

To grab for the market or to bide one's time: a dynamic model of entry

Dan Levin*

and

James Peck*

We consider a simultaneous-move, dynamic-entry game. The fixed cost of entry is private information. Entering earlier increases the likelihood of being the monopolist but also increases the likelihood of coordination failure and simultaneous entry. We consider general continuous distributions for the fixed cost, and we characterize the unique symmetric sequential equilibrium in pure strategies. Comparative-statics results are derived. As the time between rounds approaches zero, all of the “action” occurs during an arbitrarily small amount of time. For the Bertrand model, we extend the analysis to allow for n firms.

1. Introduction

■ In some markets, a firm's choice of whether to enter a market is the most important choice it faces. Entry might involve a huge sunk investment up front, yielding large profits if the firm manages to acquire a significant chunk of the market, but yielding large losses if the firm finds itself embroiled in close competition. Consider the market for the next generation of microprocessors. Suppose that a scientific breakthrough occurs, making a better technology possible and widely available. However, a multibillion dollar facility must be created to produce the chip, after which chips can be produced at negligible marginal cost. If a firm is lucky enough to be the monopolist, then it will dominate the industry for several years until the next generation of microprocessors is invented. On the other hand, suppose that duopoly will make it impossible for the firm to recoup its investment (for example, if the postentry game is Bertrand competition). We model this entry choice as a dynamic game played between two rivals, each of which privately observes its cost of entry. Time is broken into discrete rounds, in which firms that have not yet entered must decide whether to enter or remain out of the market for another round. Specifying a dynamic process is important, because a one-shot game predicts a positive probability that no one enters,

* Ohio State University; levin.36@osu.edu, peck.33@osu.edu.

We thank Steve Spear for helping us fix some problems in an earlier draft. We thank Joe Harrington, John Kagel, Howard Marvel, Steve Matthews, and two referees for helpful suggestions, Eitan Gurari for help with the numerical solutions, and Dave Terman of the OSU Math Department for advice. Any errors are our responsibility.

even if monopoly is extremely profitable. The potential for entry continues if no one enters at the first opportunity, so the static formulation ignores an important part of the story.¹ In the unique symmetric equilibrium of our dynamic model, what emerges is an interesting tension between a desire to grab the market aggressively and a desire to feel out one's rival cautiously. Entering earlier increases the likelihood of being the monopolist but also increases the likelihood of coordination failure, in which both firms enter in the same round. In equilibrium, a positive interval of entry-cost types enters in each round, if no firm has entered in a previous round.

This tension between profit opportunities and coordination failure inspired a small but valuable literature a few years ago. Dixit and Shapiro (1986) consider a dynamic-entry game, where time is broken into discrete rounds. In each round, firms that have not yet entered the market decide whether to enter, as a function of the number of firms that can still enter profitably. All firms have the identical, sunk cost of entry. Dixit and Shapiro run simulations to find symmetric mixed-strategy equilibria.² Their justification for focusing on mixed strategies is that it is unclear how firms coordinate on which firms should enter in the asymmetric pure-strategy equilibria. However, we show in Section 5 that the mixed-strategy approach generates implausible predictions when we perturb the model to allow firms to have different costs of entry. We show by example that the symmetric mixed-strategy equilibria of both the one-shot and the dynamic models of Dixit-Shapiro typically have firms with *higher* entry costs mix with a higher probability of entry than those with lower entry costs. On the other hand, one can also perturb Dixit-Shapiro by introducing a small amount of uncertainty about entry costs. For the limiting case of our model in which the support of the entry-cost distribution shrinks to a point, we show that the equilibrium entry probabilities of the symmetric *pure*-strategy equilibrium converge to Dixit-Shapiro's mixed-strategy entry probabilities. Away from the limit, however, firms with higher entry costs enter weakly later than firms with lower entry costs. The distinction is that with mixed strategies, a firm must make the other firms indifferent between entering and not, while with pure strategies, the marginal firm must itself be indifferent between entering and not.

Bolton and Farrell (1990) introduce private information about entry costs and analyze an example in which costs are either low or high. Their mixed-strategy equilibrium is the "decentralized" market outcome, which is compared to the outcome of a central planner who randomly picks one firm to enter. Decentralization efficiently sorts firms, so that lower-cost firms enter, but there may be coordination failure or costly delays. When the situation is urgent and private information is relatively unimportant, the central planner provides higher welfare by avoiding duplication and delay. Fudenberg and Tirole (1985) model the choice of when to introduce a new product, where the cost of adoption declines over time. They find that in equilibrium, rents can be dissipated through preemptive entry. In their "diffusion" equilibrium, one firm enters early and the other firm enters much later, so the probability of coordination failure, in which both firms preempt simultaneously, is zero.³

We revisit the topic of dynamic entry because the underlying tension is simple and flexible, and more work needs to be done. Bolton and Farrell (1990) observe that the model extends far beyond oligopoly markets. Within an organization, when a manager assigns tasks, duplication and delay is avoided, but allowing individual initiative could allocate tasks to those best able to handle them. This model is bound to be rediscovered in other contexts. Here, we consider a general version of Bolton and Farrell, with arbitrary continuous distributions of entry costs. Bolton and Farrell's analysis centers on an example with a two-point distribution where low-cost types enter in round 1.

¹ There is a literature analyzing static-entry choice, such as whether to join an auction. See, for example, Milgrom (1981), Levin and Smith (1994), Smith and Levin (2001), and Harstad (1990). Reinganum (1981) analyzes the choice of when to adopt a new technology, but where the adoption time is decided *ex ante*, with perfect commitment.

² See also Vettas (2000a) for a careful analysis of the "reverse monotonicity" problem in Dixit and Shapiro. It is possible that incumbents are better off when more firms have entered, because the probability of overshooting, due to miscoordination of mixed-strategy realizations, is lower.

³ There is a line of research on the unravelling of markets in time: Roth and Xing (1994), Li and Rosen (1998), and Deneckere and Peck (1999). Although this literature is far removed from the present article, it would be interesting to introduce a cost of early entry into our model.

After round 1, a symmetric mixed-strategy equilibrium is selected for the corresponding game of complete information, à la Dixit-Shapiro. In their equilibrium, only one cost type can enter in any given round. We consider arbitrary continuous distributions, which restores pure-strategy equilibrium. In equilibrium, an interval of cost types enter in any given round. We characterize the unique symmetric sequential equilibrium and demonstrate that the coordination failure does not disappear as the time between rounds approaches zero. The sizes of the intervals (of types that enter in any given round) converge to positive limits, rather than shrinking to zero, as the time between rounds approaches zero. Inferences about the rival's entry cost are drawn from the fact that a round has passed, even if the length of time between rounds is small. We show that as the length of time between rounds approaches zero (and therefore the discount factor between rounds approaches one), then with probability arbitrarily close to one, the first entry occurs before an arbitrarily short amount of real time has passed. Other comparative-statics results are derived.

Fudenberg and Tirole (1985) argue that traditional continuous-time formulations are not adequate for modelling games of timing. They develop a continuous-time framework with equilibria that are the limits of discrete-time, mixed-strategy equilibria. Our model further illustrates Fudenberg and Tirole's criticism of traditional continuous-time formulations, but it differs from their model in two important ways. First, in the usual preemption game, there is a tension between the desire to be first and the desire to wait until first-mover profits are maximized. In our model, first-mover profits are maximized by entering in round 1, and the risk of coordination failure induces some firms to wait beyond the point where first-mover profits are maximized.⁴ Second, we consider private information with a continuum of types, and consequently, we find pure-strategy equilibrium (unlike Fudenberg and Tirole's mixed-strategy equilibrium).

In our model, the firms must decide when to make a grab for the monopoly profits; essentially, the game is over when at least one firm enters. The model can be contrasted with other dynamic models of incomplete information, such as the war of attrition. (See Bulow and Klemperer (1999).) In the war of attrition, firms wait each other out and incur costs as they wait. This is an exit model, in contrast to our entry model. Incurring waiting costs is crucial for types to separate themselves in the war of attrition, because otherwise no one would exit. On the other hand, our entry game allows for, but does not require, significant costs of waiting. Since our game is essentially over when the first firm enters, it is a stopping game, similar to the stopping games in the bargaining and public goods literatures. (See Cramton (1992) and Bliss and Nalebuff (1984).) Bliss and Nalebuff analyze a game in which people wait each other out, until someone gives in and provides the public good. That person enters the activity of providing the public good in some sense, but in so doing, the person exits the conflict. The semantics of what is called "entry" and what is called "exit" are arbitrary. What is important is the nature of the incentives. In the above articles, a firm prefers to have its rival stop the game. In our article, a firm prefers to be the one stopping the game.

The rest of the article is organized as follows. In Section 2 we set up the model and demonstrate some preliminary results, adapted from Bolton and Farrell (1990). In Section 3 we show the existence of a symmetric equilibrium and also that the symmetric equilibrium is unique. Higher values of the monopoly revenue or duopoly revenue cause entry to occur earlier. We also demonstrate that the first entry occurs arbitrarily quickly (in real time) as the time between rounds shrinks to zero. In Section 4 we discuss the nature of asymmetric equilibria. In Section 5 we consider the Bertrand variant of the model, in which only a monopolist receives positive revenue (marginal production cost is zero). However, the entry-cost distribution is general and the number of firms is arbitrary. Section 6 presents some concluding remarks. Several proofs are provided in the Appendix.

⁴ In this sense, our model is closer to the "grab the dollar" game described by Fudenberg and Tirole (1985) and attributed to Richard Gilbert, with the distinction being that we introduce incomplete information.

2. The model

■ We consider a market with two potential entrants, where each firm privately observes a random cost of entry, c_i . We assume that c_1 and c_2 are independent and identically distributed, according to the strictly increasing and continuous distribution function F , defined over the support, $[\underline{c}, \bar{c}]$. We assume that F is common knowledge. We assume that $\underline{c} \geq 0$ holds and normalize $\bar{c} = 1$.

