., y^{n-1})$, but the value of the object to the bidder is only $h^R(s_1, s_2, y^1, y^2, ..., y^{n-1})$, so the profits from winning are non-positive.

(Sophi) For signal realizations in cases 1-3, bidder R1 is reporting $t^* \leq \overline{x}$. The argument for Sophi is identical to the argument for English, due to the result (Proposition 6) that t^* in English does not depend on the dropout history and outcomes are the same for the two formats. For Case 4, when R1 in English is willing to bid more than $h^R(\overline{x}, \underline{x}, \overline{x}, y^2, ..., y^{n-1})$, then R1 needs to report a type greater than $\overline{x}$ in Sophi. The report is such that R1 would receive zero profits upon winning the object, assuming that $y^1 = \overline{x}$ holds, and the ordinary bidder with signal y^1 reports more than $\overline{x}$. This condition satisfies

$$h^R(s_1, s_2, \overline{x}, y^2, ..., y^{n-1}) = h^R(\overline{x}, \underline{x} - \overline{x} + t^*, \overline{x}, y^2, ..., y^{n-1}), \tag{4.17}$$

and by Assumption S, the same t^* satisfies (4.17) for any $y^2, ..., y^{n-1}$. At $t^* = \overline{x}$, the left side of (4.17) is weakly greater than the right side, because we are in Case 4. If $t^* = 2\overline{x} - \underline{x}$ holds, then the right side of (4.17) equals $h^R(\overline{x}, \overline{x}, \overline{x}, y^2, ..., y^{n-1})$, so the left side of (4.17) is less than the right side. Because h^R is continuous, (4.17) must have a solution for t^*.

An ordinary bidder with signal y^j (for $j \geq 1$) does not want to obtain the object. To see this, by (4.17), the price a bidder with signal y^1 would have to pay is $h^R(s_1, s_2, \overline{x}, y^2, ..., y^{n-1})$, and a different ordinary bidder would have to pay more. However, the value of the object to the bidder is only $h^R(s_1, s_2, y^1, y^2, ..., y^{n-1})$, so the profits from winning are non-positive.

■

5. Stability of the Ring

The model of the previous section presumes that the ring members can commit to sharing the payoff. In this section, we suppose that the ring members cannot commit to sharing the payoff, in the event that they win the auction, until after they receive their signals. We show that the ring is stable, and the results of the previous section apply, by considering the following game. First, each of the bidders, including the potential ring members (bidders 1 and 2), observes their signal. Second, the potential ring members simultaneously decide whether to be part of the ring or not. If both members say "yes," then the ring forms, and the ring members commit to share equally the profit from the auction (either English or Sophi), in which case the ring members report their signals to each other, continuation of the game and the analysis of the previous section applies. If one or more potential ring members says "no," then these decisions are communicated to each other (become common knowledge to the potential ring members), the ring does not form, and we proceed to the auction. Proposition 8, below, is stated and proved for the case of the English auction when the ring does not form, each potential ring member can identify whether a drop-out bidder is an ordinary bidder or the other potential ring member. The stability result also holds when the identity of drop-out bidders is not observed (see Remark 2).[16]

5.1. Stability of the Ring in the English Auction

Consider the following strategy profile for the English auction:[17]

The two potential ring members say "yes," and along the equilibrium path, the game continues as analyzed in the previous section.

If one potential ring member (w.l.o.g., bidder R1) says "yes" and the other potential ring member (bidder R2) says "no," then bidder R1 believes, and continues to believe throughout the auction, that bidder R2's signal is $\overline{x}$. In the auction, bidder R2 drops immediately, after every history in which dropping is not weakly dominated; if dropping is weakly dominated, bidder R2 remains. Bidder R1's continuation strategy is the following. R1 drops with several remaining bidders if there are no signal realizations for the remaining ordinary bidders such that winning is profitable, given beliefs about R2. If bidder R2 is the only remaining bidder along with R1 (so that $y^1, y^2, ..., y^{n-1}$ have been revealed), then bidder R1 remains in the auction until the price reaches R1's willingness to pay, $\widehat{V}^{NR}(s_1, y^1) \equiv h^R(s_1, \overline{x}, y^1, y^2, ..., y^{n-1})$. If the only remaining bidder is an

[16] Marshall and Marx (2009) show that whether or not the auctioneer releases the identities of registered bidders can affect the viability of collusion in private value auctions.

