Dynamic Competition with Random Demand and Costless Search:
A Theory of Price Posting
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Online Appendix

1 Proof of Proposition 4

Proposition 4. *Suppose that the regularity condition*

$$\frac{\partial^2}{\partial p \partial \alpha} D_t(p, \alpha_t) < 0$$

holds for all α_t and all $c \leq p \leq \alpha_t$. Then if there is a single demand parameter, i.e. $T = 2$, there is a unique sequential equilibrium.

Proof of Proposition 4.
Since there is only one generation of consumers (in period 1), we suppress the period subscript whenever no confusion can arise. We prove Proposition 4 through a series of claims.

Claim 1. *In any equilibrium, we have $a_1^{\max} > 0$.*

Proof. First, we show that in any equilibrium we must have $q^* > 0$. Suppose to the contrary that we had $q^* = 0$. Then by posting a price $p' \in (c, \bar{v})$ in period 1, an inactive firm would be sure to sell, making the deviation profitable, a contradiction to equilibrium. Hence we must have $q^* > 0$.

Next, suppose to the contrary to the statement of Claim 1 we had $a_1^{\max} = 0$. We will argue that $p_2(a_2; 0)$ then cannot be constant on $[0, q^*]$. If $p_2(a_2; 0)$ were constant on $[0, q^*]$, then in period 2 unit q^* would sell in state $\underline{\alpha}$, or else firms could increase revenues by setting $p_2(a_2; 0) - \varepsilon$ for sufficiently small ε. Since q^* sells in all states, if $p_2(a_2; 0) < \bar{v}$ held, a firm could increase revenues by raising its price and continuing to sell in all states $\alpha > \underline{\alpha}$. If $p_2(a_2; 0) = \bar{v}$ held, a firm not producing could profitably enter, establishing a contradiction.

Thus a range of prices is offered in period 2, i.e. $p_2(q^*; 0) > p_2(0; 0)$ must hold. But then a firm posting $p_2(0; 0)$ in period 2 would have a profitable deviation to post $p_2(0; 0) + \delta(\bar{v})/2$ in period 1, because a range of consumers including type $\bar{v}$ would be willing to purchase at that price. This contradiction establishes that $a_1^{\max} > 0$.■

Claim 2. *In any equilibrium, there exists $\widetilde{\alpha} \leq \bar{\alpha}$ and a C_1 function $\bar{a}_1(\alpha)$ defined on $[\underline{\alpha}, \widetilde{\alpha}]$ satisfying $\frac{d\bar{a}_1}{d\alpha} > 0$ and $\bar{a}_1(\widetilde{\alpha}) = a_1^{\max}$.*

Proof. First, we prove the result if ψ_1^v is a simple function for every v, i.e. the range of ψ_1^v consists of finitely many values. Let q denote a position in the period 1 queue, so that $0 \leq q \leq D(0, \alpha)$. Also let

$$\mu_1^v(\alpha) = -\frac{\frac{\partial D(v,\alpha)}{\partial p}}{D(0,\alpha)}$$

denote the proportion of type $(v, 1)$ customers present in period 1 when the state is α. By the strong law of large numbers, the proportion of type $(v, 1)$ customers in any interval of length Δq in the queue when the state is α equals $\mu_1^v(\alpha)$. Hence on any interval on which $\psi_1^v(a_1)$ is constant in a_1, we have

$$\frac{\Delta a_1}{\Delta q} = \int_{\underline{v}}^{\bar{v}} \mu_1^v(\alpha)\psi_1^v(a_1)dv \tag{1}$$

for Δq sufficiently small. Note that because $a_1 < a_1^{\max}$ the right side of (1) must be strictly positive. Upon taking limits as $\Delta q \to 0$, and separating the resulting differential equation by variables, we obtain

$$dq = \frac{da_1}{\int_{\underline{v}}^{\bar{v}} \mu_1^v(\alpha)\psi_1^v(a_1)dv} \tag{2}$$

For any α that results in $\bar{a}_1(\alpha) < a_1^{\max}$, integrating (2) and dividing by $D(0, \alpha)$, yields

$$1 = \int_0^{\bar{a}_1(\alpha)} \frac{da_1}{\int_{\underline{v}}^{\bar{v}} -\frac{\partial D(v,\alpha)}{\partial p}\psi_1^v(a_1)dv} \tag{3}$$

Totally differentiating this expression w.r.t α yields

$$\frac{1}{\int_{\underline{v}}^{\bar{v}} -\frac{\partial D(v,\alpha)}{\partial p}\psi_1^v(a_1)dv}\frac{d\bar{a}_1(\alpha)}{d\alpha} - \int_0^{\bar{a}_1(\alpha)} \frac{\int_{\underline{v}}^{\bar{v}} -\psi_1^v(a_1)\frac{\partial^2 D(v,\alpha)}{\partial p \partial \alpha}dv}{\left[\int_{\underline{v}}^{\bar{v}} -\frac{\partial D(v,\alpha)}{\partial p}\psi_1^v(a_1)dv\right]^2}da_1 = 0 \tag{4}$$

By assumption we have $\frac{\partial^2 D(v,\alpha)}{\partial p \partial \alpha} < 0$ whenever $v \geq c$. Furthermore, since firm optimality requires $p_1(a_1) \geq c$, consumer optimality requires $\psi_1^v(a_1) = 0$ for all $v < c$. It follows that the second term in (4) is strictly positive, and hence that $\bar{a}_1(\alpha)$ is strictly increasing in α. Thus we may define $\beta_1 = \bar{a}_1^{-1}$.

Next, for general measurable $\psi_1^v(a_1)$, there exist simple functions $\psi_1^{v,n}(a_1) \leq \psi_1^v(a_1)$, such that $\psi_1^{v,n}(a_1) \uparrow \psi_1^v(a_1)$ for every a_1. Hence for each n, there exists a strictly increasing function $\bar{a}_1^n(\alpha)$ solving the analogue of (4), with inverse function $\beta_1^n(a_1)$. Furthermore, from (4) we have

$$\frac{d\beta_1^n}{da_1}(\bar{a}_1^n(\alpha)) = \left(\left(\int_{\underline{v}}^{\bar{v}} -\frac{\partial D(v,\alpha)}{\partial p}\psi_1^{v,n}(\bar{a}_1^n(\alpha))dv\right)\int_0^{\bar{a}_1^n(\alpha)} \frac{\int_{\underline{v}}^{\bar{v}} -\psi_{n1}^v(a_1)\frac{\partial^2 D(v,\alpha)}{\partial p \partial \alpha}}{\left[\int_{\underline{v}}^{\bar{v}} -\frac{\partial D(v,\alpha)}{\partial p}\psi_{n1}^v(a_1)dv\right]^2}da_1\right)^{-1}$$

Upon taking limits as $n \to \infty$, we therefore obtain

$$\frac{d\beta_1}{da_1}(\bar{a}_1(\alpha)) = \left(\left(\int_{\underline{v}}^{\bar{v}} -\frac{\partial D(v,\alpha)}{\partial p}\psi_1^v(\bar{a}_1(\alpha))dv\right)\int_0^{\bar{a}_1(\alpha)} \frac{\int_{\underline{v}}^{\bar{v}} -\psi_1^v(a_1)\frac{\partial^2 D(v,\alpha)}{\partial p \partial \alpha}}{\left[\int_{\underline{v}}^{\bar{v}} -\frac{\partial D(v,\alpha)}{\partial p}\psi_1^v(a_1)dv\right]^2}da_1\right)^{-1}$$

so β_1 is a strictly increasing function. It follows that for $a_1^r < a_1^{\max}$, the state $\alpha \in [\underline{\alpha}, \widetilde{\alpha})$ is revealed. ■