Time is broken into discrete intervals or rounds. We interpret a round to be the length of time that elapses between the moment a firm decides to enter the market and the moment the entry decision is observed by the firm's rival. Denote the length of time that elapses between rounds as Δ . Before defining the game formally, we offer the following description of the game played by the two firms. In each round, firms that have not yet entered the market observe the history (i.e., whether the rival has entered previously) and decide whether or not to enter. If a firm never enters the market, its profits are zero. When firm i enters, it incurs the sunk cost, c_i , it receives monopoly revenues during rounds in which its rival has not entered, and it receives duopoly revenues during rounds in which its rival has also entered. We assume that monopoly and duopoly revenues are independent of entry costs, and that marginal production costs are normalized to zero. Our framework includes, as special cases, Cournot competition with symmetric cost functions, Bertrand competition with symmetric cost functions and perfect substitutes, and symmetric Bertrand competition with heterogeneous products. For example, under pure Bertrand competition, duopoly revenues are zero.

Let $r > 0$ denote the discount rate per unit of time, which we fix throughout. Let the present value of a permanent flow of monopoly revenues be R_m , and let the present value of duopoly revenues be R_d . We restrict attention to the interesting case in which $R_m > 1 > R_d \geq 0$ holds.⁵ Thus, the discount factor between rounds, δ , is given by $\delta = e^{-r\Delta}$. It follows that the revenue received, in round t only, is $\delta^{t-1}(1 - \delta)R_m$ for a monopolist and $\delta^{t-1}(1 - \delta)R_d$ for a duopolist. Therefore, the profits of a firm that enters in round t and is a monopolist forever are $(R_m - c_i)\delta^{t-1}$, and the profits of a firm that enters in round t and is a duopolist forever are $(R_d - c_i)\delta^{t-1}$.

More formally, we denote the action of firm i in round t as $e_i^t \in \{0, 1\}$, where action 0 represents not having yet entered and action 1 represents having entered (either this round or previously). Also, let $e^t = (e_1^t, e_2^t)$, with $e^0 = (0, 0)$ representing the fact that firms cannot enter before round 1. Denote the history of length t as $h^t = (e^0, e^1, \dots, e^t)$, and let h denote the set of histories (of any length). A strategy for firm i is a mapping from types and histories into moves, $\sigma_i : [\underline{c}, 1] \times h \rightarrow \{0, 1\}$, satisfying the restriction that once a firm enters, it must stay in the market.⁶ The payoffs of the round- t stage game are $(\delta^{t-1}(1 - \delta)R_m, 0)$ if $e^t = (1, 0)$, the payoffs are $(0, \delta^{t-1}(1 - \delta)R_m)$ if $e^t = (0, 1)$, the payoffs are $(\delta^{t-1}(1 - \delta)R_d, \delta^{t-1}(1 - \delta)R_d)$ if $e^t = (1, 1)$, and the payoffs are $(0, 0)$ if $e^t = (0, 0)$. Our solution concept is sequential equilibrium.⁷

Lemma 1 shows that in any sequential equilibrium, a firm never enters if its rival has previously entered, and that all firms whose entry cost is less than the duopoly revenue will enter in round 1 (higher-cost firms may also choose round 1). Lemma 2 shows that entry decisions can be characterized by increasing sequences, $\{\alpha_1^t\}$ and $\{\alpha_2^t\}$, where for $i = 1, 2$, firm i enters in round t if and only if the rival has not entered through round $t - 1$ and we have $\alpha_i^{t-1} \leq c_i < \alpha_i^t$. That is, lower-cost intervals of types for firm i enter before higher-cost intervals.

Lemma 1. In any sequential equilibrium, if firm i 's cost is equal to the duopoly revenue or lower, $c_i \leq R_d$, it enters in round 1. A firm will never enter after its rival has entered.

⁵ The case in which $R_d > 1$ holds is not interesting, because then all firms would enter in round 1. If $R_m \leq 1$ holds, we are essentially back to our model if the distribution of costs is truncated at R_m and probabilities renormalized. Of course, firms with entry costs greater than R_m would never enter.

⁶ That is, we have that $\sigma_i(c_i, h^{t-1}) = 1$ implies $\sigma_i(c_i, [h^{t-1}, e^t]) = 1$ for all c_i, h^{t-1} , and e^t . Our symmetric equilibrium remains an equilibrium without this restriction, where firms could exit and reenter the market. This restriction rules out folk theorem possibilities, such as alternating who is the monopolist.

⁷ Sequential equilibrium requires specification of firm i 's beliefs, μ_i , which maps firm i 's information set, $[\underline{c}, 1] \times h$, into the set of distribution functions over $[\underline{c}, 1]$.

Proof. Let $(\sigma_1, \sigma_2, \mu_1, \mu_2)$ be a sequential equilibrium. Suppose that firm 1 has entry cost less than or equal to the duopoly revenue, $c_1 \leq R_d$, and consider three possibilities for firm 2:

- (i) Given σ_2 , if c_2 is such that firm 2 enters in round 1, firm 1 receives higher profits by entering in round 1 and receiving duopoly revenues immediately, rather than postponing entry.
- (ii) If c_2 is such that firm 2 does not enter in round 1 and we have $c_2 \leq R_d$, then firm 1 receives higher profits entering in round 1, rather than postponing entry. The reason is that if firm 1 enters in round $t > 1$ and firm 2 has not yet entered, then sequential rationality requires firm 2 to enter in round $t + 1$. Thus, the best possible outcome for firm 1 in this circumstance would be to enter in round 1, receive monopoly revenues in round 1, and duopoly revenues thereafter.
- (iii) If c_2 is such that firm 2 does not enter in round 1 and has cost, $c_2 > R_d$, then firm 1 can again do no better than to enter in round 1 and receive monopoly revenues, since sequential rationality requires that firm 2 remain out of the market. We have shown that firm 1 is better off entering in round 1 than waiting, independent of beliefs μ_1 .

Now suppose that for some history h^t , we have $e^t = (0, 1)$. From the previous paragraph, it follows that $c_1 > R_d$ holds, since firm 1 did not enter in round 1. Sequential rationality requires that firm 1 never enter, independent of beliefs μ_1 , because entry yields negative profits. A symmetric argument applies to firm 2. *Q.E.D.*

From Lemma 1, firms only enter after histories equal to the zero vector, $h^t = 0$. Furthermore, beliefs off the equilibrium path are irrelevant for the characterization of equilibrium. To see this, let \hat{t}_i be the first round for which almost all cost types for firm i will have entered, conditional on no entry by firm i 's rival. Without loss of generality, assume $\hat{t}_1 \geq \hat{t}_2$ holds. For $t < \hat{t}_2$, we observe $h^t = 0$ with positive probability, so conditional probabilities are determined by Bayes' rule. Conditional on no entry before round \hat{t}_2 , firm 1 must assign probability 1 to firm 2 entering in round \hat{t}_2 . Sequential rationality then requires firm 1 not to enter in round \hat{t}_2 (since duopoly is unprofitable). Therefore, firm 1's beliefs following a deviation by firm 2 (to enter after round \hat{t}_2) are irrelevant, since firm 2 is better off entering in round \hat{t}_2 and guaranteeing monopoly revenues. We can thus characterize the equilibrium strategy of firm i as a mapping from types into the round in which firm i enters, conditional on its rival not having yet entered.

Lemma 2. Let $(\sigma_1, \sigma_2, \mu_1, \mu_2)$ be a sequential equilibrium. If firm i enters in round t (conditional on its rival not having entered) when it has cost $c_i = c'$, and if firm i enters in round $t + 1$ (conditional on its rival not having entered) when it has cost $c_i = c''$, then $c' < c''$ holds.

Proof. Let $I_t \equiv \{c : \text{firm } i\text{'s rival enters in round } t, \text{ given } c_{-i} = c \text{ and } h^{t-1} = 0\}$, and let $\mu_i(I_t)$ denote firm i 's assessment of the probability that its rival enters in round t , conditional on no entry before round t . From the fact that firm i enters in round t rather than $t + 1$ with cost c' , we have

$$\mu_i(I_t)R_d + (1 - \mu_i(I_t))R_m - c' \geq \delta(1 - \mu_i(I_t))[\mu_i(I_{t+1})R_d + (1 - \mu_i(I_{t+1}))R_m - c']. \quad (1)$$

From the fact that firm i enters in round $t + 1$ rather than t with cost c'' , we have

$$\mu_i(I_t)R_d + (1 - \mu_i(I_t))R_m - c'' \leq \delta(1 - \mu_i(I_t))[\mu_i(I_{t+1})R_d + (1 - \mu_i(I_{t+1}))R_m - c'']. \quad (2)$$

Inequalities (1) and (2) imply

$$c'[1 - \delta(1 - \mu_i(I_t))] \leq c''[1 - \delta(1 - \mu_i(I_t))]. \quad (3)$$

Since we have $0 < \delta < 1$ and $0 \leq \mu_i(I_t) \leq 1$, the term in brackets in (3) is positive. Thus, we have $c' \leq c''$, but since we restrict attention to pure strategies, a firm with a given entry cost ($c' = c''$) cannot enter in both rounds t and $t + 1$, so we have $c' < c''$. *Q.E.D.*

Lemma 2 is a modification of Bolton and Farrell (1990, Proposition 1), allowing for positive duopoly revenues and focusing on pure-strategy equilibrium. It shows that, without loss of generality, a sequential equilibrium can be characterized by nondecreasing sequences, $\{\alpha_1^t\}$ and $\{\alpha_2^t\}$, with the interpretation that firm i will enter in round t if there has been no entry until that point and if $c_i \leq \alpha_i^t$. If these sequences are increasing and we have $c_i = \alpha_i^t$, then firm i is indifferent between entering in round t and entering in round $t + 1$ (unless its rival enters in round t , in which case firm i does not enter).⁸ In Section 3, we restrict attention to the symmetric equilibrium, in which $\alpha_1^t = \alpha_2^t$ for all t . Then in Section 4, we consider the possibility of asymmetric equilibria.

