

# Ambiguity Aversion, Games Against Nature, and Dynamic Consistency\*

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## Abstract

Several papers, adopting an axiomatic approach to study decision making under ambiguity aversion, have produced conflicting predictions about how decision makers would behave in simple dynamic urn problems. We explore the concepts of ambiguity aversion and dynamic consistency, with examples of dynamic games against nature. Basically, a malevolent nature puts balls into the urn, and a fair nature draws them out. Depending on the game, various choices that seem inconsistent with static notions of ambiguity aversion or dynamic consistency are consistent with subgame perfection. In the dynamic 3-color Ellsberg urn problem with 30 red balls and 60 blue or green balls, the decision maker could strictly prefer to bet on blue-green at time 0, and to switch to red-green after learning that the ball is not green.

Keywords: Ambiguity Aversion, Uncertainty Aversion, Dynamic Consistency, Ellsberg, Malevolent Nature

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## 1. Introduction

Consider the 3-color Ellsberg (1961) urn experiment. There are 30 balls that are red and 60 that are either blue or green in an urn. A ball is drawn from the urn at random. The state space  $\Omega$  consists of possible colors of the ball, i.e.,  $\Omega = \{R, B, G\}$ . Let  $f_R$  denote the lottery paying \$1 if a red ball is drawn, and zero otherwise. Let  $f_{RG}$  denote the lottery paying \$1 if a red or green ball is drawn, zero otherwise, and so on. Most decision makers prefer  $f_R$  to  $f_B$ . At the same time they prefer  $f_{BG}$  to  $f_{RG}$ . Taken as a decision problem this pattern of choices violates Savage's axioms. One explanation for the violation of Savage's axioms is that the decision maker (*DM*) has ambiguity-averse preferences and likes to bet on events with known probabilities. One can also view the problem as a game against nature (or the experimenter), in which case similar choices may arise for a *DM* who is an expected utility maximizer.

Several papers attempt to extend the decision theoretic notion of ambiguity aversion to dynamic problems. A key question that comes up in these extensions is whether and how strongly to invoke the assumption of dynamic consistency. In this paper we shed some light on this issue by considering several examples of games against nature. Using a dynamic version of the 3-color Ellsberg urn experiment, we show that, when an expected utility maximizing agent behaves as if she is playing a game against nature, behavior that seems to be dynamically inconsistent may arise as a subgame perfect equilibrium.

Consider the following dynamic decision problem<sup>1</sup>. At time  $t = 0$  the *DM* chooses between lotteries  $f_{BG}$  and  $f_{RG}$ , and a ball is drawn at random from the urn. At time  $t = 1$  the *DM* is told whether or not the ball is green, and chooses whether to keep the lottery chosen at  $t = 0$  or to switch to the other lottery. Epstein and Schneider (2003) argue that "typical" ambiguity-averse preferences can be dynamically inconsistent. At time 0, the ranking is  $f_{BG} \succ_0 f_{RG}$ .<sup>2</sup> Now suppose that the decision maker is told that the ball is not green, so the *DM* is essentially comparing a bet on red with a bet on blue. In this case the ambiguity-averse *DM* might rank the two acts as  $f_{RG} \succ_{1, \{R, B\}} f_{BG}$ .<sup>3</sup> If the ball is revealed to be green, the decision maker wins the bet either way, so the ranking is  $f_{RG} \sim_{1, \{G\}} f_{BG}$ . Epstein

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<sup>1</sup>Versions of this problem appear in Epstein and Schneider (2003) and Siniscalchi (2004).

<sup>2</sup>We denote the preference at time 0 by  $\succeq_0$  and preference at time 1 conditional on knowing that the color of the ball is in the set  $A$  by  $\succeq_{1, A}$ .