Extend $\bar{a}_1(\alpha)$ to equal $a_1^{\max}$ for $\alpha \in [\widetilde{\alpha}, \bar{\alpha}]$. Also, let $\bar{a}_1(v, \alpha)$ denote the measure of period-1 sales to consumers with valuations greater than or equal to v in state α. Then the set of market clearing prices on residual demand, denoted as $P^*(\alpha)$, is given by:

$$P^*(\alpha) = \{p \leq \bar{v} : D(p, \alpha) - \bar{a}_1(p, \alpha) = q^* - \bar{a}_1(\alpha)\}$$

if residual supply is positive and the above set is nonempty, $P^*(\alpha) = \{0\}$ if $D(p, \alpha) - \bar{a}_1(p, \alpha) < q^* - \bar{a}_1(\alpha)$ for all p, and we adopt the convention that $P^*(\alpha) = \{\bar{v}\}$ if $\bar{a}_1(\alpha) = q^*$.

Claim 3. *We either have (i) $\widetilde{\alpha} = \bar{\alpha}$ or (ii) there exists p^* such that $P^*(\alpha) = \{p^*\}$ for almost all $\alpha \in [\widetilde{\alpha}, \bar{\alpha}]$.*

Proof. First we argue that for all $\alpha > \widetilde{\alpha}$, with the possible exception of a single state, $P^*(\alpha)$ is single-valued. From Claim 2, it follows that there are consumers of all valuations who have not yet been released from the queue when $a_1^{\max}$ units have been sold. Therefore, the residual demand curve is downward sloping and never vertical for prices between $\underline{v}$ and $\bar{v}$. For prices below $\underline{v}$, residual demand is vertical at the quantity $D(0, \alpha) - a_1^{\max}$. Thus, $P^*(\alpha)$ is single-valued, with the only exception being states in which we have $D(0, \alpha) = q^*$, which occurs for at most one state.

Suppose that the claim is false, so that we have $\widetilde{\alpha} < \bar{\alpha}$ and $P^*(\alpha)$ is not almost-everywhere constant on the interval $[\widetilde{\alpha}, \bar{\alpha}]$. Consider the event E in which we have $\bar{a}_1(\alpha) = a_1^{\max}$. Note that E has probability $1 - F(\widetilde{\alpha}) > 0$. Also, let E' denote the set of states in E for which $P^*(\alpha)$ is single-valued.

If $a_1^{\max} = q^*$, then following event E no output remains to be sold in period 2. By convention we then have $P^*(\alpha) = \bar{v}$ for all $\alpha \in E$, contradicting the supposition that $P^*(\alpha)$ is not almost-everywhere constant.

If $a_1^{\max} < q^*$, then we claim that the period 2 equilibrium price function following the event E must be nondegenerate: a positive measure of the remaining output is priced higher than $p_2(0; a_1^{\max})$. This is because if $p_2(a_2; a_1^{\max})$ was a constant function, taking on the value $\widehat{p}$ for all $a_2 \in [0, q^* - a_1^{\max}]$, then all of the output must sell in period 2 with probability one. Otherwise a firm that posted $\widehat{p}$ could gain by posting a marginally lower price, as this would guarantee a sale with probability one. Note that the period 2 quantity demanded at the price $\widehat{p}$ is greater than or equal to $q^* - a_1^{\max}$ in states where $P^*(\alpha)$ is single-valued if and only if $\widehat{p} \leq P^*(\alpha)$, so that we must have $\widehat{p} \leq \min_{\alpha \in E'} P^*(\alpha)$. However, since $P^*(\alpha)$ is not almost-everywhere constant, there then is a positive probability of excess demand at the price $\widehat{p}$. But then a firm that never sells in period 1 can profitably deviate to posting a period 1 price marginally above $\widehat{p}$, as it would be sure to sell at this price in event E. To see why this is true, consider any consumer with $v > \widehat{p}$. If this consumer arrives at the market in period 1 and faces $\widehat{p} + \varepsilon$ as the lowest remaining price, she will surely purchase when ε is sufficiently small, since her expected payoff from waiting is strictly less than $v - \widehat{p}$, due to the positive probability of being rationed. The profitable deviation contradicts equilibrium, so we have established that that whenever $a_1^{\max} < q^*$ the period-2 pricing function must be non-degenerate.

Suppose that the firm posting the price $p_2(0; a_1^{\max})$ in period 2 (who must never be selling in period 1) deviates to posting the price $p_2(0; a_1^{\max}) + \varepsilon$ in period 1, for some for some $\varepsilon > 0$. Consider any consumer with $v > p_2(0; a_1^{\max})$. If this consumer arrives at the market in period 1 and faces $p_2(0; a_1^{\max}) + \varepsilon$ as the lowest remaining price, she will surely purchase when ε is sufficiently small, since her expected payoff from waiting is strictly less than $v - p_2(0; a_1^{\max})$, because the period-2 pricing function is non-degenerate. Consequently, the deviating firm would be sure to sell in period 1 in event E, thereby securing a net expected revenue strictly greater than $p_2(0; a_1^{\max})$, which contradicts equilibrium when we have $a_1^{\max} < q^*$.

We have shown that the supposition that Claim 3 is false leads to a contradiction, establishing the desired result. ■

Claim 4. *In period 2, all remaining output is allocated efficiently. For $\alpha < \widetilde{\alpha}$, all firms set the same market clearing price, $p_2(\alpha) \in P^*(\alpha)$. For $\alpha \geq \widetilde{\alpha}$, all firms post the same price, p^*.*

Proof. First consider the case in which $\alpha < \widetilde{\alpha}$ holds. We have $\bar{a}_1(\alpha) < a_1^{\max}$, so period-1 activity reveals the state and therefore, $P^*(\alpha)$. Consider the lowest period-2 posted price, $p_2(0; \bar{a}_1(\alpha))$. We must have $p_2(0; \bar{a}_1(\alpha)) \geq \min[p : p \in P^*(\alpha)]$, because a firm posting a price of $\min[p : p \in P^*(\alpha)]$ is guaranteed to sell, no matter what prices are posted by the other firms in period 2. Consider the highest period-2 posted price in state α, which we can write as $p_2(q^* - \bar{a}_1(\alpha); \bar{a}_1(\alpha))$. If the measure of output posted at this price is zero, then the firm posting $p_2(q^* - \bar{a}_1(\alpha); \bar{a}_1(\alpha))$ does not sell in state α, since the measure of consumers with valuation at least $\min[p : p \in P^*(\alpha)]$ is $q^* - \bar{a}_1(\alpha)$, so all of the residual demand has been exhausted, a contradiction.[1] If the measure of output posted at the price $p_2(q^* - \bar{a}_1(\alpha); \bar{a}_1(\alpha))$ is positive, then either none of the output posted at this price sells, a contradiction, or a positive measure of the output posted at this price sells in state α. If a positive measure sells, then the firm posting $p_2(0; \bar{a}_1(\alpha))$ has a profitable deviation to slightly undercut $p_2(q^* - \bar{a}_1(\alpha); \bar{a}_1(\alpha))$, unless the latter price equals the former. Therefore, we have shown that all firms post the same price in period 2, which we denote as $p_2(\alpha)$. We cannot have $p_2(\alpha) > \max[p : p \in P^*(\alpha)]$, because not all the output

[1] The contradiction occurs if $p_2(0; \bar{a}_1(\alpha)) > 0$, since positive revenues are possible. If $p_2(0; \bar{a}_1(\alpha)) = 0$ occurs, then residual supply must exceed residual demand, and all transactions must occur at a price of 0. Posted prices on the excess supply are irrelevant, and we identify all equilibria that differ only on this irrelevant dimension.

would sell, and a firm could increase expected revenues by slightly undercutting the price. Thus, we have $p_2(\alpha) \in P^*(\alpha)$.