3. The symmetric equilibrium

■ We now characterize the symmetric equilibrium entry intervals, $\{\alpha^t\}_{t=1}^\infty$. A necessary condition is that a “marginal” firm with entry cost $c_i = \alpha^t$ should be indifferent between entering in round t and waiting until round $t + 1$. Intuitively, the tradeoffs are as follows. By entering in round t , firm i receives duopoly revenues if its rival enters in round t , while it receives monopoly revenues if its rival plans to enter in round $t + 1$ or later (because the rival will observe firm i 's entry and stay out). On the other hand, by waiting until round $t + 1$, firm i receives zero revenue but avoids incurring entry costs if its rival enters in round t (because firm i will not enter), while it receives duopoly revenues if its rival enters in round $t + 1$ and monopoly revenues if its rival plans to enter later. Also, the revenue flow associated with monopoly or duopoly is discounted. This allows us to derive a difference equation whose solution gives us $\{\alpha^t\}_{t=1}^\infty$. Conditional on no entry before round t , a firm with cost $c_i = \alpha^t$ that enters in round t receives expected profits of

$$\left(\frac{1 - F(\alpha^t)}{1 - F(\alpha^{t-1})} \right) R_m(\delta)^t + \left(\frac{F(\alpha^t) - F(\alpha^{t-1})}{1 - F(\alpha^{t-1})} \right) R_d(\delta)^t - \alpha^t(\delta)^t. \tag{4}$$

If the firm waits until round $t + 1$ and enters if its rival has not yet entered, its expected profits (conditional on no entry before round t) are given by

$$\left(\frac{1 - F(\alpha^{t+1})}{1 - F(\alpha^{t-1})} \right) R_m(\delta)^{t+1} + \left(\frac{F(\alpha^{t+1}) - F(\alpha^t)}{1 - F(\alpha^{t-1})} \right) R_d(\delta)^{t+1} - \alpha^t(\delta)^{t+1} \left(\frac{1 - F(\alpha^t)}{1 - F(\alpha^{t-1})} \right). \tag{5}$$

For $t = 0, 1, \dots$, let $F(\alpha^t)$ be denoted by F_t . Equating (4) and (5), and simplifying, yields the difference equation

$$(1 - F_t)R_m + (F_t - F_{t-1})R_d - (1 - F_{t-1})\alpha^t = (1 - F_{t+1})R_m\delta + (F_{t+1} - F_t)R_d\delta - (1 - F_t)\alpha^t\delta. \tag{6}$$

Equation (6) is a second-order difference equation. The sequence $\{\alpha^t\}_{t=1}^\infty$ must be strictly increasing whenever we have $\alpha^t < 1$. Otherwise there is a round in which no one enters, but then a firm that has not yet entered is not behaving optimally. The firm could instead enter, knowing that its rival would not enter, either simultaneously or afterward. Entering would yield higher profits, because monopoly revenue exceeds the highest entry cost, $R_m > 1$. We cannot have $\alpha^t = 1$ for finite t , because then there is a round in which both firms are sure to enter (if there has been no entry previously), but duopoly cannot be profitable for a firm that does not enter in round 1. There are two boundary conditions. The first condition is $\alpha^0 = \underline{c}$. The second condition is $\lim_{t \rightarrow \infty} \alpha^t = 1$. The second condition follows from the fact that the sequence is increasing and must converge; if

⁸ Because the distribution of entry costs is nonatomic, it does not matter for our characterization of equilibrium whether or not firm i enters in round t or $t + 1$ when $c_i = \alpha_i^t$.

the limit is not one, then eventually a firm that has not yet entered would be guaranteed to be a monopolist and should enter.

Proposition 1. A symmetric equilibrium exists, satisfying (6) and the boundary conditions $\alpha^0 = \underline{c}$ and $\lim_{t \rightarrow \infty} \alpha^t = 1$. Moreover, the symmetric equilibrium is unique and α^t varies continuously with R_m , R_d , and δ for each t .

Proof. See the Appendix.

This second-order difference equation defies analytical solution. Proposition 1 is proved by transforming the problem into a two-dimensional dynamical system and using a contraction argument to show uniqueness.⁹ Because the system is well behaved, we can numerically solve for the equilibrium when the parameters are specified.¹⁰

Proposition 1 allows us to present some comparative-statics results. Increasing either R_m or R_d makes both firms more aggressive, so the probability of a firm entering in round 1 increases. In fact, we show in Proposition 2 below that *each* α^t is higher if either R_m or R_d is higher. As a result of this more aggressive behavior, firms are more willing to risk the coordination failure in which both firms enter in the same period.

Proposition 2. Holding other parameters constant, $\bar{R}_m > R_m$ implies that the corresponding equilibria satisfy $\bar{\alpha}^t > \alpha^t$ for all $t > 0$. Similarly, $\bar{R}_d > R_d$ implies $\bar{\alpha}^t > \alpha^t$ for all $t > 0$.

Proof. See the Appendix.

Proposition 2 shows that the market settles into its final configuration (either monopoly or duopoly) faster when monopoly revenues or duopoly revenues are higher. Our next result demonstrates, for the class of generalized uniform distribution functions over $[0,1]$, $F(\alpha) = (\alpha)^\lambda$, that the firms are more willing to risk duopoly in round 1 when λ is lower. Lower λ corresponds to lower costs, in the sense of first-order stochastic dominance. The lower tail of the entry-cost distribution is thicker. Lower λ also leads to lower values of α^1 , so the interval of entry-cost types choosing round 1 is narrower. The net effect is that the *probability* of entry in round 1 is unambiguously higher.

Proposition 3. Let $F(\alpha) = (\alpha)^\lambda$, where $\lambda > 0$ holds. Holding other parameters constant, $\bar{\lambda} > \lambda$ implies that the corresponding equilibria satisfy $\bar{\alpha}^1 > \alpha^1$ and $(\bar{\alpha}^1)^{\bar{\lambda}} < (\alpha^1)^\lambda$.

Propositions 2 and 3 allow us to characterize the effect of R_m , R_d , and (for generalized uniform distributions) λ on the probability of duopoly in round 1. Higher R_m or R_d causes firms to be more aggressive, and lower λ causes firms to be more aggressive, since costs are lower, according to first-order stochastic dominance. We do not have comparative-statics results about the probability of duopoly over all rounds, but simulations indicate that the impact on round 1 tends to dominate.

Corollary (of Propositions 2 and 3). Holding other parameters constant, $\bar{R}_m > R_m$ implies that for the corresponding equilibria, the probability of duopoly in round 1 is higher when we have \bar{R}_m . Similarly, $\bar{R}_d > R_d$ implies that the probability of duopoly in round 1 is higher when we have \bar{R}_d . Let $F(\alpha) = (\alpha)^\lambda$, where $\lambda > 0$ holds. Holding other parameters constant, $\bar{\lambda} > \lambda$ implies that for the corresponding equilibria, the probability of duopoly in round 1 is lower when we have $\bar{\lambda}$.

Proposition 4 draws a striking conclusion from the fact that α^t varies continuously with δ . As δ approaches one, each α^t converges, so the probability of either firm entering in any given

⁹ We thank Dave Terman from the Ohio State University Mathematics Department for suggesting this approach to us, although he should not be blamed for any of our mistakes.

¹⁰ Vettas (2000b) considers a problem quite different from ours, but performs a local analysis using the same techniques that we do. However, our contraction argument allows us to analyze the properties of the stable manifold away from the steady state.

round t also converges. For example, the first round, t^* , for which entry will have occurred with probability .999, converges as well. Letting the time between rounds approach zero, it follows that δ is approaching one and the length of time before round t^* is approaching zero. In other words, as the time between rounds approaches zero, all of the action occurs within an arbitrarily small amount of time. In the limit, when $\Delta = 0$, all cost types enter before any real time has passed. Let us contrast this limiting outcome with a continuous-time specification of the model, in which firms choose when to “stop the clock” and enter, as a function of cost type. Clearly, it cannot be an equilibrium for everyone to enter at time zero. We conclude that the traditional continuous-time model is not the appropriate framework for our game of incomplete information, as argued by Fudenberg and Tirole (1985) for mixed-strategy equilibria.

Proposition 4. Hold time preference, r , fixed and let the time between rounds, Δ , approach zero. Then, in equilibrium, we have $\Pr(\text{entry occurs before time } T) \rightarrow 1$ for all $T > 0$.

Proof. Let ν index the sequence of equilibria, $\{\alpha_\nu^t\}_{t=1}^\infty$, in which the time between rounds approaches zero, and let Δ_ν denote the time between rounds. The round that takes place at time T ,¹¹ denoted by $t^*(\nu)$, is given by $t^*(\nu) = T/\Delta_\nu$. Then, as $\nu \rightarrow \infty$, we have $\Delta_\nu \rightarrow 0$, $\delta \rightarrow 1$, and $t^*(\nu) \rightarrow \infty$. The probability of entry before time T is the probability of entry through round $t^*(\nu)$, given by the expression $1 - (1 - F(\alpha_\nu^{t^*(\nu)}))^2$.

Suppose that $1 - (1 - F(\alpha_\nu^{t^*(\nu)}))^2$ does not converge to one as $\nu \rightarrow \infty$. Then there is a subsequence of equilibria, $\{\alpha_\nu^t\}_{t=1}^\infty$, such that $\lim_{\nu \rightarrow \infty} 1 - (1 - F(\alpha_\nu^{t^*(\nu)}))^2 = \rho < 1$. Then we have $\lim_{\nu \rightarrow \infty} F(\alpha_\nu^{t^*(\nu)}) = 1 - (1 - \rho)^{1/2} < 1$. Now consider the case in which $\delta = 1$ holds. The proof of Proposition 1 also implies the existence of a solution to (6) when we have $\delta = 1$, which we denote by $\{\alpha^t\}$. It follows that there exists t^{**} such that $F(\alpha^t) > 1 - (1 - \rho)^{1/2}/2$ for all $t \geq t^{**}$.¹² By continuity, there exists $\bar{\nu}$ such that $\nu > \bar{\nu}$ implies

$$\left| F(\alpha_\nu^{t^{**}}) - \left(1 - \frac{\sqrt{1-\rho}}{2}\right) \right| < \frac{\sqrt{1-\rho}}{2}.$$

Therefore, $\nu > \bar{\nu}$ implies $F(\alpha_\nu^{t^{**}}) > 1 - (1 - \rho)^{1/2}$, a contradiction. *Q.E.D.*

Since in the limit, when we have $\delta = 1$, entry occurs before any real time has passed, it is interesting to compare the properties of our equilibrium with the symmetric equilibrium of the static game, where firms have only one opportunity to enter the market, and the firms decide simultaneously.¹³ Symmetric equilibrium in the static game is characterized by a scalar, α , where firms with entry cost less than or equal to α enter the market, and firms with higher entry cost stay out. Whereas in the dynamic model the cutoff firm’s profits must equal the profits from waiting, in the static model the cutoff firm’s profits must equal zero, yielding

$$F(\alpha^*)R_d + (1 - F(\alpha^*))R_m - \alpha^* = 0. \tag{7}$$

We now show that, for given parameters, $\alpha^* > \alpha^1$, so the probability of entry in round 1 is greater for the static model than for the dynamic model. The intuition is that the opportunity cost of entry is zero in the static model, while the opportunity cost of entry is the (positive) continuation profits in the dynamic model. Of course, the probability of eventual entry in the dynamic model is one, which must exceed $F(\alpha^*)$.

