Ambiguity Aversion, Games Against Nature, and Dynamic Consistency*

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Abstract

Several papers, adopting an axiomatic approach to study decision making under ambiguity aversion, have produced conflicting predictions about how decision makers would behave in simple dynamic urn problems. We explore the concepts of ambiguity aversion and dynamic consistency in the context of dynamic games against nature. Basically, a malevolent nature puts balls into the urn, and a fair nature draws them out. Depending on the game, various choices that seem inconsistent with static notions of ambiguity aversion or dynamic consistency are consistent with subgame perfection. In the dynamic 3-color Ellsberg urn problem with 30 red balls and 60 blue or green balls, the decision maker could strictly prefer to bet on blue-green at time 0, and to switch to red-green after learning that the ball is not green.

1. Introduction

Consider the 3-color Ellsberg (1961) urn experiment. There are 30 balls that are red and 60 that are either blue or green in an urn. A ball is drawn from the urn at random. The state space Ω consists of possible colors of the ball, i.e., $\Omega = \{R, B, G\}$. Let f_R denote the lottery paying \$1 if a red ball is drawn, and

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zero otherwise. Let f_{RG} denote the lottery paying \$1 if a red or green ball is drawn, zero otherwise, and so on. Most decision makers prefer f_R to f_B. At the same time they prefer f_{BG} to f_{RG}. One explanation for the violation of Savage's axioms is that the decision maker has ambiguity-averse preferences and likes to bet on events with known probabilities.

Several papers, attempting to extend the notion of ambiguity aversion to dynamic problems, have produced conflicting predictions about how decision makers would behave. Suppose, as described by Epstein and Schneider (2003), that at time $t = 0$ a ball is drawn at random from the urn, and at time $t = 1$ the decision maker (DM) is told whether or not the ball is green. Epstein and Schneider argue that "typical" ambiguity-averse preferences can be dynamically inconsistent. At time 0, the ranking is $f_{BG} \succ_0 f_{RG}$. Now suppose that the decision maker is told that the ball is not green, so DM is essentially comparing a bet on red with a bet on blue. In this case the ambiguity-averse DM might rank the two acts as $f_{RG} \succ_{1,\{R,B\}} f_{BG}$.[1] If the ball is revealed to be green, the decision maker wins the bet either way, so the ranking is $f_{RG} \sim_{1,\{G\}} f_{BG}$. Epstein and Schneider (2003) argue that these choices are dynamically inconsistent, since f_{RG} is preferred to f_{BG} no matter what information is revealed. They argue that a dynamically consistent DM, with ambiguity-averse preferences at $t = 1$, must have the ranking $f_{RG} \succ_0 f_{BG}$, which they view as problematic.

Epstein and Le Breton (1993) show that when conditional preferences are "based on beliefs" in a dynamically consistent way then the DM must be probabilistically sophisticated and has a Bayesian prior. This rules out Ellsberg type behavior. Therefore, the conflict between dynamic consistency and Ellsberg type behavior observed in the example above is general.

It is well known that, in static environments, ambiguity-averse behavior arises when an expected utility maximizing agent believes that she is playing a game against nature[2]. In this paper we argue that dynamically inconsistent behavior also arises. Thus, the DM[3] may display behavior that, from the perspective of the modeler, is Ellsberg type and dynamically inconsistent, when in fact she is a dynamically consistent expected utility maximizer, but believes that she is playing

[1] For example, a DM whose preferences can be represented using maxmin expected utility and who uses prior by prior Bayesian updating would behave this way.

[2] For example, Maccheroni, Marinacci and Rustichini (2004) provide a representation for ambiguity averse preferences which they explicitly interpret as an agent playing a game against a malevolent nature.

[3] We continue to use the terminology DM, although perhaps the DM in our model should be the human player in a two-player game.

a game against nature. Therefore, we believe that some of the behavior that seems problematic at first may not be problematic if the modeler could learn the game against nature that the DM might be playing.