Next consider the case in which $\alpha \geq \widetilde{\alpha}$ holds (event E occurs). From Claim 3, we know that for almost all $\alpha \in E$, there exists a unique market clearing price, p^*. Also, p^* cannot be zero, because the argument in Claim 3 would allow us to conclude that $D(0,\alpha) < q^*$ holds for all $\alpha < \bar{\alpha}$; sequential rationality on the part of firms requires all period-2 transactions to take place at a price of zero for all $\alpha < \bar{\alpha}$, but then no consumer would pay a positive price in period 1 either, contradicting the requirement that firms receive an expected revenue of at least c. Thus, we conclude that $p^* > \underline{v}$ holds, and residual demand at p^* equals residual supply for almost all $\alpha \in E$. We must have $p_2(0; a_1^{\max}) \geq p^*$, because a firm posting this price sells with probability one, no matter what prices are posted by the other firms in period 2. Consider the highest period-2 posted price in event E, which we can write as $p_2(q^* - a_1^{\max}; a_1^{\max})$. If $p_2(q^* - a_1^{\max}; a_1^{\max}) > p_2(0; a_1^{\max})$ holds and the measure of output posted at the higher price is zero, then the firm posting $p_2(q^* - a_1^{\max}; a_1^{\max})$ does not sell for almost all $\alpha \in E$. This is because residual demand at p^* equals residual supply for almost all $\alpha \in E$, and is exhausted at that price, a contradiction. If, on the other hand, $p_2(q^* - a_1^{\max}; a_1^{\max}) > p_2(0; a_1^{\max})$ holds and the measure of output posted at $p_2(q^* - a_1^{\max}; a_1^{\max})$ is positive, then either (i) none of the output posted at this price sells for almost all $\alpha \in E$, in which case a firm posting this price has a profitable deviation to post p^* instead, a contradiction, or (ii) a positive measure of the output posted at $p_2(q^* - a_1^{\max}; a_1^{\max})$ sells for a positive-measure subset of E. In the latter case, a firm posting $p_2(q^* - a_1^{\max}; a_1^{\max})$ has a profitable deviation to undercut its price slightly, increasing its probability of selling from less than one to one for all states in this positive-measure subset of E. Therefore, we must have $p_2(q^* - a_1^{\max}; a_1^{\max}) = p_2(0; a_1^{\max})$, so all firms charge the same price. To see that this price equals p^*, we must rule out all firms posting the same price, p', that exceeds p^*. Since p^* is the unique market clearing price for almost all $\alpha \in E$, it follows that a firm posting p' sells with probability strictly less than one, due to rationing of excess supply. By slightly undercutting the price p', this firm increases its probability of selling in event E to one, a contradiction. ■

Henceforth we will let $p^*(\alpha)$ denote the price that all firms post in period 2 in state α, where $p^*(\alpha)$ can be an arbitrary selection from $P^*(\alpha)$ for $\alpha < \widetilde{\alpha}$ and $p^*(\alpha) = p^*$ for $\alpha \geq \widetilde{\alpha}$.

Claim 5. *(Martingale Property) For every $a_1 \leq a_1^{\max}$, we have*

$$p_1(a_1) = \frac{\int_{\beta_1(a_1)}^{\bar{\alpha}} p^*(\alpha) f(\alpha) d\alpha}{\int_{\beta_1(a_1)}^{\bar{\alpha}} f(\alpha) d\alpha}. \tag{5}$$

Proof. Consider a firm that sets the price, $p_1(a_1)$. This firm sells in period

1 if and only if $\bar{a}_1(\alpha) \geq a_1$. Hence, its probability of selling in period 1 is

$$\pi_1(a_1) = \int_{\beta_1(a_1)}^{\bar{\alpha}} f(\alpha)d\alpha.$$

When $\bar{a}_1(\alpha) < a_1$, this firm sells in period 2 at the market clearing price, $p^*(\alpha)$. Hence, expected revenue (as of the beginning of period 1) equals

$$p_1(a_1)\pi_1(a_1) + \int_{\underline{\alpha}}^{\beta_1(a_1)} p^*(\alpha)f(\alpha)d\alpha.$$

Meanwhile, consider the firm that posts $p_1(a_1^{\max})$ in period 1, and consider two cases. If $a_1^{\max} < q^*$ holds, then this firm sells in period 1 with probability zero, so its expected revenue is

$$\int_{\underline{\alpha}}^{\bar{\alpha}} p^*(\alpha)f(\alpha)d\alpha. \tag{6}$$

Since expected revenue must be equated across firms in equilibrium, we have

$$\begin{aligned} &p_1(a_1)\pi_1(a_1) + \int_{\underline{\alpha}}^{\beta_1(a_1)} p^*(\alpha)f(\alpha)d\alpha \\ &= \int_{\underline{\alpha}}^{\beta_1(a_1)} p^*(\alpha)f(\alpha)d\alpha + \int_{\beta_1(a_1)}^{\bar{\alpha}} p^*(\alpha)f(\alpha)d\alpha, \end{aligned}$$

which implies

$$p_1(a_1)\int_{\beta_1(a_1)}^{\bar{\alpha}} f(\alpha)d\alpha = \int_{\beta_1(a_1)}^{\bar{\alpha}} p^*(\alpha)f(\alpha)d\alpha.$$

Thus, the martingale condition (5) holds.

For the case, $a_1^{\max} = q^*$, the firm posting $p_1(a_1^{\max}) = \bar{v}$ in period 1 will sell in period 1 whenever $\alpha \geq \widetilde{\alpha}$. Expected revenues are

$$\bar{v}\int_{\widetilde{\alpha}}^{\bar{\alpha}} f(\alpha)d\alpha + \int_{\underline{\alpha}}^{\widetilde{\alpha}} p^*(\alpha)f(\alpha)d\alpha.$$

Under our convention that $P^*(\alpha) = \{\bar{v}\}$ if $\bar{a}_1(\alpha) = q^*$, we have $p^*(\alpha) = \bar{v}$ for $\alpha \geq \widetilde{\alpha}$. Expected revenues are given by (6), so the martingale condition (5) follows. ■

Claim 6. *We have $p^*(\alpha) = P(q^*, \alpha)$ for all α.*

Proof. Since $p_1(a_1)$ is non-decreasing, the right side of (5) must also be non-decreasing in a_1. Therefore, we must have $\max_\alpha p^*(\alpha) = p^*$. Let $\widetilde{v}$ denote the lowest valuation type that purchases in period 1. That is,

$$\widetilde{v} = \inf\{v : \psi_1^v(a_1) > 0 \text{ for some } a_1 \leq a_1^{\max}\}.$$

Consider the following two cases:

Case 1: $\widetilde{v} \geq p^*$.