Proposition 5. For given parameters, the probability of entry in round 1 is greater for the static model than for the dynamic model. That is, we have $\alpha^* > \alpha^1$.

¹¹ Assuming that T/Δ_ν is an integer is merely a matter of convenience, to avoid cluttering the notation of the proof.

¹² The only difficulty would arise if the convergence of $F(\alpha^t)$ to one became infinitely slow when we have $\delta = 1$. However, the stable manifold defined in the proof of Proposition 1 has slope less than $(1 - R_d)/(R_m - R_d)$, which is strictly below one.

¹³ See our working paper, Levin and Peck (2002), for some comparative-statics results for the one-round game.

Proof. We determine α^1 by equating the profits of entering in round 1, given by expression (4), to the profits of entering in round 2 (if the rival does not enter in round 1), given by expression (5). In equilibrium, (5) must be positive, because firms with entry costs higher than α^1 are willing to enter. By letting $t = 1$ hold in expression (4), it follows that, for some $K > 0$, α^1 must solve

$$F(\alpha^1)R_d + (1 - F(\alpha^1))R_m - \alpha^1 - K = 0. \tag{8}$$

Treating α^1 as an implicit function of K , we can calculate

$$\frac{\partial \alpha^1}{\partial K} = -\frac{1}{1 + (R_m - R_d)F'(\alpha^1)} < 0. \tag{9}$$

Since α^* solves (8) with $K = 0$, we conclude from (9) that $\alpha^* > \alpha^1$ holds. *Q.E.D.*

Let us interpret the time between rounds as the gap between the time a firm commits to enter and the time the entry decision is communicated to or learned by the rival. Then even when entry decisions can be made and communicated quickly, relative to the lifetime of the market, it is important to view the competition as dynamic. In equilibrium, there is a flurry of decision making, where the rival firms feel each other out, hesitating for fear that both firms might commit to enter at the same time. Eventually, one or both of the firms will reach a “comfort level” in which the lower chance of duopoly later is outweighed by the profit opportunity now. What is important is not the passage of real time, but the passage of decision-making rounds, since inferences are made about the rival’s cost and probability of entry. Taking the limit, as time intervals become shorter, maintains a rich dynamic process to determine which firm or firms enter. This gives a different, and more appropriate, description of behavior than the static case.

4. Asymmetric equilibria

■ In this section, we consider the possibility of asymmetric equilibria. We begin by looking for interior equilibria, characterized by strictly increasing sequences, $\{\alpha_1^t\}$ and $\{\alpha_2^t\}$. Below we motivate the conjecture that the only interior equilibrium is the symmetric equilibrium characterized in Section 3, so there are no interior asymmetric equilibria. Consider the decision of firm 1 in round t . For the marginal firm with entry cost $c_i = \alpha_1^t$, the expected profits from entering in round t should equal the expected profits of waiting to enter in round $t + 1$ (if firm 2 does not enter in round t). The profits of entering in round t are given by the expression

$$\left(\frac{1 - F(\alpha_2^t)}{1 - F(\alpha_2^{t-1})} \right) R_m(\delta)^t + \left(\frac{F(\alpha_2^t) - F(\alpha_2^{t-1})}{1 - F(\alpha_2^{t-1})} \right) R_d(\delta)^t - \alpha_1^t(\delta)^t. \tag{10}$$

The profits of waiting until round $t + 1$ are given by

$$\left(\frac{1 - F(\alpha_2^{t+1})}{1 - F(\alpha_2^{t-1})} \right) R_m(\delta)^{t+1} + \left(\frac{F(\alpha_2^{t+1}) - F(\alpha_2^t)}{1 - F(\alpha_2^{t-1})} \right) R_d(\delta)^{t+1} - \alpha_1^t(\delta)^{t+1} \left(\frac{1 - F(\alpha_2^t)}{1 - F(\alpha_2^{t-1})} \right). \tag{11}$$

Equating expressions (10) and (11) yields one difference equation, and equating the analogous expressions for firm 2 (switching the subscripts 1 and 2) yields another difference equation. Our boundary conditions are $\alpha_1^0 = \underline{c}$ and $\alpha_2^0 = \underline{c}$. As we show in Lemma 3, the terminal condition requires $\lim_{t \rightarrow \infty} \alpha_i^t = 1$ for each firm i .

Lemma 3. For any interior sequential equilibrium, characterized by strictly increasing sequences $\{\alpha_1^t\}$ and $\{\alpha_2^t\}$, we have $\lim_{t \rightarrow \infty} \alpha_i^t = 1$ for each firm i .

Proof. Suppose instead that $\lim_{t \rightarrow \infty} \alpha_2^t < 1$ holds. Then there is a t such that, conditional on no entry before round t , the probability of firm 2 entering in round t is arbitrarily small. Therefore,

firm 1's conditional expected profits are arbitrarily close to $R_m - c_1$. However, since $\{\alpha_1^t\}$ is a strictly increasing sequence, some types of firm 1 enter after round $t + \tau$, for any $\tau > 0$. These firms receive profits, conditional on no entry before round t , less than $(R_m - c_1)(\delta)^\tau$. For sufficiently large τ , profits would be higher by entering in round t , a contradiction. *Q.E.D.*

We conjecture that the only interior¹⁴ equilibrium is the symmetric equilibrium characterized in Section 3. The motivation for this conjecture, based on local analysis near the steady state, is presented in the Appendix.

Conjecture 1. Consider a sequential equilibrium with entry intervals that are strictly increasing sequences, $\{\alpha_1^t\}$ and $\{\alpha_2^t\}$. Then $\{\alpha_1^t\}$ and $\{\alpha_2^t\}$ both equal the unique entry intervals of the symmetric equilibrium.

5. Extension to n firms

■ The techniques used to analyze the duopoly model can be applied to the model with an arbitrary number of firms, n . In fact, a stronger characterization is possible when the number of potential entrants approaches infinity. We assume throughout this section that we are in the Bertrand variant of the model, where revenue is zero whenever more than one firm is active. Actually, the same analysis goes through if revenue is positive when more than one firm is active, as long as the revenue is independent of the number of active firms. We cannot solve the general oligopoly case, because a firm may consider entering after it sees that some rivals have entered, forcing us to consider many continuation paths of the game. With our Bertrand assumption, the game is essentially over after the first entry. We first derive the difference equation that determines the intervals of entry-cost types that enter in each round, conditional on no previous entry. Next, we show the existence and uniqueness of symmetric equilibrium and provide a complete characterization when the number of firms is large. We then address the robustness of Dixit-Shapiro to heterogeneity, by looking at two types of perturbations. Finally, we discuss extending our model to allow firms to avoid some of their entry costs by suspending activity.

Conditional on no entry before round t , the highest entry-cost type that enters in round t , α^t , receives profits

$$\left(\frac{1 - F(\alpha^t)}{1 - F(\alpha^{t-1})} \right)^{n-1} R_m(\delta)^t - \alpha^t(\delta)^t. \tag{12}$$

If this firm waits until round $t + 1$ and enters if none of its rivals have entered, its expected profits (conditional on no entry before round t) are given by

$$\left(\frac{1 - F(\alpha^{t+1})}{1 - F(\alpha^{t-1})} \right)^{n-1} R_m(\delta)^{t+1} - \alpha^t(\delta)^{t+1} \left(\frac{1 - F(\alpha^t)}{1 - F(\alpha^{t-1})} \right)^{n-1}. \tag{13}$$

Equating expressions (12) and (13), and simplifying, yields the difference equation

$$[1 - F(\alpha^t)]^{n-1} R_m - \alpha^t [1 - F(\alpha^{t-1})]^{n-1} = [1 - F(\alpha^{t+1})]^{n-1} R_m \delta - \alpha^t \delta [1 - F(\alpha^t)]^{n-1}. \tag{14}$$

To simplify further, define $G(\alpha^t)$ as follows:

$$G(\alpha^t) \equiv [1 - F(\alpha^t)]^{n-1}. \tag{15}$$

The economic interpretation of $G(\alpha^t)$ is the unconditional probability that all $n - 1$ of a firm's

¹⁴ There is a unique asymmetric corner equilibrium, at least when δ is close enough to one. See Levin and Peck (2002) for details.

rivals remain out of the market through round t . Then we can rewrite (14) as

$$G(\alpha^{t+1}) = \frac{G(\alpha^t)[R_m + \alpha^t \delta] - \alpha^t G(\alpha^{t-1})}{R_m \delta}. \tag{16}$$

Proposition 6. There exists a unique symmetric equilibrium, satisfying (16) for all t and the boundary conditions $\alpha^0 = \underline{c}$ and $\lim_{t \rightarrow \infty} \alpha^t = 1$.

Proof. See the Appendix.

The proof of Proposition 6 follows the same technique used to prove Proposition 1. We omit the straightforward derivation of comparative-statics results analogous to Propositions 2–4. From (16), we can derive a closed-form expression for the limiting equilibrium, as $n \rightarrow \infty$.