[17] We continue to assume that the ordinary bidders are oblivious to the presence of the (potential) ring, so for all intents and purposes, this is a game being played by the two ring members.

ordinary bidder, and bidder R2 has already dropped at a price such that the remaining ordinary bidder infers $S_2 = x'$, then the remaining ordinary bidder's maximum willingness to pay is given by $V^{NR}(y^1) \equiv h^R(s_1, x', y^1, y^2, ..., y^{n-1})$. Thus, bidder R1's drop-out problem is to "pretend" to be an ordinary bidder with signal $t^* > s_1$, where t^* is a maximizer of

$$\int_{\max[x',y^2]}^{t} \left[\widehat{V}^{NR}(s_1, y^1) - V^{NR}(y^1)\right] dF_{Y^1|Y^2,...,Y^{n-1},S_1,S_2}(y^1|y^2, ..., y^{n-1}, s_1, \overline{x}). \tag{5.1}$$

Finally, if both potential ring members say "no," then bidders 1 and 2 each believe that the other bidder's signal is $\overline{x}$. Bidder R1 drops with several remaining bidders if there are no signal realizations for the remaining ordinary bidders such that winning is profitable, given beliefs about R2. Otherwise, if bidder R2 is the only remaining bidder, then bidder R1 will drop at the price $\widehat{V}^{NR}(s_1, y^1)$. If, instead, bidder R2 has already dropped, at a price such that the remaining ordinary bidder infers $S_2 = x'$, then bidder R1's drop-out problem is to "pretend" to be an ordinary bidder with signal $t^* > s_1$, where t^* is a maximizer of (5.1). Bidder R2 chooses the same continuation strategy.[18]

Proposition 8. *The strategy profile described above constitutes a PBE for the English auction.*

Proof. If bidder R1 and bidder R2 both say "yes," then truthfully revealing their signals to each other, and bidding as specified in the previous section, is sequentially rational.

Suppose bidder R1 says "yes" and bidder R2 says "no." Certainly it is sequentially rational for bidder R1 to drop whenever remaining cannot be profitable, and otherwise to remain until there is only one other remaining bidder. Consider the case in which the only remaining bidder besides R1 is bidder R2. Given bidder R1's belief that bidder R2 is type $\overline{x}$, when the price is greater than $\widehat{V}^{NR}(s_1, y^1)$, it is sequentially rational for bidder R1 to drop immediately. If the price is less than $\widehat{V}^{NR}(s_1, y^1)$, it is sequentially rational for bidder R1 to remain until $\widehat{V}^{NR}(s_1, y^1)$. Now consider the case in which the only remaining bidder besides R1 is an ordinary bidder. Bidder R1's expected profit from pretending to be type t is (5.1), so his continuation strategy is sequentially rational.

[18] That is, we can relabel 2 as 1 and 1 as 2, and choose the continuation strategy of bidder R1 described above.

Whenever bidder R2 has a continuation strategy that weakly dominates dropping out, remaining in the auction is sequentially rational. Otherwise, it is sequentially rational for bidder R2 to drop immediately. To see this, suppose that bidder R2 wins the auction when bidder R1 is the last to drop. Then the profit received by bidder R2 is

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(s_1, \overline{x}, y^1, y^2, ..., y^{n-1}),$$

which is non-positive (strictly negative unless we have $s_2 = \overline{x}$). Suppose that bidder R2 wins the auction when an ordinary bidder is the last to drop. Then the fact that bidder R1 dropped implies $h^R(s_1, \overline{x}, y^1, y^2, ..., y^{n-1})$ is less than the price at which bidder R1 dropped. Therefore, the price paid by bidder R2 is more than $h^R(s_1, \overline{x}, y^1, y^2, ..., y^{n-1})$ and the value to bidder R2 is less than $h^R(s_1, \overline{x}, y^1, y^2, ..., y^{n-1})$, so winning cannot be profitable.