In Case 1, all consumers who purchase in period 1 have valuation greater than or equal to $p^*(\alpha)$, so rationing is efficient and we have $p^*(\alpha) = P(q^*, \alpha)$ for all $\alpha < \widetilde{\alpha}$ and almost all $\alpha \in E$.[2] Since we have $p^*(\alpha) = p^*$ for all $\alpha \in E$ and $P(q^*, \alpha)$ is strictly increasing in α, it follows that under Case 1 the interval E must be degenerate, i.e., $\widetilde{\alpha} = \bar{\alpha}$ holds. From the argument of Claim 4, $p^*(\bar{\alpha}) = P(q^*, \bar{\alpha})$ must hold as well.

Case 2: $\widetilde{v} < p^*$.

We start by calculating $\bar{a}_1(v, \alpha)$. Let $a(v, q, \alpha)$ denote the measure of period 1 sales made to consumers with valuations $v' \geq v$ in state α when position q in the queue has been reached. Analogously to the derivation of (1) we have

$$\frac{da}{dq}(v, q, \alpha) = \int_{v}^{\bar{v}} \psi(v'; a)\mu_1^{v'}(\alpha)dv' \tag{7}$$

Let $a(q, \alpha)$ denote the measure of period 1 sales made in state α when position q in the queue has been reached. It follows from (1) that

$$\frac{da}{dq}(q, \alpha) = \int_{\underline{v}}^{\bar{v}} \psi(v'; a)\mu_1^{v'}(\alpha)dv' \tag{8}$$

Combining (7) and (8) yields

$$\frac{da(v, q, \alpha)}{da(q, \alpha)} = \frac{\int_{v}^{\bar{v}} \psi(v'; a)\frac{\partial}{\partial p}D(v', \alpha)dv'}{\int_{\underline{v}}^{\bar{v}} \psi(v'; a)\frac{\partial}{\partial p}D(v', \alpha)dv'}$$

and hence that

$$\bar{a}_1(v, \alpha) = \int_0^{\bar{a}_1(\alpha)} \frac{\int_{v}^{\bar{v}} \psi(v'; z)\frac{\partial}{\partial p}D(v', \alpha)dv'}{\int_{\underline{v}}^{\bar{v}} \psi(v'; z)\frac{\partial}{\partial p}D(v', \alpha)dv'}dz \tag{9}$$

It follows from (3) and (9) that for all α and all $v \leq \widetilde{v}$

$$\bar{a}_1(\alpha) - \bar{a}_1(v, \alpha) = \int_0^{\bar{a}_1(\alpha)} \left(\frac{\int_{\underline{v}}^{v} \psi_1^{v'}(z)\frac{-\partial}{\partial p}D(v', \alpha)dv'}{\int_{\underline{v}}^{\bar{v}} \psi_1^{v'}(z)\frac{-\partial}{\partial p}D(v', \alpha)dv'} \right) dz = 0 \tag{10}$$

The final equality follows because $\psi_1^{v'}(a_1) = 0$ for all $v' < \widetilde{v}$ and all a_1.

Define $\widetilde{A}_+(a_1) \equiv \{\alpha \geq \beta_1(a_1) : p^*(\alpha) > \widetilde{v}\}$. Notice that we must also have $P(q^*, \alpha) > \widetilde{v}$ for almost all $\alpha \in \widetilde{A}_+(a_1)$. With $P(q^*, \alpha) \leq \widetilde{v}$, we would have $D(\widetilde{v}, \alpha) \leq q^*$, and from (10), $D(\widetilde{v}, \alpha) - \bar{a}_1(\widetilde{v}, \alpha) \leq q^* - \bar{a}_1(\alpha)$, implying a price

[2] There is at most one state for which $P(q^*, \alpha)$ is the entire interval $[0, \underline{v}]$, leading to a trivial sort of multiple equilibria, based on the period-2 price in this state. We identify all equilibria that differ only on which market-clearing price is posted in period 2 in this state.

clearing the residual market in period 2 that is weakly below $\widetilde{v}$. This cannot occur because $p^*(\alpha)$ is the market clearing price for almost all α, and $p^*(\alpha) > \widetilde{v}$ holds for all $\alpha \in \widetilde{A}_+(a_1)$. Therefore, we have

$$p^*(\alpha) \geq P(q^*, \alpha) > \widetilde{v} \quad \text{for almost all } \alpha \in \widetilde{A}_+(a_1) \tag{11}$$

Similarly, define $\widetilde{A}_-(a_1) \equiv \{\alpha \geq \beta_1(a_1) : p^*(\alpha) \leq \widetilde{v}\}$. Therefore, for almost all $\alpha \in \widetilde{A}_-(a_1)$, output is rationed efficiently,

$$p^*(\alpha) = P(q^*, \alpha) \leq \widetilde{v} \;\text{ for almost all } \alpha \in \widetilde{A}_-(a_1). \tag{12}$$

Denote the utility of purchasing in period 1, net of the utility of waiting, by $\Delta(v, a_1)$. Consider now a sequence (v^n, a_1^n) such that $\psi_1^{v^n}(a_1^n) > 0$, and such that $v^n \downarrow \widetilde{v}$. Since $[0, a_1^{\max}]$ is compact, the sequence a_1^n has a convergent subsequence, whose limit we shall denote by $\widetilde{a}_1$. Renumbering the subsequence, if necessary, we may assume that $a_1^n \to \widetilde{a}_1$. Since $\psi_1^{v^n}(a_1^n) > 0$, we must have $\Delta(v^n, a_1^n) \geq 0$. Because Δ is a continuous function, it then follows that $\Delta(\widetilde{v}, \widetilde{a}_1) \geq 0$.

Let the beliefs of type $\widetilde{v}$, conditional on arriving at the queue in period 1 when the measure of transactions is a_1, be denoted by $f(\alpha|\widetilde{v}, a_1)$. Then we have

$$\begin{aligned}
\Delta(\widetilde{v}, \widetilde{a}_1) &= \delta(\widetilde{v}) - \frac{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} p^*(\alpha) f(\alpha) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha) d\alpha} + \frac{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} \min\{p^*(\alpha), \widetilde{v}\} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha} \\
&\leq \delta(\widetilde{v}) - \frac{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha) d\alpha} + \frac{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} \min\{p^*(\alpha), \widetilde{v}\} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha} \\
&= \delta(\widetilde{v}) - \frac{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha) d\alpha} + \\
&\quad \frac{\int_{\widetilde{A}_-(\widetilde{a}_1)} \min\{p^*(\alpha), \widetilde{v}\} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha + \int_{\widetilde{A}_+(\widetilde{a}_1)} \min\{p^*(\alpha), \widetilde{v}\} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha}
\end{aligned}$$

Using (11) and (12), we have

$$\begin{aligned}
\Delta(\widetilde{v}, \widetilde{a}_1) &\leq \delta(\widetilde{v}) - \frac{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha) d\alpha} + \\
&\quad \frac{\int_{\widetilde{A}_-(\widetilde{a}_1)} P(q^*, \alpha) f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha + \int_{\widetilde{A}_+(\widetilde{a}_1)} \widetilde{v} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha} \\
&< \delta(\widetilde{v}) - \frac{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha) d\alpha} + \frac{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} P(q^*, \alpha) f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha}{\int_{\beta_1(\widetilde{a}_1)}^{\bar{\alpha}} f(\alpha|\widetilde{v}, \widetilde{a}_1) d\alpha}.
\end{aligned}$$