Proposition 7. Taking the limit of symmetric equilibria, as $n \rightarrow \infty$, we have, for each t ,

$$G(\alpha^t) = \left(\frac{\underline{c}}{R_m} \right)^t. \tag{17}$$

The limiting probability of monopoly is

$$\frac{-\left(\frac{\underline{c}}{R_m}\right) \log\left(\frac{\underline{c}}{R_m}\right)}{1 - \left(\frac{\underline{c}}{R_m}\right)}.$$

If we have $\underline{c} = 0$, then the limiting probability of monopoly is zero, so the probability of some coordination failure is one.

Proof. Let t be the first round for which we do not have $\lim_{n \rightarrow \infty} \alpha^t = \underline{c}$. Thus, we have $\lim_{n \rightarrow \infty} \alpha^\tau = \underline{c}$ for $\tau = 1, \dots, t - 1$, and $\lim_{n \rightarrow \infty} \alpha^t > \underline{c}$. Then, conditional on no entry until round t , the probability of having no rival enter in round t approaches zero as $n \rightarrow \infty$. Thus, expected revenues approach zero, but firms entering in round t incur strictly positive cost of entry. For the marginal firm, expected profits from entering in round t must be negative for large-enough n , a contradiction. We conclude that, for any fixed t , we have

$$\lim_{n \rightarrow \infty} \alpha^t = \underline{c}. \tag{18}$$

From (16) and (18), we can characterize the limiting difference equation (as $n \rightarrow \infty$) as¹⁵

$$G(\alpha^{t+1}) = G(\alpha^t) \left[\frac{1}{\delta} + \frac{\underline{c}}{R_m} \right] - G(\alpha^{t-1}) \frac{\underline{c}}{R_m \delta}. \tag{19}$$

From the boundary conditions, $G(\alpha^0) = 1$ and $\lim_{t \rightarrow \infty} G(\alpha^t) = 0$, the solution to (19) is given by (17).

The probability of monopoly originating in round t is $G(\alpha^t)n[F(\alpha^t) - F(\alpha^{t-1})]$. From (15) and (17), we have, for large n ,

$$F(\alpha^t) = 1 - \left(\frac{\underline{c}}{R_m} \right)^{t/(n-1)}.$$

¹⁵ Note that while (18) implies that the probability that any particular firm enters approaches zero, the probability that at least one firm enters need not (and will not) approach zero.

The limiting probability of monopoly originating in round t is therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{c}{R_m}\right)^t n \left(\frac{c}{R_m}\right)^{(t-1)/(n-1)} \left[1 - \left(\frac{c}{R_m}\right)^{1/(n-1)}\right] &= \left(\frac{c}{R_m}\right)^t \lim_{n \rightarrow \infty} n \left[1 - \left(\frac{c}{R_m}\right)^{1/(n-1)}\right] \\ &= -\left(\frac{c}{R_m}\right)^t \log\left(\frac{c}{R_m}\right). \end{aligned} \tag{20}$$

From (20), the limiting probability of monopoly in any round is the sum of the geometric series,

$$\sum_{t=1}^{\infty} -\left(\frac{c}{R_m}\right)^t \log\left(\frac{c}{R_m}\right) = \frac{-\left(\frac{c}{R_m}\right) \log\left(\frac{c}{R_m}\right)}{1 - \left(\frac{c}{R_m}\right)}. \tag{21}$$

From (21), we see that the probability of monopoly is zero if $c = 0$ holds. *Q.E.D.*

We now compare our model to the analogous version of Dixit-Shapiro. The maximum number of firms that the industry can accommodate is one (in their notation, $M = 1$), and exit is not allowed. Our purpose is to show that Dixit-Shapiro is best understood as the limiting case of our model, as the support for the entry-cost distribution shrinks. We first show that the mixed-strategy equilibrium in Dixit-Shapiro yields perverse predictions when the model is perturbed to allow observable but heterogeneous costs. Firms with higher entry cost mix with a higher probability of entry than firms with lower entry cost. As a result, firms with higher entry cost are more likely to enter. Next, we show that for the limiting symmetric pure-strategy equilibrium of our model as $c \rightarrow 1$, the probability of entry in any round converges to the probability of entry in the Dixit-Shapiro mixed-strategy equilibrium. In these pure-strategy equilibria, however, firms with higher entry costs always enter with (weakly) lower probability. Thus, the “right” generalization of Dixit-Shapiro is to introduce heterogeneity with uncertainty and private information, and to consider pure-strategy equilibria. The intuition is that a firm’s mixing probability in Dixit-Shapiro makes the *other* entrants indifferent between entering and not, while in our model the marginal type (which determines the entry probability) must *itself* be indifferent between entering and not.

Consider the game with n firms, where firm 1 is known to have an entry cost of \hat{c} , and firms 2 through n are known to have an entry cost of one. Monopoly revenue is $R_m > 1$, and revenue is zero if more than one firm enters. This is the Dixit-Shapiro model with deterministic heterogeneity added. Since no firm will enter after at least one firm enters, we will solve for the type-symmetric mixed-strategy equilibrium characterized by the probability that each type of firm enters in any round, conditional on no previous entry. Denote the (conditional) probability that firm 1 enters as \hat{q} , and denote the (conditional) probability that firm i enters (for $i = 2, \dots, n$) as q .

Proposition 8. If we have $\hat{c} < 1$, then $\hat{q} < q$ holds, and if we have $\hat{c} > 1$, then $\hat{q} > q$ holds. In both cases, the firm or firms with the higher entry costs enter with higher probability.

Proof. Because of the stationarity of the environment, and the requirement for mixed-strategy equilibrium that a firm is indifferent between entering and waiting, it follows that all firms receive expected profits of zero in equilibrium. Therefore, we have

$$(1 - q)^{n-2}(1 - \hat{q})R_m - 1 = 0 \tag{22}$$

and

$$(1 - q)^{n-1}R_m - \hat{c} = 0. \tag{23}$$

Solving (22) and (23), we have

$$q = 1 - \left(\frac{\hat{c}}{R_m}\right)^{1/(n-1)} \tag{24}$$

and

$$\hat{q} = 1 - \left(\frac{1}{R_m}\right) \left(\frac{\hat{c}}{R_m}\right)^{-(n-2)/(n-1)}. \tag{25}$$

Rearranging (24) and (25) and simplifying, we have

$$\frac{1 - q}{1 - \hat{q}} = \hat{c},$$

from which the result follows. *Q.E.D.*

Notice that because the continuation payoff is zero in the mixed-strategy equilibrium, the same entry probabilities constitute a mixed-strategy equilibrium to the static game. The perverse result, that firms with higher entry costs enter with *higher* probability, therefore holds in the static game as well.¹⁶

Now we return to the model with privately observed entry costs, but consider the case in which $\underline{c} \rightarrow 1$. This can be interpreted as a perturbation of Dixit-Shapiro to introduce a small amount of uncertainty about entry costs.

Proposition 9. In the limit, as $\underline{c} \rightarrow 1$, the probability of entry in any round converges to the probability of entry in the Dixit-Shapiro mixed-strategy equilibrium.

Proof. It is clear that for \underline{c} sufficiently close to one, each α^t must be within a small neighborhood of one, so the difference equation characterizing equilibrium, (16), becomes

$$G(\alpha^{t+1}) = \frac{G(\alpha^t)[R_m + \delta] - G(\alpha^{t-1})}{R_m \delta}. \tag{26}$$

Equation (26), along with the boundary conditions, $G(\alpha^0) = 1$ and $\lim_{t \rightarrow \infty} G(\alpha^t) = 0$, has the closed-form solution

$$G(\alpha^t) = \left(\frac{1}{R_m}\right)^t. \tag{27}$$

From (27) and (15), we have

$$F(\alpha^t) = 1 - \left(\frac{1}{R_m}\right)^{t/(n-1)}. \tag{28}$$

Based on (28), we can write the probability of a firm entering in round t , conditional on no entry before round t , as

$$\frac{F(\alpha^t) - F(\alpha^{t-1})}{1 - F(\alpha^{t-1})} = 1 - \left(\frac{1}{R_m}\right)^{1/(n-1)} = 1 - (R_m)^{-1/(n-1)}. \tag{29}$$

The right-hand expression in (29) is easily seen to be the probability of entry in the Dixit-Shapiro mixed-strategy equilibrium for the corresponding game with known entry cost equal to one. *Q.E.D.*

Cabral (1993) extends Dixit-Shapiro by allowing a firm’s payoff to increase with the number of rounds in which it has been active. Experience advantages considerably change the nature of equilibrium when exit is taken into account. The game can alternate between a “grab the dollar”

¹⁶ For simplicity and brevity, our example is for the Bertrand variant where the maximum number of firms the market can accommodate is one. We have analyzed another example in which the market can accommodate two firms, and again we find that firms with higher entry cost mix with a higher probability of entry at each stage of the game. Details are available upon request.

regime, which occurs when the market can profitably accommodate more firms, and a “war of attrition” regime, which occurs when the least-experienced firms cannot profitably remain in the market.

A similar phenomenon can arise in our framework. Suppose that when firm i enters, it incurs an entry cost, c_i/T , for each of the first T rounds that it is active. Think of the firm as having the ability to halt construction and save part of its entry costs. Now the equilibrium will have the following properties. There continues to be a sequence defining the cutoff types that enter in each round, conditional on no previous entry, $\{\alpha^t\}_{t=1}^\infty$. If only one firm is the first to enter, then it is known to have the lowest entry cost and would win any subsequent war of attrition; that firm remains active and becomes the monopolist. If two or more firms are the first to enter in round t , then those firms are known to have costs in the interval $[\alpha^{t-1}, \alpha^t]$, and a war of attrition ensues. There is a cutoff entry cost α_1^t , above which firms become inactive in round $t + 1$, a cutoff entry cost α_2^t , such that firms with entry cost in the interval $[\alpha_2^t, \alpha_1^t]$ become inactive in round $t + 2$, and so on until we find the interval $[\alpha_T^t, \alpha_{T-1}^t]$ of types that remain active throughout.¹⁷ Once a firm becomes inactive, it is known to have entry cost higher than any active firm, so the firm remains inactive forever. If the war of attrition reduces to one active firm, that firm remains active and becomes the monopolist. If all firms that were active in round $t + \tau - 1$ become inactive in round $t + \tau$, then these firms play a new “grab the dollar” game, where initial entry costs are known to be in the interval $[\alpha_{\tau+1}^t, \alpha_\tau^t]$.¹⁸ Of course, simultaneous resumption of activity leads to another war of attrition, and so on.