<sup>3</sup>For example, a *DM* whose preferences can be represented using maxmin expected utility and who uses prior by prior Bayesian updating would behave this way.

and Schneider (2003) argue that these choices are dynamically inconsistent, since  $f_{RG}$  is preferred to  $f_{BG}$  no matter what information is revealed. They argue that a dynamically consistent  $DM$ , with ambiguity-averse preferences at  $t = 1$ , must have the ranking  $f_{RG} \succ_0 f_{BG}$ , which they view as problematic.<sup>4</sup>

We consider five games against nature built around this dynamic decision problem.<sup>5</sup> We denote these games by  $\Gamma_1, \dots, \Gamma_5$ . In all these games, think of a malevolent nature (or experimenter) putting balls into the urn at time 0, and a fair nature drawing them out at time 1. Once the ball is drawn, the  $DM$  learns whether the ball is green or not. Just like in the decision problem, the  $DM$  initially picks  $f_{BG}$  or  $f_{RG}$ , and may later be given a chance to switch. The games that we consider vary in terms of the timing of these decisions as well as the choices available to the malevolent nature. We selected the games to show that a variety of behaviors may arise, depending on how the  $DM$  perceives nature to be manipulating her. In particular, in games  $\Gamma_1$  and  $\Gamma_2$ , the  $DM$  chooses  $f_{BG}$  at  $t = 0$ , and chooses not to switch at  $t = 1$ . In game  $\Gamma_3$ , the  $DM$  is indifferent between  $f_{BG}$  and  $f_{RG}$  at  $t = 0$ , but is required to mix over whether to switch or not at  $t = 1$ . In game  $\Gamma_4$ , the  $DM$  chooses  $f_{RG}$  at  $t = 0$ , and chooses not to switch at  $t = 1$ . In game  $\Gamma_5$ , the  $DM$  chooses  $f_{BG}$  at  $t = 0$ , and chooses to switch at  $t = 1$ . The problematic, apparently dynamically inconsistent, behavior discussed above arises in game  $\Gamma_3$  with positive probability, and always occurs when the  $DM$  is given the opportunity to switch in game  $\Gamma_5$ . As far as we know, we are the first to provide a Bayesian, subjective expected utility framework in which choosing  $f_{BG}$  at  $t = 0$  and switching at  $t = 1$  is dynamically consistent.

Beginning with Epstein and Schneider (2003), there is an axiomatic literature investigating these issues. Maccheroni, Marinacci, and Rustichini (2006) extend the multiple priors model by generalizing nature's constraint to include costs of choosing or altering probability distributions. They characterize the  $DM$ 's preferences as being dynamically consistent if and only if the  $DM$  believes that nature is dynamically consistent. Their model can be interpreted as a game against nature, where the  $DM$  chooses a plan and nature learns the plan and manipulates probabilities over time. Hanany and Klibanoff (2005) propose an updating rule for which the consistent choice for an ambiguity-averse  $DM$  is  $f_{BG}$  at  $t = 0$  and

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<sup>4</sup>In fact, the conflict between dynamic consistency and Ellsberg type behavior observed in the above example is very general. Epstein and Le Breton (1993) show that when conditional preferences are "based on beliefs" in a dynamically consistent way then the  $DM$  must be probabilistically sophisticated and has a Bayesian prior. This rules out Ellsberg type behavior.

<sup>5</sup>In game  $\Gamma_5$ , the  $DM$  is only given the opportunity to switch with probability one half.

at  $t = 1$  as in our games  $\Gamma_1$  and  $\Gamma_2$ .