Because type $\widetilde{v}$ is the lowest valuation that buys in period 1, $f(\alpha)$ dominates $f(\alpha|\widetilde{v},\widetilde{a}_1)$ in the monotone likelihood ratio order, and hence in the order of first order stochastic dominance. Since $P(q^*,\alpha)$ is increasing in α, we have

$$\Delta(\widetilde{v},\widetilde{a}_1) < \delta(\widetilde{v}).$$

We have shown above that $\Delta(\widetilde{v},\widetilde{a}_1) \geq 0$, which implies $\delta(\widetilde{v}) > 0$. Therefore, $\widetilde{v} > \widehat{v}$. Since $\bar{\alpha} \in \widetilde{A}_+(\widetilde{a}_1)$, it follows from (11) that $P(q^*,\bar{\alpha}) > \widetilde{v}$. Thus, we have

$$P(q^*,\bar{\alpha}) > \widetilde{v} > \widehat{v} \geq P(q^*,\bar{\alpha}),$$

a contradiction. Therefore, Case 2 cannot arise. ■

Claim 7. The equilibrium quantity is the efficient quantity, $q^* = q^e$.

Proof. From Claims 5 and 6, we have

$$p_1(0) = \int_{\underline{\alpha}}^{\bar{\alpha}} P(q^*,\alpha) f(\alpha) d\alpha. \tag{13}$$

If $q^* > q^e$ holds, then we have $p_1(0) < c$, and the firm posting $p_1(0)$ has a profitable deviation not to produce. If $q^* < q^e$ holds, then we have $p_1(0) > c$. If a firm not producing deviates and produces, this has a negligible effect on q^*, so sequential rationality and the previous claims implies that the pricing function following a unilateral deviation satisfies (13). Thus, a firm not producing could produce, post the price $p_1(0)$, and receive positive profits. ■

Lemmas 11-13 establish the existence of a unique cutoff equilibrium satisfying $v^*(a_1) \geq \hat{v}$, for the subgame following the output choice, q^e. This completes the proof of Proposition 4.

2 Proof of Proposition 5

Proposition 5. *For the model with multiplicative uncertainty, there is an equilibrium that is Pareto optimal, characterized as follows:*

(i) $q^* = q^e$

(ii) If the state equals $(\alpha_1, ..., \alpha_{T-1})$, then $a_t^r = \bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r)$ for all $t = 1, ..., T-1$, where

$$\bar{a}_t(\alpha_t; a_1^r, ..., a_{t-1}^r) = q_{t-1}^L(a_1^r, ..., a_{t-1}^r) + \alpha_t D_t(\bar{p}_t(a_1^r, ..., a_{t-1}^r))$$

(iii) For all t, all $t' \leq t$, and all equilibrium private histories a_t^p we have $\psi_t^{v,t'}(a_t; a_1^r, ..., a_{t-1}^r, a_t^p) = 1$ if and only if $v \geq \bar{p}_t(a_1^r, ..., a_{t-1}^r)$;

(iv) For all $t = 1, ..., T$, and all histories $(a_1^r, ..., a_{t-1}^r)$ we have

$$p_t(a_t; a_1^r, ..., a_{t-1}^r) = E[P(q^e, \alpha_1, ..., \alpha_{T-1})|\alpha_1 = \alpha_1^r, ..., \alpha_{t-1} = \alpha_{t-1}^r, \alpha_t \geq \beta_t(a_t; a_1^r, ..., a_{t-1}^r)]$$

where

$$\beta_t(a_t; a_1^r, ..., a_{t-1}^r) = \begin{cases} \underline{\alpha}_t, & \text{if } a_t \leq q_{t-1}^L(a_1^r, ..., a_{t-1}^r) + \underline{\alpha}_t D_t(\bar{p}_t(a_1^r, ..., a_{t-1}^r)) \\ \frac{a_t - q_{t-1}^L(a_1^r, ..., a_{t-1}^r)}{D_t(\bar{p}_t(a_1^r, ..., a_{t-1}^r))}, & \text{if } a_t \geq q_{t-1}^L(a_1^r, ..., a_{t-1}^r) + \underline{\alpha}_t D_t(\bar{p}_t(a_1^r, ..., a_{t-1}^r)) \end{cases}$$

(v) For all $t = 1, ..., T$, and all histories $(a_1^r, ..., a_{t-1}^r)$ we have

$$q_{t-1}^L(a_1^r, ..., a_{t-1}^r) = \sum_{\tau=1}^{t-1} \alpha_\tau^r [D_\tau(\bar{p}_t(a_1^r, ..., a_{t-1}^r)) - D_\tau(\bar{p}_{t-1}(a_1^r, ..., a_{t-2}^r))]$$

Proof of Proposition 5. First, note that, if consumers behave according to (iii) and firms behave according to (iv), then $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$ and $q_{t-1}^L(a_1^r, ..., a_{t-1}^r)$ are well defined. The reason is that purchases by newly active consumers in period t in state α_t are

$$\alpha_t D_t(P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})), \tag{14}$$

and purchases by consumers who became active in previous periods, q_{t-1}^L, are by those with valuations between $P(q^e, \alpha_1^r, ..., \alpha_{t-2}^r, \bar{\alpha}_{t-1}, ..., \bar{\alpha}_{T-1})$ and $P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})$,

$$\sum_{\tau=1}^{t-1} \alpha_\tau^r \left[D_t(P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})) - D_t(P(q^e, \alpha_1^r, ..., \alpha_{t-2}^r, \bar{\alpha}_{t-1}, ..., \bar{\alpha}_{T-1})) \right]. \tag{15}$$

Equating the sum of (14) and (15) to a_t, and solving for α_t, yields the formula for $\beta_t(a_t; a_1^r, ..., a_{t-1}^r)$. If the solution is less than $\underline{\alpha}_t$, then for every demand state α_t purchases will necessarily continue beyond a_t, and the lowest possible value of α_t is $\underline{\alpha}_t$.

Second, note that for all $t \leq T - 1$ we have $P(q^e, \alpha_1, ..., \alpha_{T-1}) \leq \bar{p}_t \leq \bar{p} < \hat{v}$. This implies that any generation t consumer for which $\delta(v) > 0$ purchases in period t, and that no consumer with valuation $v \leq P(q^e, \alpha_1, ..., \alpha_{T-1})$ purchases before period T. Given that for each realization of demand we have $(\alpha_1^r, ..., \alpha_{T-1}^r) = (\alpha_1, ..., \alpha_{T-1})$, it then follows that

$$p_T(a_T; a_1^r, ..., a_{T-1}^r) = P(q^e, \alpha_1, ..., \alpha_{T-1}),$$

i.e. all firms with output remaining in period T set the market clearing price for the realized demand state. It follows from (iii) that all consumers with valuation above $P(q^e, \alpha_1, ..., \alpha_{T-1})$ purchase in period T if they have not already done so, that consumers with a lower valuation do not purchase, and that all output is sold. Because output is allocated efficiently, and because no consumer experiences positive delay costs in equilibrium, it follows that the allocation is Pareto optimal. We now show that sequential rationality is satisfied.

We have already shown in our general existence argument that prices are martingales and that sequential rationality by firms is satisfied. To show sequential rationality on the part of consumers, multiplicative uncertainty implies that when arriving at the market in period t all consumers from generations t or earlier share the same beliefs about $(\alpha_1, ..., \alpha_{T-1})$ as firms. Consider a consumer with valuation v, who has not purchased before period t and observes the price, $p_t(a_t; a_1^r, ..., a_{t-1}^r)$. By the martingale property associated with (iv), all continuation strategies in which she purchases with probability one in some period yield the same expected payment.