Suppose bidder R1 and bidder R2 both say "no." Given bidder R1's belief that bidder R2 is type $\overline{x}$, by the arguments given above, bidder R1's continuation strategy is sequentially rational. Since bidder R2 is adopting the same continuation strategy as bidder R1, bidder R2's continuation strategy is sequentially rational as well.

It is sequentially rational for bidder R1 and bidder R2 to say "yes." After saying "no" the best continuation play is to drop immediately, while saying "yes" leads to positive expected profit. ■

Remark 2: Consider the continuation of the game when bidder R1 says "yes" and bidder R2 says "no." If we were to extend the earlier equilibrium selection (for an established ring) of dropping when remaining in the auction cannot be profitable, then bidder R2 would drop immediately after every history. The construction in Proposition 8 guarantees that the equilibrium does not involve a player choosing a weakly dominated strategy. For Proposition 8, we consider the model in which the potential ring members observe the identities of the bidders dropping from the auction, so bidder R1 knows the point at which bidder R2 drops, and vice versa. A similar proposition could be proved for the model in which the identities of those dropping from the auction is not observed. In that case, bidder R1 believes that bidder R2 has signal $\overline{x}$, and that the first bidder to drop is bidder R2. With identities unobservable, bidder R2 is even less willing to compete with bidder R1 than when identities are observable. The reason is that, if bidder R2 deviates to remain in the auction, bidder R1 believes that the ordinary bidders are remaining in the auction longer, and revealing higher signals, than they actually are.

5.2. Stability of the Ring in the Sophi Auction

Consider the following strategy profile for the Sophi auction:

The two potential ring members say "yes," and along the equilibrium path, the game continues as analyzed in the previous section.

If one potential ring member (w.l.o.g., bidder R1) says "yes" and the other potential ring member (bidder R2) says "no," then bidder R1 believes that bidder R2's signal is $\overline{x}$. In the auction, bidder R1 reports type $\overline{x}$ and bidder R2 reports type $\underline{x}$.

If both potential ring members say "no," then bidders 1 and 2 each believe that the other bidder's signal is $\overline{x}$. In the auction, both bidders truthfully report their types.

Proposition 9. *The strategy profile described above constitutes a PBE for the Sophi auction.*

Proof. If bidder R1 and bidder R2 both say "yes," then truthfully revealing their signals and bidding as specified in the previous section is sequentially rational.

Suppose bidder R1 says "yes" and bidder R2 says "no." Then, given the beliefs of bidder R1, for arbitrary $y^1, y^2, ..., y^{n-1}$, bidder R1 wins the auction and receives expected profit

$$h^R(s_1, \overline{x}, y^1, y^2, ..., y^{n-1}) - h^R(y^1, \underline{x}, y^1, y^2, ..., y^{n-1}).$$

By symmetry, the expected profit equals

$$h^R(s_1, \overline{x}, y^1, y^2, ..., y^{n-1}) - h^R(\underline{x}, y^1, y^1, y^2, ..., y^{n-1}).$$

Therefore, since $s_1 > \underline{x}$ and $\overline{x} > y^1$ hold with probability one, the expected profit is positive for every s_1 and every realization of $y^1, y^2, ..., y^{n-1}$. Since expected profit from winning is positive and the price paid by bidder R1 does not depend on his report (as long as he wins), reporting $\overline{x}$ is sequentially rational. For bidder R2, reporting $\underline{x}$ is sequentially rational, since the only way bidder R2 can win is to report $\overline{x}$, in which case bidder R2's profit, when he wins the tie-break coin flip, is

$$h^R(s_1, s_2, y^1, y^2, ..., y^{n-1}) - h^R(\overline{x}, \overline{x}, y^1, y^2, ..., y^{n-1}), \tag{5.2}$$

which is non-positive.