The previous literature specifies preferences as being over *acts*, or state-time contingent consumption.<sup>6</sup> Choosing  $f_{BG}$  at  $t = 0$ , and switching to  $f_{RG}$  at  $t = 1$ , corresponds to the same act as choosing  $f_{RG}$  at  $t = 0$ , and keeping  $f_{RG}$  at  $t = 1$ . Thus, the previous literature cannot formally distinguish between these behaviors. When the DM believes that nature could be manipulating probabilities in response to her choices, however, then preferences should depend on the history of her choices, rather than the induced state-time contingent consumption. For example, in game  $\Gamma_5$ , the DM strictly prefers choosing  $f_{BG}$  at  $t = 0$ , and switching to  $f_{RG}$  at  $t = 1$ , over choosing  $f_{RG}$  at  $t = 0$ , and sticking with  $f_{RG}$  at  $t = 1$ , even though both strategies correspond to the same act. Choosing  $f_{BG}$  at  $t = 0$ , and switching to  $f_{RG}$  at  $t = 1$ , is dynamically consistent because there is no revision to the original plan. Now consider a comparison of behavior across two *different* decision problems: (i) the DM chooses a lottery at  $t = 0$ , and cannot switch after learning whether the ball is green, and (ii) the DM observes whether the ball is green, and chooses a lottery at  $t = 1$ . If the domain of preferences is state-time contingent consumption, then choosing  $f_{BG}$  in decision problem (i) and  $f_{RG}$  in decision problem (ii) violates the definition of dynamic consistency. However, if the DM acts as if she is playing a game against nature, then decision problems (i) and (ii) could well correspond to different games and different manipulations by nature. Choosing  $f_{BG}$  in decision problem (i) and  $f_{RG}$  in decision problem (ii) does not show inconsistency of preferences across time, because the DM in decision problem (ii) is planning to choose  $f_{RG}$  all along.

Although this paper does not attempt an axiomatic analysis, there is certainly no conflict between our examples and the axiomatic approach. Maccheroni, Marinacci, and Rustichini (2006) provide an axiomatic analysis of behavior that can be interpreted as the DM playing a game against nature. In their game, the DM chooses a plan of state-time contingent consumptions, and malevolent nature responds by choosing probabilities over time. One can view our contribution as demonstrating that, by expanding the class of admissible games, the set of observed behavior is expanded. In particular, when the DM perceives that nature is manipulating her, it is not without loss of generality to assume that she is directly choosing state-time contingent consumption.

We take an *as if* approach, in the sense that we think that the DM behaves

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<sup>6</sup>Siniscalchi (2004) takes a different approach and allows the DM to be dynamically inconsistent but assumes that she is sophisticated in the sense that the DM can correctly anticipate her future choices.

as if she is playing a game against nature. In doing so, our main goal is to point out that dynamic consistency (of preferences over state-time contingent consumption) may be a problematic axiom. It seems important to note that the *DM*'s preferences may change with the environment and still be consistent with her behaving as if she is playing a game against nature. Thus our approach is consistent with more transparency in the environment leading to less ambiguity averse choices. Moreover, it suggests the types of experiments that can be set up to test our resolution of the dynamic consistency puzzle. For example, controlling for (i) whether the urn is filled before or after lottery choice; (ii) whether a ball is physically and transparently drawn, or randomization is done by a computer; or (iii) whether the experimenter "wants" the subject to win will have predictable effects on the decisions made in the dynamic version of the Ellsberg problem. The extreme circumstance, of allowing the subject to have a trusted attorney verify that the announcements are made truthfully after a ball is drawn, would also lead to less ambiguity averse choices. A more direct type of experiment would be to ask subjects to submit a plan of what lottery to choose at  $t = 0$  and whether to switch at  $t = 1$ . If subjects plan to switch from the outset, then switching cannot be characterized as dynamically inconsistent.

Our approach can also be viewed as relaxing the assumption of *consequentialism*. Machina (1989) points out that dynamic inconsistency arguments rely on consequentialism which means that a *DM* would behave in the continuation of a decision tree exactly as if the continuation were the entire decision tree. He argues that consequentialism is often an unreasonable assumption, because nonseparabilities cause previous choices to affect continuation preferences.<sup>7</sup> We would add that, even when the modeler thinks the continuation of a decision tree is identical to the full decision tree of a different problem, a *DM* that behaves as if she is playing a game against nature might view these two situations as different games.

## 2. The Games

In this section we present a series of two-player zero-sum games between *DM* and a malevolent nature,  $N^M$ . The power of  $N^M$  is limited, of course. The malevolent

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<sup>7</sup>Machina gives the example of a mother, who strictly prefers flipping a coin to see which child receives a treat, as opposed to giving the treat to one of the children. However, after the coin flip has been chosen, the mother strictly prefers giving the treat to the winner, as opposed to flipping a second coin.

nature chooses how to put the balls into the urn, but a fair nature,  $N^F$ , draws a ball out of the urn. The games presented below reflect different assumptions about the timing of nature's moves and the ability of the malevolent nature to manipulate the  $DM$ .