Think of a malevolent nature (or experimenter) putting balls into the urn, and a fair nature drawing them out. Then DM, in the static 3-color Ellsberg urn experiment will choose f_R over f_B, but will choose f_{BG} over f_{RG}. Our contribution is to explore the concepts of ambiguity aversion and dynamic consistency in the context of dynamic games against nature, by presenting a series of games centered around the dynamic 3-color Ellsberg urn experiment described by Epstein and Schneider (2003). Depending on the choices available to the malevolent nature, various choices by DM that seem dynamically inconsistent are shown to occur in a subgame perfect equilibrium. In particular, in games Γ_1 and Γ_2, DM chooses f_{BG} at $t = 0$, and chooses not to switch at $t = 1$. In game Γ_3, DM is indifferent between f_{BG} and f_{RG} at $t = 0$, but is required to mix over whether to switch or not at $t = 1$. In game Γ_4, DM chooses f_{RG} at $t = 0$, and chooses not to switch at $t = 1$. In game Γ_5, DM chooses f_{BG} at $t = 0$, and chooses to switch at $t = 1$. As far as we know, we are the first to provide a Bayesian framework in which choosing f_{BG} at $t = 0$ and switching at $t = 1$ is dynamically consistent.

There is an axiomatic literature investigating similar issues. Hanany and Klibanoff (2005) propose an updating rule for which the consistent choice for an ambiguity-averse DM is f_{BG} at $t = 0$ and at $t = 1$ as in our games Γ_1 and Γ_2. Siniscalchi (2004) takes a different approach and allows the DM to be dynamically inconsistent but assumes that she is sophisticated in the sense that the DM can correctly anticipate her future choices.

Our approach can also be viewed as relaxing the assumption of *consequentialism.* Machina (1989) points out that dynamic inconsistency arguments rely on consequentialism which means that a DM would behave in the continuation of a decision tree exactly as if the continuation were the entire decision tree. He argues that consequentialism is often an unreasonable assumption, because non-separabilities cause previous choices to affect continuation preferences.[4] We would add that two decision trees that are identical from the perspective of the modeler may be different from the perspective of a DM that believes she is playing a game against nature.

[4] Machina gives the example of a mother, who strictly prefers flipping a coin to see which child receives a treat, as opposed to giving the treat to one of the children. However, after the coin flip has been chosen, the mother strictly prefers giving the treat to the winner, as opposed to flipping a second coin.

2. The Games

2.1. Game Γ_1: Nature moves first

We will consider two-player zero-sum games between DM and a malevolent nature, N^M. The power of N^M is limited, of course. The malevolent nature chooses how to put the balls into the urn, but a fair nature, N^F, draws a ball out of the urn. In our first game, Γ_1,the stages are as follows:

1. N^M fills the urn with 30 red balls, n green balls, and $60 - n$ blue balls, where $n \in \{0, ..., 60\}$.
2. DM chooses either f_{BG} or f_{RG} ($t = 0$).
3. N^F draws a ball out of the urn at random, and announces whether it is green or not. If the announcement is "green," the game ends.
4. If the announcement is "not green," DM decides whether to **stay** with the chosen act or **switch** ($t = 1$).
5. If DM wins the bet without switching, payoffs are $(1, -1)$; if DM loses the bet without switching, payoffs are $(0, 0)$; if DM wins the bet after switching, payoffs are $(1-\varepsilon, -1+\varepsilon)$; if DM loses the bet after switching, payoffs are $(-\varepsilon, \varepsilon)$.

We assume that ε is small and nonnegative. A mixed strategy for N^M is a probability distribution over the number of green balls, denoted by $p(n)$. DM's set of pure strategies is

$$\{(f, s, \overline{s}) : f \in \{f_{BG}, f_{RG}\}, s \in \{stay, switch\}, \overline{s} \in \{stay, switch\}\},$$

where s represents the choice about whether to keep the lottery chosen at $t = 0$, after learning that the ball is not green, and $\overline{s}$ represents whether DM would have kept the lottery *not chosen* at $t = 0$, after learning that the ball is not green. Since $\overline{s}$ is chosen at a decision node that is impossible to reach, given DM's choice at $t = 0$, we focus on the set of reduced strategies,

$$\{(f_{BG}, stay), (f_{BG}, switch), (f_{RG}, stay), (f_{RG}, switch)\}.$$

The payoff to DM for each (reduced) strategy is given by:

$$\begin{aligned}
U\left(f_{BG}, stay\right) &= \sum_{n=0}^{60} p\left(n\right)\frac{2}{3} = \frac{2}{3} \\
U\left(f_{BG}, switch\right) &= \sum_{n=0}^{60} p\left(n\right)\left[\frac{n}{90} + \frac{90-n}{90}\left(\frac{30}{90-n}\right)\left(1-\varepsilon\right)\right] \\
&= \frac{\left(1-\varepsilon\right)}{3} + \frac{1}{90}\sum_{n=0}^{60} p\left(n\right)n \\
U\left(f_{RG}, stay\right) &= \sum_{n=0}^{60} p\left(n\right)\frac{30+n}{90} = \frac{1}{3} + \frac{1}{90}\sum_{n=0}^{60} p\left(n\right)n \\
U\left(f_{RG}, switch\right) &= \sum_{n=0}^{60} p\left(n\right)\left[\frac{n}{90} + \frac{90-n}{90}\left(\frac{60-n}{90-n}\right)\left(1-\varepsilon\right)\right] \\
&= \frac{2}{3} - \varepsilon\left[\sum_{n=0}^{60} p\left(n\right)\left(\frac{60-n}{90}\right)\right]
\end{aligned}$$

Note that for $\varepsilon > 0$ we have $U\left(f_{RG}, stay\right) > U\left(f_{BG}, switch\right)$ and $U\left(f_{BG}, stay\right) > U\left(f_{RG}, switch\right)$. DM weakly prefers $\left(f_{BG}, stay\right)$ over $\left(f_{RG}, stay\right)$ if and only if

$$30 \geq \sum_{n=0}^{60} p\left(n\right)n \equiv E(n)$$

holds. That is, DM will choose $\left(f_{BG}, stay\right)$ over $\left(f_{RG}, stay\right)$ if and only if the expected number of green balls is less than 30. It is easy to see that any pure or mixed strategy for N^M satisfying $E(n) \leq 30$ is consistent with subgame perfect equilibrium. In any Nash equilibrium, DM ends the game with f_{BG} and receives a payoff of $\frac{2}{3}$. For $\varepsilon > 0$, we can further state that DM must assign probability one to the pure strategy, $\left(f_{BG}, stay\right)$.

2.2. Game Γ_2: The Decision Maker Moves First

Here we consider a game with the following timing:

1. DM chooses either f_{BG} or f_{RG} $(t = 0)$.

2. N^M observes DM's choice and fills the urn with 30 red balls, n green balls, and $60 - n$ blue balls, where $n \in \{0, ..., 60\}$.

3. N^F draws a ball out of the urn at random, and announces whether it is green or not. If the announcement is "green," the game ends.

4. If the announcement is "not green," DM decides whether to **stay** with the chosen act or **switch** ($t = 1$).

The payoff structure is the same as in Γ_1. Now a mixed strategy for N^M is a probability distribution over the number of green balls after DM chooses f_{BG}, and after DM chooses f_{RG}. Without going into details, it is easy to see that equilibrium outcomes are the same as in Γ_1. That is, in any Nash equilibrium, DM ends the game with f_{BG} and receives a payoff of $\frac{2}{3}$. For $\varepsilon > 0$, we can further state that DM must assign probability one to the pure strategy, $(f_{BG}, stay)$.

2.3. Game Γ_3: Strategic Announcement; Urn is Filled Before the Switching Decision

Here we depart from the previous games by assuming that N^M is able to announce whether or not the ball is green *before* N^F draws the ball. By announcing that the ball is not green, this forces N^F to remove all of the green balls from the urn before drawing. Thus, Γ_3 has the following timing.

1. DM chooses either f_{BG} or f_{RG} ($t = 0$).

2. N^M observes DM's choice and fills the urn with 30 red balls, n green balls, and $60 - n$ blue balls, where $n \in \{0, ..., 60\}$.

3. N^M announces "green" or "not green." If the announcement is "green," the game ends.

4. If the announcement is "not green," N^F removes all of the green balls from the urn. Then N^F draws a ball out of the urn at random.