If bidder R1 and bidder R2 both say "no," then each bidder believes that the other bidder is type $\overline{x}$, in which case they will lose the auction unless they bid $\overline{x}$. However, bidder R1's profit from bidding $\overline{x}$ is

$$h^R(s_1, \overline{x}, y^1, y^2, ..., y^{n-1}) - h^R(\overline{x}, \overline{x}, y^1, y^2, ..., y^{n-1}),$$

which is non-positive, and bidder R2's profit from bidding $\overline{x}$ is

$$h^R(\overline{x}, s_2, y^1, y^2, ..., y^{n-1}) - h^R(\overline{x}, \overline{x}, y^1, y^2, ..., y^{n-1}),$$

which also is non-positive. Thus, truthful reporting, after bidder R1 and bidder R2 both say "no," is sequentially rational.

It is sequentially rational for bidder R1 and bidder R2 to both say "yes." The reason is that, after saying "no," the best continuation play leads to zero profit, while saying "yes" leads to positive expected profit.

Clearly, the beliefs are consistent. ■

Remark 3: The construction of a PBE in Proposition 9 uses a continuation strategy for bidder R2, after bidder R1 says "yes" and bidder R2 says "no," that might be weakly dominated. If we impose Assumption S, then an alternate construction is available with strategies that cannot be weakly dominated. The idea is that bidder R2 reports x', determined such that bidder R1 receives zero expected profits by winning when $(s_1, s_2) = (\underline{x}, \overline{x})$. Bidder R1 reports a type such that expected profits from winning are zero when tying with the ordinary bidder with the highest signal, based on the belief that bidder R2 has signal $\overline{x}$. This ensures sequential rationality, and that bidder R2 could receive negative profits by bidding more. The proof is fairly intricate. We conjecture that a construction is available without Assumption S.

5.3. Robustness to Larger Rings

If the number of ring members is $m > 2$, the optimal bidding behavior would have all but one ring member drop immediately in the English auction, and report $\underline{x}$ in the Sophi auction. The optimal bid by the remaining ring member requires slightly modifying the analysis of Section 4, to take into account that there are multiple ring members mimicking type $\underline{x}$. For the question of stability, we consider the following game. First, each of the bidders, including the potential ring members, observes their signal. Second, the potential ring members simultaneously decide whether to be part of the ring or not. If a subset of at least two members say "yes," then a ring consisting of that subset forms, and the ring members commit to share equally the profit from the auction (either English or Sophi). It is assumed that the makeup of the ring is common knowledge among the m potential ring members.

We now sketch the argument that the full-size ring is stable for the English auction. If one potential ring member deviates and says "no," then the $m-1$ ring members believe he is type $\overline{x}$. All but one of the ring members drop immediately, but the remaining ring member waits until expected profit for the ring is zero. The deviator drops immediately. After the deviation, it is sequentially rational for the deviator to drop immediately, because outbidding the ring member guarantees non-positive profits. Given that a lone deviator receives zero profits, everyone saying "yes" is sequentially rational. (If several potential ring members say "no," then they are all believed to be type $\overline{x}$. Choose any continuation strategies that are sequentially rational, given the beliefs.)

For the Sophi auction, there is a potential problem with stability when $m > 2$ holds. After a single potential ring member says "no," sequential rationality requires all but one of the $m-1$ ring members to report $\underline{x}$, in order to minimize the price paid when the ring wins. However, rather than being part of a full-size ring, a potential ring member could then deviate to "no," and report $\overline{x}$. Even if one ring member also reports $\overline{x}$, the profit could be substantial if m is large, and a 0.5 probability of winning the tie-breaking coin flip could make a deviation worthwhile. However, suppose that a functional form for the expectation of the common value, as a function of the signal realizations, exists and is well defined (e.g., under Assumption Z). Then the Sophi auction could allow bidders to report signals higher than $\overline{x}$, similar to the way the English auction does not limit how high the price can go. In this case, the ring could have one of its members submit a report, $\overline{x}' > \overline{x}$, such that the expected profit from winning is zero, given that the deviator also reports $\overline{x}'$. This would guarantee that it would be sequentially rational for the deviator to report $\underline{x}$, and therefore that the deviation to say "no" cannot be beneficial.