Moving beyond this qualitative characterization is extremely difficult, even for the case $n = 2$. For starters, there are many, many “grab the dollar” and “war of attrition” games to be solved, based on how many rounds firms have been active and the current interval of entry-cost types. Moreover, the fact that difference equation (6) has no closed-form solution when $T = 1$ implies that the analog of equation (6) cannot even be specified when $T > 1$. For example, the expected revenue received when two firms simultaneously enter, which is zero when $T = 1$ (since R_d is set to zero), is the solution to a war of attrition game for which there may be no closed-form solution when $T > 1$.

6. Concluding remarks

■ A subject for future research is to introduce many potential entrants, where there is room for several firms to be profitable, more in line with Dixit-Shapiro. Then there must be a positive probability that so many firms will enter in round 1 as to make the market unprofitable for those that do not enter. However, unlike Dixit-Shapiro, the number of firms that the market can support depends on the realization of entry costs, so firms learn over time whether entry is viable for them. The number of entrants in a given round causes the interval of entrants for the next round to fluctuate. We conjecture that there are two effects determining the decision to enter, similar to, but not quite the same as, those in Bulow and Klemperer (1994). Any new entry in round t means that the market is closer to being “full,” so this effect would tend to shrink the interval of types that enter in round $t + 1$. On the other hand, the number of firms that enter in round t can be greater than expected or less than expected. If few firms enter in round t , firms update their beliefs about where their entry costs rank among those that have not yet entered, and this effect would encourage entry in round $t + 1$. If many firms enter in round t , then both effects would discourage entry.

Bolton and Farrell (1990) perform a welfare analysis in which gross social benefits of entry are assumed to depend on whether entry occurs, but not the number of entrants. For the example they study, in which costs are either high or low, an explicit expression for welfare is derived. The

¹⁷ When we have $n > 2$, then α_τ^t depends on the history of how many firms become inactive in which rounds, which enormously complicates computing the equilibrium. Also, depending on parameter values, it is possible to have a corner solution in which any firm active for τ rounds remains active for all T rounds and beyond.

¹⁸ Remaining entry costs are in the interval $[\frac{T-\tau}{T}\alpha_{\tau+1}^t, \frac{T-\tau}{T}\alpha_\tau^t]$, and these costs are incurred over $T - \tau$ rounds.

outcome of this “decentralized” game is compared to random assignment by a central planner. For some parameter values, welfare is higher under the central planner. The advantage of the planner is that decisions can be made quickly and in a coordinated fashion. The advantage of the decentralized mechanism is that market incentives select the lower-cost entrant. Bolton and Farrell make a brilliant application of the model to the famous Lange-Lerner-Hayek debate on decentralized versus centralized systems. We merely add that the comparison extends beyond their example. Once the parameters and cost distribution are specified, and the equilibrium is computed by solving the dynamical system numerically, it is a simple matter to compute welfare under decentralization and centralization. For the examples presented in Tables 1 and 2 of Levin and Peck (2002), the central planner outperforms the decentralized competition. It would be interesting to perform such a welfare comparison after separately specifying consumer surplus associated with monopoly and duopoly. That is, suppose that the planner can choose which firms enter but cannot affect the postentry competition. Since duopoly is a coordination failure for the firms but beneficial to consumers, it is possible that there is too little entry under decentralization.

Appendix

■ Proofs of Propositions 1–3 and 6, and the motivation for Conjecture 1 follow.

Proof of Proposition 1. Consider the dynamical system in \mathbb{R}^2 , where the coordinates are denoted by (x, y) . Let $x = F(\alpha^{t-1})$ and $y = F(\alpha^t)$. Then, from equation (6), the transition of the system is given by

$$\begin{aligned} x &\rightarrow y \\ y &\rightarrow g(x, y), \end{aligned}$$

where $g(x, y)$ is given by the expression

$$\frac{(F^{-1}(y) - R_m)(1 - \delta) + y[R_m - (1 + \delta)R_d + F^{-1}(y)\delta] - x(F^{-1}(y) - R_d)}{\delta(R_m - R_d)}.$$

The boundary condition we require is $\alpha^0 = c$, and the terminal condition is $\lim_{t \rightarrow \infty} \alpha^t = 1$. We will show that there is a stable manifold of the system converging to $(1, 1)$, so that the terminal condition can be satisfied. Evaluated at $(1, 1)$, we can compute

$$\frac{\partial g(x, y)}{\partial x} = \frac{R_d - 1}{\delta(R_m - R_d)}$$

and

$$\frac{\partial g(x, y)}{\partial y} = \frac{R_m - (1 + \delta)R_d + \delta}{\delta(R_m - R_d)},$$

which yields the Jacobian matrix,

$$\begin{pmatrix} 0 & 1 \\ \frac{R_d - 1}{\delta(R_m - R_d)} & \frac{R_m - (1 + \delta)R_d + \delta}{\delta(R_m - R_d)} \end{pmatrix}.$$

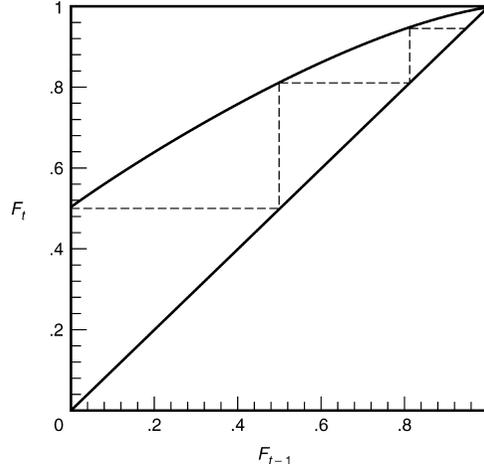
This matrix has eigenvalues, $1/\delta$ and $(1 - R_d)/(R_m - R_d)$. Because one eigenvalue is greater than one and the other eigenvalue is between zero and one, this establishes that the system has a stable manifold such that if we start on the manifold, we converge to the steady state $(1, 1)$. (See, for example, Stokey and Lucas (1989).)

We now show that there is some value of α^1 such that $(0, F(\alpha^1))$ is on the stable manifold. Looking at the backward dynamics, we can always solve for F_{t-1} , given F_t and F_{t+1} , so the manifold must either: cross the y-axis between 0 and 1, cross the line segment between $(0, 1)$ and $(1, 1)$, or cross the line segment between $(0, 0)$ and $(1, 1)$. (See Figure A1.) From equation (6), we can derive an expression for the slope of the segment connecting two consecutive points,

$$\frac{(F_{t+1} - F_t)}{(F_t - F_{t-1})} = \frac{\alpha^t - R_d}{\delta(R_m - R_d)} - \frac{(R_m - \alpha^t)(1 - \delta)(1 - F_t)}{\delta(R_m - R_d)(F_t - F_{t-1})}. \tag{A1}$$

First, we show that the stable manifold cannot cross the line segment between $(0, 1)$ and $(1, 1)$. Suppose that we have

FIGURE 1



$F_{t-1} < 1$ and $F_t = \alpha^t = 1$. Then it follows from (A1) that we have

$$\frac{(F_{t+1} - F_t)}{(F_t - F_{t-1})} = \frac{1 - R_d}{\delta(R_m - R_d)} > 0, \tag{A2}$$

which implies $F_{t+1} > 1$.¹⁹ Iterating forward, $F_{t+1} > 1$ implies $(F_{t+2} - F_{t+1})/(F_{t+1} - F_t) > 0$, since the last term being subtracted in (A1) is negative. Therefore, we have $F_{t+2} > F_{t+1}$. Continuing in this way, it is clear that we are not on the stable manifold, since F_t does not converge to one.

Second, we show that the stable manifold cannot cross the line segment between (0,0) and (1,1). If $\delta = 1$ holds, then any point on the line $x = y$ is a fixed point, so it cannot be on the manifold converging to (1,1). If we have $\delta < 1$, then any point on the line $x = y$ implies $g(x, y) = -\infty$, contradicting the fact that $(y, g(x, y))$ must be on the stable manifold.²⁰ The only remaining possibility is that the stable manifold crosses the y -axis at some point, $(0, F(\alpha^1))$. This establishes existence of equilibrium.

To show uniqueness, we will show that the stable manifold, denoted as $y = H(x)$, is strictly monotonic. It follows that we can rewrite equation (6) as

$$(1 - H(x))R_m + (H(x) - x)R_d - (1 - x)F^{-1}(H(x)) \\ = (1 - H(H(x)))R_m\delta + (H(H(x)) - H(x))R_d\delta - (1 - H(x))F^{-1}(H(x))\delta. \tag{A3}$$

Differentiating (A3) with respect to x and solving implicitly for $H'(x)$, we have

$$H'(x) = \frac{F^{-1}(H(x)) - R_d}{R_m - R_d + (F^{-1}(H(x)) - R_d)\delta + \frac{1 - \delta + \delta H(x) - x}{f(H(x))} - (R_m - R_d)\delta H'(H(x))}. \tag{A4}$$

We will use a contraction argument, showing that whenever $H'(H(x))$ is bounded between two constants, then $H'(x)$ is bounded between the same two constants. Therefore, we suppose that $0 < H'(H(x)) < 1/\delta$ holds. The denominator in (A4) is positive for all values of $H'(H(x))$ between zero and $1/\delta$, and the numerator is positive because we can restrict attention to $\alpha^t > R_d$. This establishes that $H'(x) > 0$ holds. Since the denominator in (A4) is decreasing in $H'(H(x))$, we have

$$H'(x) < \frac{F^{-1}(H(x)) - R_d}{(F^{-1}(H(x)) - R_d)\delta + \frac{1 - \delta + \delta H(x) - x}{f(H(x))}} < \frac{1}{\delta}. \tag{A5}$$

¹⁹ Of course, $F_{t+1} > 1$ does not make sense from our knowledge that F_{t+1} represents a distribution function, but we must show that the difference equation yields a sensible solution.