5. DM decides whether to **stay** with the chosen act or **switch** ($t = 1$).

The payoff structure is the same as in Γ_1. Subgame perfect equilibrium is characterized in the following proposition.

Proposition 1: *In any subgame perfect equilibrium of Γ_3, N^M announces "not* *green," and mixes over how it fills the urn, such that we have*

$$
\begin{aligned}
\textit{pr(blue ball drawn, } f_{BG} \textit{ subgame)} &= \frac{1-\varepsilon}{2}, \\
\textit{pr(blue ball drawn, } f_{RG} \textit{ subgame)} &= \frac{1+\varepsilon}{2}.
\end{aligned}
$$

DM's choice at $t = 0$ is arbitrary. At $t = 1$, DM mixes, staying with probability $\frac{1}{2}$ and switching with probability $\frac{1}{2}$. DM's payoff is $\frac{1-\varepsilon}{2}$.

Proof. Obviously, N^M strictly prefers to announce "not green" in all circumstances. Consider the subgame after DM chooses f_{BG} at $t = 0$. Because payoffs depend on $p(n)$ only through the induced probability of a blue ball being drawn, denoted by p_B, we characterize the equilibrium p_B. DM payoffs as a function of p_B are $U(f_{BG}, stay) = p_B$ and $U(f_{BG}, switch) = p_B(-\varepsilon) + (1 - p_B)(1 - \varepsilon)$. Letting S denote the probability that DM stays, N^M payoffs as a function of S are $S + (1 - S)(-\varepsilon)$ if a blue ball is drawn, and $(1 - S)(1 - \varepsilon)$ if a red ball is drawn. The solution must involve mixing, so by equating the payoff expressions, we have

$$p_B = \frac{1-\varepsilon}{2} \text{ and } S = \frac{1}{2}. \tag{2.1}$$

From (2.1), DM receives a payoff of $\frac{1-\varepsilon}{2}$.

Now consider the subgame after DM chooses f_{RG} at $t = 0$. We have $U(f_{RG}, stay) = (1-p_B)$ and $U(f_{RG}, switch) = p_B(1-\varepsilon)+(1-p_B)(-\varepsilon)$. N^M payoffs as a function of S are $(1 - S)(1 - \varepsilon)$ if a blue ball is drawn, and $S + (1 - S)(-\varepsilon)$ if a red ball is drawn. The solution must involve mixing, so by equating the payoff expressions, we have

$$p_B = \frac{1+\varepsilon}{2} \text{ and } S = \frac{1}{2}. \tag{2.2}$$

From (2.2), DM receives a payoff of $\frac{1-\varepsilon}{2}$. □

In Γ_3, DM mixes over whether to switch. If instead DM were to stay with f_{BG} with probability one, then N^M could choose $n = 60$, announce "not green," and guarantee that a red ball is selected. If DM were to switch from f_{BG} with probability one, then N^M could choose $n = 0$, announce "not green," and hold DM to a payoff below $\frac{1}{3}$. Either pure action by DM gives an advantage to N^M. The proof of Proposition 1 presumes that N^M can feasibly choose any probability of a blue ball being drawn. In fact, N^M cannot induce $p_B > \frac{2}{3}$, but this restriction does not bind. There are many subgame perfect equilibria yielding the probabilities specified in Proposition 1. For example, N^M could choose

$$\begin{aligned} p(0) &= \frac{3(1-\varepsilon)}{4} \text{ and } p(60) = \frac{1+3\varepsilon}{4} \text{ in the } f_{BG} \text{ subgame,} \\ p(0) &= \frac{3(1+\varepsilon)}{4} \text{ and } p(60) = \frac{1-3\varepsilon}{4} \text{ in the } f_{RG} \text{ subgame.} \end{aligned}$$

2.4. Game Γ_4: Strategic Announcement; Urn is Filled After the Switching Decision

In this game, N^M is given tremendous power to manipulate DM, by announcing "not green" and waiting until the switching decision to fill the urn. The timing in Γ_4 is the following.