Case 1. Suppose $v \geq P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})$ and $\delta(v) = 0$ hold. Sequential rationality requires her to purchase eventually since her valuation exceeds the highest price she could face in period T, and from the martingale property, she is indifferent between purchasing in period t and delaying purchase. In particular, purchasing in period t is optimal.

Case 2. Suppose $v \geq P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})$ and $\delta(v) > 0$ hold. Purchasing in period t yields the same utility as deviations in which she eventually purchases with probability one, if we were to ignore the delay costs incurred. Deviations in which she does not always purchase yield strictly lower utility than purchasing in period t, so all such deviations yield strictly lower utility when the delay cost is taken into account. Thus, purchasing in period t is strictly preferred to the best alternative.

Case 3. Suppose $v < P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1})$ holds. Because we have $P(q^e, \alpha_1^r, ..., \alpha_{t-1}^r, \bar{\alpha}_t, ..., \bar{\alpha}_{T-1}) \leq \bar{p}$, our assumptions imply $\delta(v) = 0$. Therefore, she is indifferent between purchasing in period t and *always* purchasing in period T. Since there is a positive probability that $P(q^e, \alpha_1, ..., \alpha_{T-1}) > v$ holds, waiting until period T and only purchasing when her valuation exceeds the price yields strictly higher utility. Thus, a deviation to purchasing in period t strictly lowers utility.[3] ■

3 Sales Bound under Multiplicative Uncertainty

We assume that demand is multiplicative, i.e. that $D_t(p, \alpha_t) = \alpha_t D_t(p)$ for all $t = 1, ..., T-1$. We make two additional assumptions:

Assumption A.1 : There exists $0 < L < K < \infty$ such that for all t and $p \in [0, \bar{v}]$ we have:

$$L \leq |\frac{dD_t}{dp}| \leq K$$

Assumption A.2 : There exists $0 < \underline{\alpha} < \bar{\alpha} < \infty$ such that for all t we have:

$$\underline{\alpha} \leq \underline{\alpha}_t < \bar{\alpha}_t \leq \bar{\alpha}$$

Note that A.1 implies that $\underline{v} = 0$. Furthermore, it follows from A.1 and A.2 that $D_t(p, \alpha_t) > 0$ for all $p < \bar{v}$.

[3] Following a unilateral deviation by a firm to post a price greater than $p_t(a_t^{\max}; a_1^r, ..., a_{t-1}^r)$, the continuation strategy given in (iii) remains sequentially rational. After a deviation by a firm to post a price less than $p_t(0; a_1^r, ..., a_{t-1}^r)$, then purchasing is sequentially rational if $v \geq P(q^*, \alpha_1^r, ..., \alpha_{t-1}^r, \overline{\alpha}_t, ..., \overline{\alpha}_{T-1})$. For lower valuations, it may become optimal for a consumer to purchase, and one could compute the new cutoff for each firm deviation. We skip this detail because the firm's deviation cannot be optimal.

Consider the number of people who purchase in some period $t < T$. Old consumers entering period t who purchase in that period have valuations between $\bar{p}_{t-1}$ and $\bar{p}_t$. Thus the number of such people equals

$$\sum_{\tau=1}^{t-1} \alpha_\tau \left[D_\tau(\bar{p}_t) - D_\tau(\bar{p}_{t-1})\right]$$

Next, consider the generation t consumers who purchase in period t, i.e. those whose valuation exceeds $\bar{p}_t$. The number of such consumers equals

$$\alpha_t D_t(\bar{p}_t)$$

The number of consumers purchasing in period t, expressed as a fraction of total sales over the demand season, therefore equals:

$$g_t(\alpha_1, ..., \alpha_{T-1}) = \frac{\sum_{\tau=1}^{t-1} \alpha_\tau \left[D_\tau(\bar{p}_t) - D_\tau(\bar{p}_{t-1})\right] + \alpha_t D_t(\bar{p}_t)}{\sum_{\tau=1}^{T-1} \alpha_\tau D_\tau(P(q^*, \alpha_1, ..., \alpha_{T-1}))}$$

The same expression is also valid in period T, provided we use the convention that $\bar{p}_T = P(q^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1})$ and $D_T \equiv 0$.

Let q_T^* denote the equilibrium output when the number of periods is T. We may then prove:

Lemma 1 *Suppose that there exists $\varepsilon > 0$ such that for all T we have $P(q_T^*, \bar{\alpha}_1, ..., \bar{\alpha}_{T-2}, \bar{\alpha}_{T-1}) \leq \bar{v} - \varepsilon$. Suppose also that Assumptions A.1-A.2 hold. Then $g_t(\alpha_1, ..., \alpha_{T-1}) \to 0$ as $T \to \infty$.*

Proof. Let us first bound $\bar{p}_{t-1} - \bar{p}_t$. Note that by definition $P(q_T^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1})$ solves

$$\sum_{t=1}^{T-1} \alpha_t D_t \left(P(q_T^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1})\right) = q_T^*$$

Applying the implicit function theorem, we have

$$\frac{\partial P(q_T^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1})}{\partial \alpha_t} = \frac{D_t \left(P(q_T^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1})\right)}{\sum_{\tau=1}^{T-1} \alpha_\tau \left|\frac{\partial D_t}{\partial p}(P(q_T^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1}))\right|}$$

Using Assumption A.1 it follows that

$$D_t \left(P(q_T^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1})\right) \leq D_t(0) \leq K\bar{v}$$

Furthermore, it follows from Assumptions A.1 and A.2 that

$$\sum_{\tau=1}^{T-1} \alpha_\tau \left|\frac{\partial D_t}{\partial p}(P(q_T^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1}))\right| \geq (T-1)\underline{\alpha} L$$

and so

$$\frac{\partial P(q_T^*, \alpha_1, ..., \alpha_{T-2}, \alpha_{T-1})}{\partial \alpha_t} \leq \frac{K\bar{v}}{(T-1)\underline{\alpha} L}$$

Hence for any t such that $2 \leq t \leq T$ holds we have

$$\bar{p}_{t-1}-\bar{p}_t = \int_{\alpha_t}^{\bar{\alpha}_t} \frac{\partial P(q_T^*, \alpha_1^r, ..., \alpha_{t-1}^r, \alpha_t, \bar{\alpha}_{t+1}, ..., \bar{\alpha}_{T-1})}{\partial \alpha_t} d\alpha_t \leq \frac{K\bar{v}}{(T-1)\underline{\alpha}L}(\bar{\alpha}_t-\alpha_t) \leq \frac{\bar{\alpha}-\underline{\alpha}}{\underline{\alpha}} \frac{K\bar{v}}{(T-1)L} \tag{16}$$

It follows from the mean value theorem that

$$D_\tau(\bar{p}_t) - D_\tau(\bar{p}_{t-1}) = \ |\frac{\partial D_t}{\partial p}(\widetilde{p}_t)| \ (\bar{p}_{t-1} - \bar{p}_t) \leq K \ (\bar{p}_{t-1} - \bar{p}_t) \tag{17}$$

for some $\widetilde{p}_t \in (\bar{p}_t, \bar{p}_{t-1})$. Using (17) and (16) we therefore have

$$\sum_{\tau=1}^{t-1} \alpha_\tau \left[D_\tau(\bar{p}_t) - D_\tau(\bar{p}_{t-1})\right] \leq \frac{\bar{\alpha}}{\underline{\alpha}} \frac{K^2}{L}(\bar{\alpha}-\underline{\alpha})\bar{v} \tag{18}$$