## 2.1. Game $\Gamma_1$ : Nature moves first

In our first game,  $\Gamma_1$ , the stages are as follows:

1.  $N^M$  fills the urn with 30 red balls,  $n$  green balls, and  $60 - n$  blue balls, where  $n \in \{0, \dots, 60\}$ .
2.  $DM$  chooses either  $f_{BG}$  or  $f_{RG}$  ( $t = 0$ ).
3.  $N^F$  draws a ball out of the urn at random, and announces whether it is green or not. If the announcement is "green," the game ends.
4. If the announcement is "not green,"  $DM$  decides whether to **stay** with the chosen act or **switch** ( $t = 1$ ).
5. If  $DM$  wins the bet without switching, payoffs are  $(1, -1)$ ; if  $DM$  loses the bet without switching, payoffs are  $(0, 0)$ ; if  $DM$  wins the bet after switching, payoffs are  $(1 - \varepsilon, -1 + \varepsilon)$ ; if  $DM$  loses the bet after switching, payoffs are  $(-\varepsilon, \varepsilon)$ .

We assume that  $\varepsilon$  is small and nonnegative. A mixed strategy for  $N^M$  is a probability distribution over the number of green balls, denoted by  $p(n)$ .  $DM$ 's set of pure strategies is

$$\{(f, s, \bar{s}) : f \in \{f_{BG}, f_{RG}\}, s \in \{stay, switch\}, \bar{s} \in \{stay, switch\}\},$$

where  $s$  represents the choice about whether to keep the lottery chosen at  $t = 0$ , after learning that the ball is not green, and  $\bar{s}$  represents whether  $DM$  would have kept the lottery *not chosen* at  $t = 0$ , after learning that the ball is not green. Since  $\bar{s}$  is chosen at a decision node that is impossible to reach, given  $DM$ 's choice at  $t = 0$ , we focus on the set of reduced strategies,

$$\{(f_{BG}, stay), (f_{BG}, switch), (f_{RG}, stay), (f_{RG}, switch)\}.$$

The payoff to  $DM$  for each (reduced) strategy is given by:

$$\begin{aligned}
U(f_{BG}, stay) &= \sum_{n=0}^{60} p(n) \frac{2}{3} = \frac{2}{3} \\
U(f_{BG}, switch) &= \sum_{n=0}^{60} p(n) \left[ \frac{n}{90} + \frac{90-n}{90} \left( \frac{30}{90-n} \right) (1-\varepsilon) \right] \\
&= \frac{(1-\varepsilon)}{3} + \frac{1}{90} \sum_{n=0}^{60} p(n) n \\
U(f_{RG}, stay) &= \sum_{n=0}^{60} p(n) \frac{30+n}{90} = \frac{1}{3} + \frac{1}{90} \sum_{n=0}^{60} p(n) n \\
U(f_{RG}, switch) &= \sum_{n=0}^{60} p(n) \left[ \frac{n}{90} + \frac{90-n}{90} \left( \frac{60-n}{90-n} \right) (1-\varepsilon) \right] \\
&= \frac{2}{3} - \varepsilon \left[ \sum_{n=0}^{60} p(n) \left( \frac{60-n}{90} \right) \right]
\end{aligned}$$

Note that for  $\varepsilon > 0$  we have  $U(f_{RG}, stay) > U(f_{BG}, switch)$  and  $U(f_{BG}, stay) > U(f_{RG}, switch)$ .  $DM$  weakly prefers  $(f_{BG}, stay)$  over  $(f_{RG}, stay)$  if and only if

$$30 \geq \sum_{n=0}^{60} p(n) n \equiv E(n)$$

holds. That is,  $DM$  will choose  $(f_{BG}, stay)$  over  $(f_{RG}, stay)$  if and only if the expected number of green balls is less than 30. From this the following result follows immediately:

**Proposition 1.** *Any pure or mixed strategy for  $N^M$  satisfying  $E(n) \leq 30$  is consistent with subgame perfect equilibrium. In any Nash equilibrium,  $DM$  ends the game with  $f_{BG}$  and receives a payoff of  $\frac{2}{3}$ . With a small switching cost,  $\varepsilon > 0$ , the  $DM$  must assign probability one to the pure strategy,  $(f_{BG}, stay)$ .*

## 2.2. Game $\Gamma_2$ : The Decision Maker Moves First

The game  $\Gamma_2$  is very similar to game  $\Gamma_1$  except that the  $DM$  moves first, and the nature moves next observing the  $DM$ 's move. We see that the main conclusion of Proposition 1, namely that the  $DM$  assigns probability one to the pure strategy  $(f_{BG}, \textit{stay})$ , is robust to the timing of the moves. The stages in  $\Gamma_2$  are as follows:

1.  $DM$  chooses either  $f_{BG}$  or  $f_{RG}$  ( $t = 0$ ).
2.  $N^M$  observes  $DM$ 's choice and fills the urn with 30 red balls,  $n$  green balls, and  $60 - n$  blue balls, where  $n \in \{0, \dots, 60\}$ .
3.  $N^F$  draws a ball out of the urn at random, and announces whether it is green or not. If the announcement is "green," the game ends.
4. If the announcement is "not green,"  $DM$  decides whether to **stay** with the chosen act or **switch** ( $t = 1$ ).

The payoff structure is the same as in  $\Gamma_1$ . Now a mixed strategy for  $N^M$  is a probability distribution over the number of green balls after  $DM$  chooses  $f_{BG}$ , and after  $DM$  chooses  $f_{RG}$ . Without going into details, it is easy to see that equilibrium outcomes are the same as in  $\Gamma_1$ . That is:

**Proposition 2.** *In any Nash equilibrium,  $DM$  ends the game with  $f_{BG}$  and receives a payoff of  $\frac{2}{3}$ . With a small switching cost,  $\varepsilon > 0$ , the  $DM$  must assign probability one to the pure strategy,  $(f_{BG}, \textit{stay})$ .*

## 2.3. Game $\Gamma_3$ : Strategic Announcement; Urn is Filled Before the Switching Decision

Here we depart from the previous games by assuming that  $N^M$  is able to announce whether or not the ball is green *before*  $N^F$  draws the ball. By announcing that the ball is not green, this forces  $N^F$  to remove all of the green balls from the urn before drawing. Thus,  $\Gamma_3$  has the following timing.

1.  $DM$  chooses either  $f_{BG}$  or  $f_{RG}$  ( $t = 0$ ).
2.  $N^M$  observes  $DM$ 's choice and fills the urn with 30 red balls,  $n$  green balls, and  $60 - n$  blue balls, where  $n \in \{0, \dots, 60\}$ .
3.  $N^M$  announces "green" or "not green." If the announcement is "green," the game ends.
4. If the announcement is "not green,"  $N^F$  removes all of the green balls from the urn. Then  $N^F$  draws a ball out of the urn at random.



5.  $DM$  decides whether to **stay** with the chosen act or **switch** ( $t = 1$ ).

The payoff structure is the same as in  $\Gamma_1$ . Subgame perfect equilibrium is characterized in the following proposition.

**Proposition 3.** *In any subgame perfect equilibrium of  $\Gamma_3$ ,  $N^M$  announces "not green," and mixes over how it fills the urn, such that we have*

$$\begin{aligned} pr(\text{blue ball drawn}, f_{BG} \text{ subgame}) &= \frac{1 - \varepsilon}{2}, \\ pr(\text{blue ball drawn}, f_{RG} \text{ subgame}) &= \frac{1 + \varepsilon}{2}. \end{aligned}$$

$DM$ 's choice at  $t = 0$  is arbitrary. At  $t = 1$ ,  $DM$  mixes, staying with probability  $\frac{1}{2}$  and switching with probability  $\frac{1}{2}$ .  $DM$ 's payoff is  $\frac{1-\varepsilon}{2}$ .