²⁰ The economic intuition behind a negative slope is that when $\delta < 1$ holds, starting at a point (x, y) too close to the 45° line will cause a firm to strictly prefer to enter in round t rather than round $t + 1$. The firm would rather risk the slight chance of duopoly in round t than wait for the discounted flow of monopoly profits one round later. Thus, equation (6) is inconsistent with $F_{t+1} \geq F_t$.

We know that $0 < H'(x) < 1/\delta$ holds in the neighborhood of the steady state, so it must hold everywhere. Because H is monotonic, the stable manifold can intersect the y -axis only once, so there is a unique equilibrium.

To show that each α^t varies continuously with the parameters, we show that the stable manifold varies continuously with the parameters. This is accomplished by first showing continuity within a neighborhood of $(1,1)$ and then showing continuity outside the neighborhood. Let r denote the parameter in question, either R_m , R_d , or δ , and let the stable manifold as a function of r be denoted by $y = H(x, r)$.

Since $H(x, r)$ is tangent to the stable eigenvector at $x = 1$, and the eigenvalues are continuous in r , it follows that $H(x, r)$ is continuous in (x, r) for some neighborhood of the steady state. Specifically, for all $\varepsilon' > 0$, there exists $\gamma > 0$ and $\rho > 0$ such that²¹

$$1 - \gamma \leq x_1 \leq 1, 1 - \gamma \leq x_2 \leq 1, \text{ and } |r_1 - r_2| < \rho \text{ implies } |H(x_1, r_1) - H(x_2, r_2)| < \varepsilon'. \tag{A6}$$

Let $\Gamma(x, y, r) = (y, g(x, y; r))$ hold, and let $\Gamma^N(x, y, r)$ denote the N -fold composition of Γ with itself, representing the forward dynamics of the system. From equation (6), we can construct the mapping corresponding to the backward iterate, $Q(x, y, r) = (\psi(x, y; r), x)$, where we have

$$\psi(x, y; r) = x - \frac{\delta(R_m - R_d)(y - x)}{F^{-1}(x) - R_d} - \frac{(1 - \delta)(R_m - F^{-1}(x))(1 - x)}{F^{-1}(x) - R_d}. \tag{A7}$$

Let $Q^N(x, y, r)$ denote the N -fold composition of Q with itself, representing the backward dynamics of the system. Since ψ is continuous, it follows that Q is continuous, which implies that $Q^N(x, y, r)$ is continuous.

For all r_1 and $\gamma' > 0$, there must exist N such that $\Gamma^N(0, H(0, r_1), r_1) \equiv (x_1, y_1)$ is contained in the ball of radius γ' around $(1, 1)$.²² Fix ε' in (A6), and consider r_2 such that $|r_1 - r_2| < \rho$. For sufficiently small γ' , $1 - \gamma \leq x_1 \leq 1$ must hold. It follows from (A6) that $|H(x_1, r_1) - H(x_1, r_2)| < \varepsilon'$ holds. By continuity of Q^N , for all $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that

$$\|(0, H(0, r_1)) - Q^N(x_1, H(x_1, r_2), r_2)\| < \delta\varepsilon.$$

Since the slope of $H(x_1, r_2)$ is bounded below $1/\delta$, it follows that we have

$$\|(0, H(0, r_1)) - (0, H(0, r_2))\| < \varepsilon. \tag{A8}$$

Since $H(0, r_1)$ is α^1 when the parameter is r_1 and $H(0, r_2)$ is α^1 when the parameter is r_2 , this establishes the continuity of α^1 , from which the continuity of $\alpha^t, t > 1$, follows trivially. *Q.E.D.*

Proof of Proposition 2. Equation (A1) can be rewritten as follows:

$$\frac{\delta \left[\frac{F_{t+1}}{F_t} - 1 \right] + (1 - \delta) \left[\frac{1}{F_t} - 1 \right] \left(\frac{R_m - \alpha^t}{R_m - R_d} \right)}{\left[1 - \frac{F_{t-1}}{F_t} \right]} = \frac{\alpha^t - R_d}{R_m - R_d}. \tag{A9}$$

We start with $\bar{R}_m > R_m$, and first show that $\bar{\alpha}^1 > \alpha^1$ holds. Suppose instead that we have $\bar{\alpha}^1 \leq \alpha^1$. Depending on whether we are looking at the economy with parameter R_m or \bar{R}_m , we introduce the following notation:

$$\begin{aligned} \left[\frac{F_{t+1}}{F_t} - 1 \right] &\equiv A_t \text{ (respectively, } \bar{A}_t) \\ \left[\frac{1}{F_t} - 1 \right] &\equiv B_t \text{ (respectively, } \bar{B}_t) \\ \left[\frac{R_m - \alpha^t}{R_m - R_d} \right] &\equiv C_t \text{ (respectively, } \bar{C}_t) \\ \left[1 - \frac{F_{t-1}}{F_t} \right] &\equiv D_t \text{ (respectively, } \bar{D}_t) \\ \left[\frac{\alpha^t - R_d}{R_m - R_d} \right] &\equiv E_t \text{ (respectively, } \bar{E}_t). \end{aligned} \tag{A10}$$

²¹ To be precise, γ and ρ must be independent of r_1 and r_2 . This is not a problem, because we can restrict parameters (in particular, R_m) to lie in a compact set.

²² We use the Euclidean norm.

From $\bar{\alpha}^1 \leq \alpha^1$ follows $\bar{F}_1 \leq F_1$. Also, we have $0 < B_1 \leq \bar{B}_1, 0 \leq C_1 < \bar{C}_1, D_1 = \bar{D}_1 = 1$, and $0 \leq \bar{E}_1 < E_1$. In order for equation (A9) to hold for both economies (with parameters R_m or \bar{R}_m), this implies

$$\frac{\bar{F}_2}{\bar{F}_1} < \frac{F_2}{F_1}. \tag{A11}$$

Since $\bar{F}_1 \leq F_1$ holds, $\bar{F}_2 < F_2$ and $\bar{\alpha}^2 < \alpha^2$ must hold as well. Thus, we can show that $B_2 < \bar{B}_2, C_2 < \bar{C}_2$, and $\bar{D}_2 < D_2$. Therefore, we have

$$\frac{\bar{F}_3}{\bar{F}_2} < \frac{F_3}{F_2}, \bar{F}_3 < F_3, \text{ and } \bar{\alpha}^3 < \alpha^3. \tag{A12}$$

Rearranging inequalities (A11) and (A12), and proceeding inductively, we have, for all t ,

$$\frac{\bar{F}_t}{F_t} < \frac{\bar{F}_{t-1}}{F_{t-1}} < \dots < \frac{\bar{F}_2}{F_2} < 1. \tag{A13}$$

However, (A13) contradicts the fact that, in equilibrium, $\lim_{t \rightarrow \infty} F_t = 1$ and $\lim_{t \rightarrow \infty} \bar{F}_t = 1$. We conclude that $\bar{\alpha}^1 > \alpha^1$ holds.

Next, we show that $\bar{\alpha}^t > \alpha^t$ holds for all t . Suppose not, and let τ be the *first* round for which the reverse inequality holds, $\bar{\alpha}^\tau \leq \alpha^\tau$. Thus, we have

$$\bar{F}_\tau \leq F_\tau \text{ and } \bar{F}_t > F_t \text{ for all } t < \tau. \tag{A14}$$

From (A14), we can show

$$\frac{\bar{F}_{\tau-1}}{\bar{F}_\tau} > \frac{F_{\tau-1}}{F_\tau}.$$

Thus, we know that we have $B_\tau \leq \bar{B}_\tau, 0 \leq C_\tau < \bar{C}_\tau, \bar{D}_\tau < D_\tau$, and $\bar{E}_\tau < E_\tau$. For (A9) to be satisfied for both economies, we must have

$$\frac{\bar{F}_{\tau+1}}{\bar{F}_\tau} < \frac{F_{\tau+1}}{F_\tau}. \tag{A15}$$

Since $\bar{F}_\tau \leq F_\tau$ holds, inequality (A15) implies $\bar{F}_{\tau+1} < F_{\tau+1}$ and $\bar{\alpha}^{\tau+1} < \alpha^{\tau+1}$. Proceeding inductively as above, we show

$$\dots \frac{\bar{F}_{\tau+2}}{F_{\tau+2}} < \frac{\bar{F}_{\tau+1}}{F_{\tau+1}} < \frac{\bar{F}_\tau}{F_\tau} \leq 1,$$

contradicting $\lim_{t \rightarrow \infty} F_t = 1$ and $\lim_{t \rightarrow \infty} \bar{F}_t = 1$. This establishes that $\bar{R}_m > R_m$ implies $\bar{\alpha}^t > \alpha^t$ holds for all t . To show that $\bar{R}_d > R_d$ implies $\bar{\alpha}^t > \alpha^t$ holds for all t , we repeat the same argument as above, since the inequalities (relating B_t and \bar{B}_t, C_t and \bar{C}_t , and so on) are unchanged. *Q.E.D.*

Proof of Proposition 3. Equation (A1) can be rewritten as follows:

$$\frac{\delta \left[\left(\frac{\alpha^{t+1}}{\alpha^t} \right)^\lambda - 1 \right] + (1 - \delta) \left[\left(\frac{1}{\alpha^t} \right)^\lambda - 1 \right] \left(\frac{R_m - \alpha^t}{R_m - R_d} \right)}{\left[1 - \left(\frac{\alpha^{t-1}}{\alpha^t} \right)^\lambda \right]} = \frac{\alpha^t - R_d}{R_m - R_d}. \tag{A16}$$

We start with $\bar{\lambda} > \lambda$, and first show that $\bar{\alpha}^1 > \alpha^1$ holds. Suppose instead that we have $\bar{\alpha}^1 \leq \alpha^1$. Then (A16) implies

$$\delta \left[\left(\frac{\bar{\alpha}^2}{\bar{\alpha}^1} \right)^{\bar{\lambda}} - 1 \right] + (1 - \delta) \left[\left(\frac{1}{\bar{\alpha}^1} \right)^{\bar{\lambda}} - 1 \right] \left(\frac{R_m - \bar{\alpha}^1}{R_m - R_d} \right) \leq \delta \left[\left(\frac{\alpha^2}{\alpha^1} \right)^\lambda - 1 \right] + (1 - \delta) \left[\left(\frac{1}{\alpha^1} \right)^\lambda - 1 \right] \left(\frac{R_m - \alpha^1}{R_m - R_d} \right). \tag{A17}$$