1. DM chooses either f_{BG} or f_{RG} $(t = 0)$.
2. N^M announces "green" or "not green." If the announcement is "green," the game ends.
3. If the announcement is "not green," DM decides whether to **stay** with the chosen act or **switch** $(t = 1)$.
4. N^M observes DM's choices and fills the urn with 30 red balls, n green balls, and $60 - n$ blue balls, where $n \in \{0, ..., 60\}$.
5. If the announcement is "not green," N^F removes all of the green balls from the urn. Then N^F draws a ball out of the urn at random.

The payoff structure is the same as in Γ_1. Obviously, N^M announces "not green" in all circumstances. If DM either stays with or switches to f_{BG} at $t = 1$, then N^M chooses $p(60) = 1$ in the ensuing subgame, yielding DM a payoff of 0 or $-\varepsilon$ (depending on whether switching costs are incurred). If DM either stays with or switches to f_{RG} at $t = 1$, then N^M chooses $p(0) = 1$ in the ensuing subgame, yielding DM a payoff of $\frac{1}{3}$ or $\frac{1}{3} - \varepsilon$ (depending on whether switching costs are incurred). Therefore, in any subgame perfect equilibrium, DM winds up with f_{RG} at $t = 1$, and receives a payoff of $\frac{1}{3}$. If we impose a small switching cost, $\varepsilon > 0$, then DM must assign probability one to the pure strategy, $(f_{BG}, stay)$.

2.5. Game Γ_5: Random Opportunity for Strategic Announcement

In the previous games, $\Gamma_1 - \Gamma_4$, DM switches when the switching cost is zero (so the initial choice does not matter), or as part of a mixed strategy equilibrium in Γ_3. In the following game, every Nash equilibrium has DM playing the pure strategy $(f_{BG}, switch)$. This corresponds to the preferences Epstein and Schneider (2003) and Siniscalchi (2004) identify as ambiguity averse but dynamically inconsistent. If the DM views the problem as the following game against nature, Γ_5, it turns out that $(f_{BG}, switch)$ can be justified as dynamically consistent.

1. DM chooses either f_{BG} or f_{RG} ($t = 0$).

2a. With probability $\frac{1}{2}$, N^M observes DM's choice and fills the urn with 30 red balls, n green balls, and $60 - n$ blue balls, where $n \in \{0, ..., 60\}$.

3a. N^F draws a ball out of the urn at random. Then the game ends.

2b. With probability $\frac{1}{2}$, N^M observes DM's choice and announces "green" or "not green." If the announcement is "green," the game ends.

3b. If the announcement is "not green," DM decides whether to **stay** with the chosen act or **switch** ($t = 1$).

4b. N^M observes DM's choice and fills the urn with 30 red balls, n green balls, and $60 - n$ blue balls, where $n \in \{0, ..., 60\}$.

5b. If the announcement is "green," N^F removes all of the blue and red balls from the urn. If the announcement is "not green," N^F removes all of the green balls from the urn. Then N^F draws a ball out of the urn at random.

The payoff structure is the same as in Γ_1. This is an extensive game with perfect information, which can be solved by backwards induction. After the subgame, $(f_{BG}, 2a)$, the choice by N^M makes no difference, and the payoff is $\frac{2}{3}$. After the subgame, $(f_{RG}, 2a)$, the optimal choice by N^M is to fill the urn with blue balls, $p(0) = 1$, and the payoff is $\frac{1}{3}$. After the subgame, $(f_{BG}, 2b, not\ green, stay)$, the optimal choice by N^M is to fill the urn with green balls, $p(60) = 1$, so that all the green balls are removed, and the payoff is 0. After the subgame, $(f_{BG}, 2b, not\ green, switch)$, the optimal choice by N^M is to fill the urn with blue balls, $p(0) = 1$, and the payoff is $\frac{1}{3} - \varepsilon$. After the subgame, $(f_{RG}, 2b, not\ green, stay)$, the optimal choice by N^M is to fill the urn with blue balls, $p(0) = 1$, and the payoff is $\frac{1}{3}$. After the subgame, $(f_{RG}, 2b, not\ green, switch)$, the optimal choice by N^M is to fill the urn with green balls, $p(60) = 1$, so that all the green balls are removed, and the payoff is $-\varepsilon$.