**Proof.** Obviously,  $N^M$  strictly prefers to announce "not green" in all circumstances. Consider the subgame after  $DM$  chooses  $f_{BG}$  at  $t = 0$ . Because payoffs depend on  $p(n)$  only through the induced probability of a blue ball being drawn, denoted by  $p_B$ , we characterize the equilibrium  $p_B$ .  $DM$  payoffs as a function of  $p_B$  are  $U(f_{BG}, \text{stay}) = p_B$  and  $U(f_{BG}, \text{switch}) = p_B(-\varepsilon) + (1 - p_B)(1 - \varepsilon)$ . Letting  $S$  denote the probability that  $DM$  stays,  $N^M$  payoffs as a function of  $S$  are  $S + (1 - S)(-\varepsilon)$  if a blue ball is drawn, and  $(1 - S)(1 - \varepsilon)$  if a red ball is drawn. The solution must involve mixing, so by equating the payoff expressions, we have

$$p_B = \frac{1 - \varepsilon}{2} \text{ and } S = \frac{1}{2}. \quad (1)$$

From (1),  $DM$  receives a payoff of  $\frac{1-\varepsilon}{2}$ .

Now consider the subgame after  $DM$  chooses  $f_{RG}$  at  $t = 0$ . We have  $U(f_{RG}, \text{stay}) = (1 - p_B)$  and  $U(f_{RG}, \text{switch}) = p_B(1 - \varepsilon) + (1 - p_B)(-\varepsilon)$ .  $N^M$  payoffs as a function of  $S$  are  $(1 - S)(1 - \varepsilon)$  if a blue ball is drawn, and  $S + (1 - S)(-\varepsilon)$  if a red ball is drawn. The solution must involve mixing, so by equating the payoff expressions, we have

$$p_B = \frac{1 + \varepsilon}{2} \text{ and } S = \frac{1}{2}. \quad (2)$$

From (2),  $DM$  receives a payoff of  $\frac{1-\varepsilon}{2}$ . ■

In  $\Gamma_3$ ,  $DM$  mixes over whether to switch. If instead  $DM$  were to stay with  $f_{BG}$  with probability one, then  $N^M$  could choose  $n = 60$ , announce "not green," and guarantee that a red ball is selected. If  $DM$  were to switch from  $f_{BG}$  with probability one, then  $N^M$  could choose  $n = 0$ , announce "not green," and hold  $DM$  to a payoff below  $\frac{1}{3}$ . Either pure action by  $DM$  gives an advantage to  $N^M$ . The proof of Proposition 1 presumes that  $N^M$  can feasibly choose any probability of a blue ball being drawn. In fact,  $N^M$  cannot induce  $p_B > \frac{2}{3}$ , but this restriction does not bind. There are many subgame perfect equilibria yielding the probabilities specified in Proposition 1. For example,  $N^M$  could choose

$$p(0) = \frac{3(1 - \varepsilon)}{4} \text{ and } p(60) = \frac{1 + 3\varepsilon}{4} \text{ in the } f_{BG} \text{ subgame,}$$

$$p(0) = \frac{3(1 + \varepsilon)}{4} \text{ and } p(60) = \frac{1 - 3\varepsilon}{4} \text{ in the } f_{RG} \text{ subgame.}$$

#### 2.4. Game $\Gamma_4$ : Strategic Announcement; Urn is Filled After the Switching Decision

In this game,  $N^M$  is given tremendous power to manipulate  $DM$ , by announcing "not green" and waiting until the switching decision to fill the urn. The timing in  $\Gamma_4$  is the following.

1.  $DM$  chooses either  $f_{BG}$  or  $f_{RG}$  ( $t = 0$ ).
2.  $N^M$  announces "green" or "not green." If the announcement is "green," the game ends.
3. If the announcement is "not green,"  $DM$  decides whether to **stay** with the chosen act or **switch** ( $t = 1$ ).
4.  $N^M$  observes  $DM$ 's choices and fills the urn with 30 red balls,  $n$  green balls, and  $60 - n$  blue balls, where  $n \in \{0, \dots, 60\}$ .
5.  $N^F$  removes all of the green balls from the urn. Then  $N^F$  draws a ball out of the urn at random.