6. Concluding Remarks

We analyze, separately, the misbehavior of an auctioneer who uses shill bidding, and the misbehavior of a ring of bidders, within a common-value setting. We consider the impact of misbehavior on the allocation and the price under two auction formats: the ascending-price English auction and the Sophi auction, a static direct mechanism that is equivalent to the English auction in the absence of misbehavior. We characterize the form of the misbehavior, and rank the two action formats from the standpoints of the auctioneer, the ring (if present) and the regular bidders.

For the case of the auctioneer using a shill bidder in the English auction, we characterize when the auctioneer drops immediately after the bidder with the second highest signal drops, and when the auctioneer remains longer, thereby risking keeping the object. We show that the auctioneer is strictly better off under English than under Sophi, from which it follows that the ordinary bidders are better off under Sophi.

For the case of the bidding ring in the English auction, one ring member drops at the first opportunity, mimicking type $\underline{x}$. We characterize when the other ring member mimicks a type strictly lower than the highest possible type, and when the other ring member guarantees a win on the equilibrium path, by remaining in the auction beyond the price at which the highest possible type should drop. Generally, the ring is strictly better off under English than under Sophi. However, and somewhat surprisingly, we recognize an important special case (Assumption S) where, in spite of the additional information revealed to the ring from the dropout prices in the English auction, the ring cannot do better under English than under Sophi. Under Assumption S, ordinary bidders would not change their strategies even if the existence of the ring was common knowledge. We also show that the ring is stable, by augmenting the game to include a stage in which the potential ring members decide whether or not to join the ring after observing their private signals. A potential ring member saying "no" would be considered to be type $\overline{x}$, which would induce aggressive bidding by the other potential ring member. Therefore, potential ring members have an incentive to join the ring, to report their signals honestly to the ring-mechanism designer, and to behave in the auction according to the previous analysis.

The results, that the misbehaving auctioneer prefers English to Sophi, and that the ring prefers English to Sophi, are examples of a more general principle. Suppose that we have a dynamic indirect mechanism (such as English) that yields the same equilibrium mapping from type realizations to outcomes as a direct mechanism (such as Sophi). Now suppose that the utility functions change, either for one agent (e.g., because the agent is colluding with the auctioneer) or for a ring of agents (who are colluding with each other), but that the remaining agents are not aware of this change in incentives. Then the agent(s) with new incentives weakly prefer the dynamic mechanism to the direct mechanism. The reason is that, in the dynamic mechanism, they can mimic the behavior of the type they would have reported in the direct mechanism. However, the dynamic mechanism also allows the option of tailoring behavior to information revealed during the play. Seen in this light, Proposition 3 and Proposition 5 are not surprising, but the proofs are included because they are short and illustrate the incentives in the auctions. More surprising is Proposition 6, which shows that, under Assumption S, the ring does not benefit from the option to tailor behavior to information revealed during the English auction.

The work of Li (2017) on obviously dominant strategies suggests that dynamic auctions may have features that bring behavior closer to the Bayesian Nash equilibrium norm than their static counterparts. In the private values setting, the dominant strategy of bidding one's value is obvious in the dynamic English auction, but not so in the static second-price auction (see Kagel et al. (1987)). With multiple-unit demands, Kagel and Levin (2001) show that the dynamic Vickrey auction performs better than the static uniform price auction. In the common-values setting, Levin et al. (2016) show that behavior in the dynamic Dutch auction is closer to BNE than in the strategically equivalent static first-price auction. There are no dominant strategies, but the dynamic format of the ticking price clock gives subjects a hint about adverse selection, since no one else has stopped the clock. In the present paper, however, the Sophi rules build in a correction for adverse selection and the winner's curse, and bidding one's value is the equilibrium behavior of ordinary bidders. We find that the static Sophi format is actually better for ordinary bidders when unexpected misbehavior by the auctioneer is present. When a bidding ring is present, we can only make sharp comparisons under Assumption S. When the ring drops at the first opportunity, the two formats yield the same outcome (and knowledge of the ring does not matter). When the ring chooses a best response that involves waiting beyond t^* in the English auction, the dynamic format helps the auctioneer and hurts the ordinary bidders.

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