Working backwards, after the subgame, $(f_{BG}, 2b, not\ green)$, DM strictly prefers to switch, yielding a payoff of $\frac{1}{3} - \varepsilon$. After the subgame, $(f_{RG}, 2b, not\ green)$, DM strictly prefers to stay, yielding a payoff of $\frac{1}{3}$. Therefore, at the initial node, choosing f_{BG} yields DM a payoff of $\frac{1}{2}(\frac{2}{3}) + \frac{1}{2}(\frac{1}{3} - \varepsilon) = \frac{1-\varepsilon}{2}$. Choosing f_{RG} yields DM a payoff of $\frac{1}{2}(\frac{1}{3}) + \frac{1}{2}(\frac{1}{3}) = \frac{1}{3}$. As long as ε is small, there is a unique subgame perfect equilibrium, in which DM chooses the pure strategy, $(f_{BG}, switch)$.

3. Discussion

In discussing the issue of dynamic consistency, one should be clear about why DM's choice at $t = 0$ matters. Indeed, we included the ε switching cost to rule out trivial examples in which DM switches because it is costless to do so, and the initial choice makes no difference. Our analysis indicates two situations in which a decision maker strictly prefers to switch. In Γ_3, DM is suspicious of the announcement, "not green," believing that the relative likelihood of red vs. blue is not yet settled. Thus, DM does not want to be predictable at $t = 1$. The initial choice is not important, but DM switches with positive probability, irrespective of the initial choice. In Γ_5, DM believes that there is a chance that there will be no opportunity to switch, so he/she strictly prefers the static ambiguity-averse choice, f_{BG}, at $t = 0$. After observing "not green," DM is suspicious of the announcement, believing that the relative likelihood of red vs. blue will not be settled until after his/her switching decision. Thus, DM switches to the new ambiguity-averse choice, f_{RG}, at $t = 1$.

The usefulness of our approach extends beyond environments in which the decision maker consciously believes that he/she is or might be playing a game against nature. Just as a game against nature can be a metaphor for ambiguity aversion in static problems, it can serve the same role in dynamic problems. By analyzing games for which seemingly inconsistent behavior arises in equilibrium, we can better understand the source of the inconsistency. Thus, we see our approach as complementary to the axiomatic approach. Although the set of games against nature is endless, one could experimentally test which games work well in which environments.

We do not pretend to offer a complete theory about ambiguity aversion. Many different phenomena yield behavior one might call ambiguity aversion. This paper is based on uncertainty about the exact "decision problem," as presented by the experimenter, the market, the House, or nature. Ambiguity aversion can be seen as a healthy skepticism about the existence of other interested parties. We readily admit that there are situations in which ambiguity aversion arises out of psychological concerns. For example, ambiguity might create stress or cause sleep loss, creating harm that can be physically measured. Even these situations could be amenable to a game theoretic approach,[5] but it might be more useful to take ambiguity-averse preferences at face value. Within the framework of games against nature, it is conceivable that the game is not zero sum. If the decision maker views nature as benevolent, then ambiguity-loving preferences could be observed.

References

[1] Ellsberg, D., "Risk, Ambiguity, and the Savage Axioms," Quarterly Journal of Economics, 75(4), 1961, 643-669.

[2] Epstein, L. G. and M. Le Breton, "Dynamically Consistent Beliefs Must be Bayesian," Journal of Economic Theory, 61, 1993, 1-22.

[3] Epstein, L. G. and M. Schneider, "Recursive Multiple-priors," Journal of Economic Theory, 113, 2003, 1-31.

[4] Hanany, E. and P. Klibanoff, "Updating Preferences with Multiple Priors," March 2005.

[5] Maccheroni, F., M. Marinacci and R. Rustichini, "Ambiguity Aversion, Malevolent Nature and the Variational Representation of Preferences," September, 2004.

[6] Machina, M. J., "Dynamic Consistency and Non-Expected Utility Models of Choice Under Uncertainty," Journal of Economic Literature, 27(4), 1989, 1622-1668.

[7] Siniscalchi, M., "Dynamic Choice under Ambiguity," September 2004.

[5] Perhaps the decision maker can be modelled as a principal dealing with an agent, rather than as a game against nature.