The payoff structure is the same as in  $\Gamma_1$ . Obviously,  $N^M$  announces "not green" in all circumstances. If  $DM$  either stays with or switches to  $f_{BG}$  at  $t = 1$ , then  $N^M$  chooses  $p(60) = 1$  in the ensuing subgame, yielding  $DM$  a payoff of 0 or  $-\varepsilon$  (depending on whether switching costs are incurred). If  $DM$  either stays with

or switches to  $f_{RG}$  at  $t = 1$ , then  $N^M$  chooses  $p(0) = 1$  in the ensuing subgame, yielding  $DM$  a payoff of  $\frac{1}{3}$  or  $\frac{1}{3} - \varepsilon$  (depending on whether switching costs are incurred). This gives us the next proposition:

**Proposition 4.** *In any subgame perfect equilibrium,  $DM$  ends the game with  $f_{RG}$  at  $t = 1$ , and receives a payoff of  $\frac{1}{3}$ . With a small switching cost,  $\varepsilon > 0$ , the  $DM$  must assign probability one to the pure strategy,  $(f_{RG}, \text{stay})$ .*

## 2.5. Game $\Gamma_5$ : Random Opportunity for Strategic Announcement

In the previous games,  $\Gamma_1 - \Gamma_4$ ,  $DM$  switches when the switching cost is zero (so the initial choice does not matter), or as part of a mixed strategy equilibrium in  $\Gamma_3$ . In the following game, every Nash equilibrium has the  $DM$  playing the pure strategy  $(f_{BG}, \text{switch})$ . This corresponds to the preferences Epstein and Schneider (2003) and Siniscalchi (2004) identify as ambiguity averse but dynamically inconsistent. If the  $DM$  views the problem as the following game against nature,  $\Gamma_5$ , it turns out that  $(f_{BG}, \text{switch})$  can be justified as dynamically consistent. In game  $\Gamma_5$ , with probability one half, the game ends without the  $DM$  having an opportunity to switch lotteries (stages 2a and 3a), and with probability one half, there is an announcement and the  $DM$  can switch lotteries (stages 2b, 3b, 4b, and 5b).

1.  $DM$  chooses either  $f_{BG}$  or  $f_{RG}$  ( $t = 0$ ).
- 2a. With probability  $\frac{1}{2}$ ,  $N^M$  observes  $DM$ 's choice and fills the urn with 30 red balls,  $n$  green balls, and  $60 - n$  blue balls, where  $n \in \{0, \dots, 60\}$ .
- 3a.  $N^F$  draws a ball out of the urn at random. Then the game ends.
- 2b. With probability  $\frac{1}{2}$ ,  $N^M$  observes  $DM$ 's choice and announces "green" or "not green." If the announcement is "green," the game ends.
- 3b. If the announcement is "not green,"  $DM$  decides whether to **stay** with the chosen act or **switch** ( $t = 1$ ).
- 4b.  $N^M$  observes  $DM$ 's choice and fills the urn with 30 red balls,  $n$  green balls, and  $60 - n$  blue balls, where  $n \in \{0, \dots, 60\}$ .
- 5b.  $N^F$  removes all of the green balls from the urn. Then  $N^F$  draws a ball out of the urn at random.

The payoff structure is the same as in  $\Gamma_1$ . This is an extensive game with perfect information, which can be solved by backwards induction. After the subgame,  $(f_{BG}, 2a)$ , the choice by  $N^M$  makes no difference, and the payoff is  $\frac{2}{3}$ . After the

subgame,  $(f_{RG}, 2a)$ , the optimal choice by  $N^M$  is to fill the urn with blue balls,  $p(0) = 1$ , and the payoff is  $\frac{1}{3}$ . After the subgame,  $(f_{BG}, 2b, \text{not green}, \text{stay})$ , the optimal choice by  $N^M$  is to fill the urn with green balls,  $p(60) = 1$ , so that all the green balls are removed, and the payoff is 0. After the subgame,  $(f_{BG}, 2b, \text{not green}, \text{switch})$ , the optimal choice by  $N^M$  is to fill the urn with blue balls,  $p(0) = 1$ , and the payoff is  $\frac{1}{3} - \varepsilon$ . After the subgame,  $(f_{RG}, 2b, \text{not green}, \text{stay})$ , the optimal choice by  $N^M$  is to fill the urn with blue balls,  $p(0) = 1$ , and the payoff is  $\frac{1}{3}$ . After the subgame,  $(f_{RG}, 2b, \text{not green}, \text{switch})$ , the optimal choice by  $N^M$  is to fill the urn with green balls,  $p(60) = 1$ , so that all the green balls are removed, and the payoff is  $-\varepsilon$ .