Next, let us bound period t sales to newly arriving customers. Using the mean value theorem, we have

$$D_t(\bar{p}_t) = D_t(\bar{p}_t) - D_t(\bar{v}) = |\frac{\partial D_t}{\partial p}(\hat{p}_t)| \ (\bar{v} - \bar{p}_t)$$

for some $\hat{p}_t \in (\bar{p}_t, \bar{v})$. Using A.1 and A.2, we therefore have

$$\alpha_t D_t(\bar{p}_t) \leq \bar{\alpha}K(\bar{v} - \bar{p}_t) \leq \bar{\alpha}K\bar{v} \tag{19}$$

Finally, let us derive a lower bound to sales over the demand season. Again, using the mean value theorem and A.1, we have

$$D_\tau(P(q_T^*, \alpha_1, ..., \alpha_{T-1})) = D_\tau(P(q_T^*, \alpha_1, ..., \alpha_{T-1})) - D_\tau(\bar{v}) \geq L(\bar{v} - P(q_T^*, \alpha_1, ..., \alpha_{T-1})) \geq L\varepsilon$$

and so

$$\sum_{\tau=1}^{T-1} \alpha_\tau D_\tau(P(q_T^*, \alpha_1, ..., \alpha_{T-1})) \geq \underline{\alpha}(T-1)L\varepsilon \tag{20}$$

Combining (18), (19) and (20) we finally obtain:

$$g_t(\alpha_1, ..., \alpha_{T-1}) \leq \frac{1}{T-1} \frac{\bar{\alpha}}{\underline{\alpha}} \frac{K}{L} \frac{\bar{v}}{\varepsilon} \left[\frac{K}{L} \frac{\bar{\alpha}-\underline{\alpha}}{\underline{\alpha}} + 1\right] \tag{21}$$

The result then follows because the right side of (21) converges to zero as $T \to \infty$. ■

4 Examples

4.1 Verifying the Assumptions for the $T = 2$ Example

Here we verify that the $T = 2$ example with information effects in Section 4.1 satisfies Assumptions 1-5.

The demand specification $D(p, \alpha) = 1 - p^\alpha$, implies that $\underline{v} = 0$, $\bar{v} = 1$, and that Assumption 1 is satisfied. Assumption 2 holds vacuously, since there is only one batch of demand. Assumption 3 is satisfied whenever $A < 1$ holds. Since $\hat{v} = \bar{p} = \frac{\sqrt{3}}{2}$, Assumption 4 is satisfied. To verify Assumption 5, we compute

$$-\frac{\frac{\partial D(v,\alpha)}{\partial p}}{D(v,\alpha)} = \frac{\alpha v^{\alpha-1}}{1 - v^\alpha}.$$

Therefore,

$$\frac{\partial}{\partial \alpha}\left[\ln\left(-\frac{\frac{\partial D(v,\alpha)}{\partial p}}{D(v,\alpha)}\right)\right] = \frac{[\ln(v^\alpha) + 1 - v^\alpha]}{\alpha(1 - v^\alpha)}. \tag{22}$$

Since the term in brackets is increasing in v^α for all $\alpha \in [1, 2]$ and all $v \in (0, 1)$ and is zero at $v^\alpha = 1$, then the term in brackets must be negative and the entire expression must be negative. This establishes that Assumption 5(i) is satisfied. From (22), we have

$$\frac{\partial^2}{\partial v \partial \alpha}\left[\ln\left(-\frac{\frac{\partial D(v,\alpha)}{\partial p}}{D(v,\alpha)}\right)\right] = \frac{[v^\alpha \ln(v^\alpha) + 1 - v^\alpha]}{v(1 - v^\alpha)^2}.$$

The term in brackets is decreasing in v^α for all $\alpha \in [1, 2]$ and all $v \in (0, 1)$ and is zero at $v^\alpha = 1$, so the term in brackets must be positive and the entire expression must be positive. This establishes that Assumption 5(ii) is satisfied.

Finally, we consider the additional assumption required in our uniqueness proof, Proposition 4, that we have

$$\frac{\partial^2}{\partial v \partial \alpha} D(v, \alpha) < 0 \tag{23}$$

for all $\alpha \in [1, 2]$ and all $v \in (c, 1)$. We have

$$\frac{\partial^2}{\partial v \partial \alpha}[1 - v^\alpha] = -v^{\alpha-1}(\ln(v^\alpha) + 1). \tag{24}$$

From (24), we see that (23) holds whenever we have $v^\alpha > e^{-1} \simeq 0.36787944$. Since $c \simeq 0.819875713$, and since we have

$$\min_{\substack{v \geq c \\ \alpha \in [1,2]}} v^\alpha = c^2,$$

(23) must hold.

4.2 An Example with Multiplicative Uncertainty and $T = 3$

Demand in period $t \in \{1, 2\}$ is given by

$$D_t(p, \alpha_t) = \alpha_t(1 - p)$$

Therefore, the aggregate demand and inverse demand are given by

$$\begin{aligned} D(p,\alpha_1,\alpha_2) &= (\alpha_1+\alpha_2)(1-p) \\ P(q,\alpha_1,\alpha_2) &= 1-\frac{q}{\alpha_1+\alpha_2}. \end{aligned}$$

It follows that $\underline{v}=0$ and $\bar{v}=1$. Any specification of $\delta(v)$ satisfying our maintained assumptions will work, since all consumers with a positive $\delta(v)$ will purchase in the period in which they first arrive. Let us set

$$\begin{aligned} \delta(v) &= 0 \quad \text{for} \quad v \leq \frac{3}{4} \\ \delta(v) &= \frac{1}{100}(v-\frac{3}{4}) \quad \text{for} \quad v \geq \frac{3}{4}. \end{aligned}$$

It will be convenient to set the marginal production cost as follows

$$c = 6\ln 3 - 10\ln 2 + 1 \simeq 0.6602.$$

We assume that α_1 and α_2 are independent and identically distributed according to the uniform density on $[1,2]$. We have

$$f(\alpha_1,\alpha_2) = 1 \;\text{ for all } (\alpha_1,\alpha_2)\in[1,2]\times[1,2].$$

Then the equilibrium quantity $q^* = q^e$ solves

$$\int_{\alpha_1=1}^{2}\int_{\alpha_2=1}^{2}(1-\frac{q}{\alpha_1+\alpha_2})d\alpha_2 d\alpha_1 = c = 6\ln(3)-10\ln(2)+1,$$

yielding $q^*=q^e=1$. Since $\bar{\alpha}_1=\bar{\alpha}_2=2$, we have

$$\bar{p} = 1-\frac{1}{2+2} = \frac{3}{4}.$$

In period 1, all consumers with $v \geq \bar{p}$ purchase,[4] so total purchases in period 1 will be $\alpha_1(1-\bar{p})$. This allows us to infer the minimum possible period-1 demand state, as a function of period-1 transactions a_1, as follows:

$$\begin{aligned} \beta_1(a_1) &= 1 \quad \text{for} \quad a_1 \leq \underline{\alpha}_1(1-\bar{p}) = \frac{1}{4} \\ \beta_1(a_1) &= 4a_1 \quad \text{for} \quad \frac{1}{4} < a_1 \leq \bar{\alpha}_1(1-\bar{p}) = \frac{1}{2}. \end{aligned}$$