Working backwards, after the subgame,  $(f_{BG}, 2b, \text{not green})$ ,  $DM$  strictly prefers to switch, yielding a payoff of  $\frac{1}{3} - \varepsilon$ . After the subgame,  $(f_{RG}, 2b, \text{not green})$ ,  $DM$  strictly prefers to stay, yielding a payoff of  $\frac{1}{3}$ . Therefore, at the initial node, choosing  $f_{BG}$  yields  $DM$  a payoff of  $\frac{1}{2}(\frac{2}{3}) + \frac{1}{2}(\frac{1}{3} - \varepsilon) = \frac{1-\varepsilon}{2}$ . Choosing  $f_{RG}$  yields  $DM$  a payoff of  $\frac{1}{2}(\frac{1}{3}) + \frac{1}{2}(\frac{1}{3}) = \frac{1}{3}$ . From these observations the next result follows immediately:

**Proposition 5.** *As long as  $\varepsilon$  is small, there is a unique subgame perfect equilibrium, in which  $DM$  chooses the pure strategy,  $(f_{BG}, \text{switch})$ .*

### 3. Discussion

In discussing the issue of dynamic consistency, one should be clear about why  $DM$ 's choice at  $t = 0$  matters. Indeed, we included the  $\varepsilon$  switching cost to rule out trivial examples in which  $DM$  switches because it is costless to do so, and the initial choice makes no difference. Our analysis indicates two situations in which a decision maker strictly prefers to switch. In  $\Gamma_3$ ,  $DM$  is suspicious of the announcement, "not green," believing that the relative likelihood of red vs. blue is not yet settled. Thus,  $DM$  does not want to be predictable at  $t = 1$ . The initial choice is not important, but  $DM$  switches with positive probability, irrespective of the initial choice. In  $\Gamma_5$ ,  $DM$  believes that there is a chance that there will be no opportunity to switch, so she strictly prefers the static ambiguity-averse choice,  $f_{BG}$ , at  $t = 0$ . After observing "not green,"  $DM$  is suspicious of the announcement, believing that the relative likelihood of red vs. blue will not be settled until after her switching decision. Thus,  $DM$  switches to the new ambiguity-averse choice,  $f_{RG}$ , at  $t = 1$ .

We do not insist that the decision maker consciously believes that she is or might be playing a game against nature. Just as a game against nature can be a metaphor for ambiguity aversion in static problems, it can serve the same role in dynamic problems. By analyzing games for which seemingly inconsistent behavior arises in equilibrium, we can better understand the source of the inconsistency. We hope that our examples can suggest extensions of the axiomatic approach.

We do not pretend to offer a complete theory about ambiguity aversion. Many different phenomena yield behavior one might call ambiguity aversion. This paper is based on uncertainty about the exact "decision problem," as presented by the experimenter, the market, the House, or nature. Ambiguity aversion can be seen as a healthy skepticism about the existence of other interested parties. We readily admit that there are situations in which ambiguity aversion arises out of psychological concerns. For example, ambiguity might create stress or cause sleep loss, creating harm that can be physically measured. Even these situations could be amenable to a game theoretic approach,<sup>8</sup> but it might be more useful to take ambiguity-averse preferences at face value. Within the framework of games against nature, it is conceivable that the game is not zero sum. If the decision maker views nature as benevolent, then ambiguity-loving preferences could be observed.

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<sup>8</sup>Perhaps the decision maker can be modelled as a principal dealing with an agent, rather than as a game against nature.

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