Prices in period 1 are given by

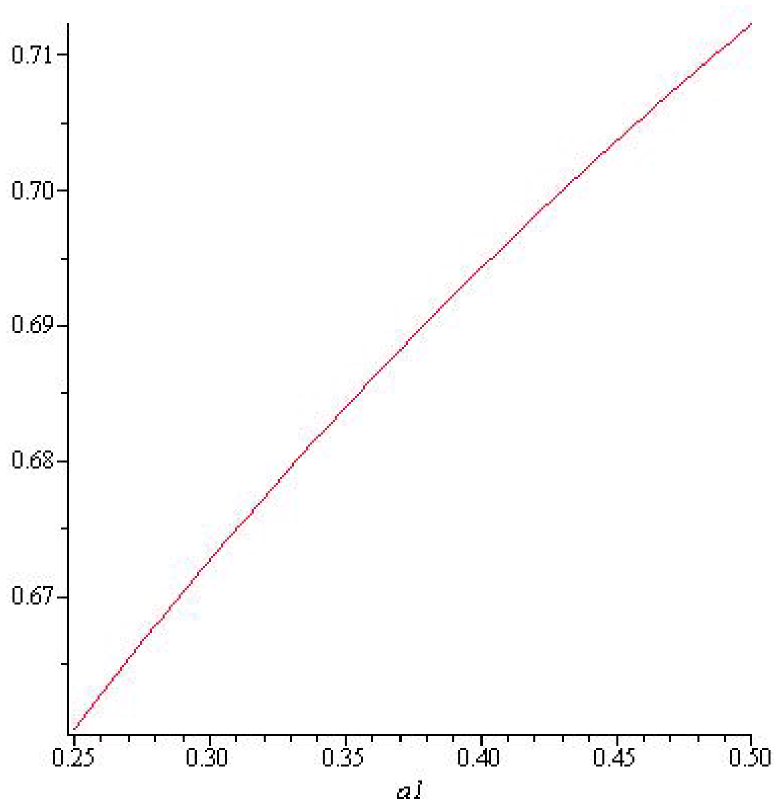
$$\begin{aligned} p_1(a_1) &= c = 6\ln(3)-10\ln(2)+1 \quad \text{for} \quad a_1 \leq \frac{1}{4} \\ p_1(a_1) &= \frac{\int_{4a_1}^{2}\int_{1}^{2}(1-\frac{1}{\alpha_1+\alpha_2})d\alpha_2 d\alpha_1}{\int_{4a_1}^{2}\int_{1}^{2} d\alpha_2 d\alpha_1} \quad \text{for} \quad \frac{1}{4} < a_1 \leq \frac{1}{2}. \end{aligned} \tag{25}$$

[4] By setting $\widehat{v} = \frac{3}{4}$, this means that all consumers with a higher valuation strictly prefer to purchase in period 1, and all consumers with a lower valuation strictly prefer to wait.

Equation (25) yields the closed-form expression

$$p_1(a_1) = 1 - \frac{(1+4a_1)\ln(1+4a_1) - 2(1+2a_1)\ln(1+2a_1) + (6-4a_1)\ln(2) - 3\ln(3)}{2-4a_1}.$$

Here is a plot of $p_1(a_1)$:



We now proceed to period 2. Based on Proposition 5 and the relations, $\alpha_1^r = 4a_1^r$ and $\bar{p}_2(a_1^r) = 1 - \frac{1}{\alpha_1^r+2} = 1 - \frac{1}{4a_1^r+2}$, we can compute the measure of consumers who are born in period 1 but purchase at the beginning of period 2, $q_1^L(a_1^r)$. These are the consumers with valuations between $\bar{p}_2(a_1^r)$ and $\bar{p}$.

$$\begin{aligned} q_1^L(a_1^r) &= \alpha_1^r[(1-\bar{p}_2(a_1^r)) - (1-\bar{p})] \\ &= \frac{a_1^r(1-2a_1^r)}{1+2a_1^r}. \end{aligned}$$

The minimum possible period-2 demand state, as a function of the period-1 history a_1^r and the period-2 transactions a_2, is given by

$$\begin{aligned} \beta_2(a_2;a_1^r) &= 1 \quad \text{for} \quad a_2 \le a_2^{\min}(a_1^r) \\ \beta_2(a_2;a_1^r) &= a_2(4a_1^r+2) - 2a_1^r(1-2a_1^r) \quad \text{for} \quad a_2^{\min}(a_1^r) < a_2 \le 1 - a_1^r \end{aligned} \tag{26}$$

where

$$a_2^{\min}(a_1^r) = q_1^L(a_1^r) + 1 - \bar{p}_2(a_1^r) = \frac{2a_1^r - 4(a_1^r)^2 + 1}{2(1 + 2a_1^r)}.$$

Prices in period 2 are given by

$$p_2(a_2; a_1^r) = \frac{\int_{\beta_2(a_2;a_1^r)}^2 (1 - \frac{1}{4a_1^r + \alpha_2}) d\alpha_2}{2 - \beta_2(a_2; a_1^r)},$$

which yields the closed-form expression

$$p_2(a_2; a_1^r) = 1 + \frac{\ln(a_2 + a_1^r)}{2 - a_2(4a_1^r + 2) + 2a_1^r(1 - 2a_1^r)}.$$

Moving on to period 3, all transactions occur at the market clearing price for the realized demand state,

$$P(1, \alpha_1^r, \alpha_2^r) = 1 - \frac{1}{\alpha_1^r + \alpha_2^r}. \tag{27}$$

We can compute the measure of consumers who are born in period 1 or period 2, but who purchase in period 3, $q_2^L(a_1^r, a_2^r)$. These are the consumers with valuations between $P(1, \alpha_1^r, \alpha_2^r)$ and $\bar{p}_2(a_1^r)$, or in terms of the revealed demand states,

$$q_2^L = 1 - \frac{\alpha_1^r + \alpha_2^r}{\alpha_1^r + 2}. \tag{28}$$

Using $\alpha_1^r = 4a_1^r$ and using (26) to derive $\alpha_2^r = a_2^r(4a_1^r + 2) - 2a_1^r(1 - 2a_1^r)$, substituting into (27), and simplifying, we have

$$\begin{aligned} p_3(a_3; a_1^r, a_2^r) &= 1 - \frac{1}{2(a_1^r + a_2^r)(1 + 2a_1^r)} \\ q_2^L(a_1^r, a_2^r) &= 1 - a_1^r - a_2^r. \end{aligned}$$

It is interesting to note that even though all consumers with an option value of waiting delay their purchases, most of the sales occur before period 3. From (28), the maximum possible quantity sold in period 3 occurs when $\alpha_1^r = 1$ and $\alpha_2^r = 1$, where one third of the output is sold in period 3.[5]

[5] Sales across the 3 periods are then $(\frac{1}{4}, \frac{5}{12}, \frac{1}{3})$. If instead the realized demand states are at their mean values, $\alpha_1^r = \frac{3}{2}$ and $\alpha_2^r = \frac{3}{2}$, sales are $(\frac{3}{8}, \frac{27}{56}, \frac{1}{7})$. Finally, if the realized demand states are at their maximum values, $\alpha_1^r = 2$ and $\alpha_2^r = 2$, sales are $(\frac{1}{2}, \frac{1}{2}, 0)$.