Since $1/\bar{\alpha}^1 > 1/\alpha^1 > 1$ and $R_m - \bar{\alpha}^1 > R_m - \alpha^1$ hold, we conclude from (A17) that we have

$$\left[\frac{\bar{\alpha}^2}{\bar{\alpha}^1} \right]^{\bar{\lambda}} \leq \left[\frac{\alpha^2}{\alpha^1} \right]^{\lambda}. \tag{A18}$$

Since we have $\bar{\lambda} > \lambda$, $\bar{\alpha}^2 > \bar{\alpha}^1$, and $\alpha^2 > \alpha^1$, we must also have

$$\frac{\bar{\alpha}^2}{\bar{\alpha}^1} < \frac{\alpha^2}{\alpha^1}. \tag{A19}$$

Since $\bar{\alpha}^1 \leq \alpha^1$ holds by assumption, we have

$$\bar{\alpha}^2 < \alpha^2. \tag{A20}$$

It follows from (A20) that the left side of (A16) for the $\bar{\lambda}$ economy is strictly less than the left side of (A16) for the λ economy. From (A18), it follows that the denominator of (A16) for the $\bar{\lambda}$ economy is less than or equal to the denominator of (A16) for the λ economy. Therefore, the numerator of (A16) is strictly smaller for the $\bar{\lambda}$ economy. Since $1/\bar{\alpha}^2 > 1/\alpha^2 > 1$ and $R_m - \bar{\alpha}^2 > R_m - \alpha^2$ hold, we conclude that

$$\left[\frac{\bar{\alpha}^3}{\bar{\alpha}^2} \right]^{\bar{\lambda}} \leq \left[\frac{\alpha^3}{\alpha^2} \right]^{\lambda} \text{ and } \frac{\bar{\alpha}^3}{\bar{\alpha}^2} < \frac{\alpha^3}{\alpha^2}$$

hold. Therefore, we have $\bar{\alpha}^3 < \alpha^3$. Proceeding inductively, we have, for all $t > 0$,

$$\frac{\bar{\alpha}^{t+1}}{\bar{\alpha}^t} < \frac{\alpha^{t+1}}{\alpha^t} \text{ and } \bar{\alpha}^{t+1} < \alpha^t. \tag{A21}$$

Rearranging inequality (A21), we have, for all t ,

$$\frac{\bar{\alpha}^t}{\alpha^t} < \frac{\bar{\alpha}^{t-1}}{\alpha^{t-1}} < \dots < \frac{\bar{\alpha}^2}{\alpha^2} < \frac{\bar{\alpha}^1}{\alpha^1} < 1. \tag{A22}$$

However, (A22) contradicts the fact that, in equilibrium, $\lim_{t \rightarrow \infty} \alpha^t = 1$ and $\lim_{t \rightarrow \infty} \bar{\alpha}^t = 1$. We conclude that $\bar{\alpha}^1 > \alpha^1$ holds.

Next, we show that we have $(\bar{\alpha}^1)^{\bar{\lambda}} < (\alpha^1)^{\lambda}$. Suppose instead that we have

$$(\bar{\alpha}^1)^{\bar{\lambda}} \geq (\alpha^1)^{\lambda}. \tag{A23}$$

Since we have already shown that $\bar{\alpha}^1 > \alpha^1$ holds, it follows that the right side of (A16) is greater for the $\bar{\lambda}$ economy, so the left side must be greater as well. From $\bar{\alpha}^1 > \alpha^1$ and (A23), we conclude that the first term on the left side of (A16) is greater for the $\bar{\lambda}$ economy, which implies

$$\left[\frac{\bar{\alpha}^2}{\bar{\alpha}^1} \right]^{\bar{\lambda}} > \left[\frac{\alpha^2}{\alpha^1} \right]^{\lambda}. \tag{A24}$$

From (A23) and (A24), we have $(\bar{\alpha}^1)^{\bar{\lambda}} > (\alpha^1)^{\lambda}$, which implies $\bar{\alpha}^2 > \alpha^2$. Proceeding inductively, and rearranging the inequalities, we have

$$\dots \frac{(\bar{\alpha}^t)^{\bar{\lambda}}}{(\alpha^t)^{\lambda}} \dots \frac{(\bar{\alpha}^2)^{\bar{\lambda}}}{(\alpha^2)^{\lambda}} > \frac{(\bar{\alpha}^1)^{\bar{\lambda}}}{(\alpha^1)^{\lambda}} \geq 1. \tag{A25}$$

However, (A25) contradicts the fact that, in equilibrium, $\lim_{t \rightarrow \infty} (\alpha^t)^{\lambda} = 1$ and $\lim_{t \rightarrow \infty} (\bar{\alpha}^t)^{\bar{\lambda}} = 1$. We conclude that $(\bar{\alpha}^1)^{\bar{\lambda}} < (\alpha^1)^{\lambda}$ holds. *Q.E.D.*

Motivation for Conjecture 1. Consider the dynamical system in \mathfrak{R}^4 , where the coordinates are denoted by (x_1, y_1, x_2, y_2) . Let $x_1 = F(\alpha_2^{t-1})$, $y_1 = F(\alpha_2^t)$, $x_2 = F(\alpha_1^{t-1})$, and $y_2 = F(\alpha_1^t)$ hold. Then the transition of the system is given by

$$\begin{aligned} x_1 &\rightarrow y_1 \\ y_1 &\rightarrow g_1(x_1, y_1, F^{-1}(y_2)) \\ x_2 &\rightarrow y_2 \\ y_2 &\rightarrow g_2(x_2, y_2, F^{-1}(y_1)), \end{aligned}$$

where $g_1(x_1, y_1, F^{-1}(y_2))$ is given by the expression

$$\frac{(F^{-1}(y_2) - R_m)(1 - \delta) + y_1[R_m - (1 + \delta)R_d + F^{-1}(y_2)\delta] - x_1(F^{-1}(y_2) - R_d)}{\delta(R_m - R_d)}$$

There is an analogous expression for $g_2(x_2, y_2, F^{-1}(y_1))$ with the subscripts reversed.

We will show that there is a stable manifold of the system converging to $(1,1,1,1)$. Evaluated at $(1,1,1,1)$, we can compute for $i = 1,2$,

$$\frac{\partial g_i}{\partial x_i} = \frac{R_d - 1}{\delta(R_m - R_d)} \quad \text{and} \quad \frac{\partial g_i}{\partial y_i} = \frac{R_m - (1 + \delta)R_d + \delta}{\delta(R_m - R_d)}$$

Cross-derivatives are zero, so we must compute the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{R_d - 1}{\delta(R_m - R_d)} & \frac{R_m - (1 + \delta)R_d + \delta}{\delta(R_m - R_d)} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{R_d - 1}{\delta(R_m - R_d)} & \frac{R_m - (1 + \delta)R_d + \delta}{\delta(R_m - R_d)} \end{pmatrix},$$

yielding eigenvalues $(1 - R_d)/(R_m - R_d)$, $1/\delta$, $(1 - R_d)/(R_m - R_d)$, $1/\delta$. Since there are two real eigenvalues greater than one and two real eigenvalues less than one, there is a stable manifold of dimension 2. From the analysis of the symmetric equilibrium, and the fact that (10) and (11) are equivalent to (4) and (5) when $\alpha_1^{t-1} = \alpha_2^{t-1}$ and $\alpha_1^t = \alpha_2^t$, we know that there exists (symmetric) α_1^1 and α_2^1 such that $(0, F(\alpha_1^1), 0, F(\alpha_2^1))$ is on the stable manifold. If the local approximation held globally, this would be the only equilibrium, because the two-dimensional stable manifold (in (x_1, y_1, x_2, y_2) space) would intersect the two-dimensional subspace (defined by $x_1 = 0$ and $x_2 = 0$) at exactly one point. (See, for example, Stokey and Lucas (1989).)

Proof of Proposition 6. Set up a dynamical system like the system in the proof of Proposition 1, where here we have

$$\begin{aligned} x &= G(\alpha^{t-1}), \\ y &= G(\alpha^t), \quad \text{and} \\ g(x, y) &= \frac{y(R_m + G^{-1}(y)\delta) - G^{-1}(y)x}{R_m \delta}. \end{aligned} \tag{A26}$$

Here the steady state occurs at $(0, 0)$, and the Jacobian matrix in the neighborhood of the steady state is given by

$$\begin{pmatrix} 0 & 1 \\ -\frac{1}{R_m \delta} & \frac{R_m + \delta}{R_m \delta} \end{pmatrix},$$

which has the eigenvalues $1/R_m$ and $1/\delta$. Therefore, there is a one-dimensional stable manifold, given by the formula $y = H(x)$. An argument along the lines given in the proof of Proposition 1 guarantees that the stable manifold remains between the 45° line and the x-axis, and must therefore cross the line, $x = 1$. To show uniqueness, we will show that the stable manifold is strictly monotonic. From the definitions of x , y , and H , we can rewrite (16) as

$$H(x)R_m - G^{-1}(H(x))x = H(H(x))R_m \delta - G^{-1}(H(x))\delta H(x). \tag{A27}$$

Implicit differentiation of (A27) yields

$$H'(x) = \frac{G^{-1}(H(x))}{R_m - \frac{x - \delta H(x)}{G'(H(x))} - H'(H(x))R_m \delta + G^{-1}(H(x))\delta} \tag{A28}$$

Suppose $0 < H'(H(x)) < 1/\delta$ holds. Then the denominator of (A28) exceeds $G^{-1}(H(x))\delta - [(x - \delta H(x))/G'(H(x))]$. We know that $H(x) < x$ and $G'(\alpha) < 0$ hold, which implies that the denominator of (A28) is positive and exceeds $G^{-1}(H(x))\delta$. Therefore, we have $0 < H'(x) < 1/\delta$, which completes the contraction argument. *Q.E.D.*

References

- BLISS, C. AND NALEBUFF, B. "Dragon-slaying and Ballroom Dancing: The Private Supply of a Public Good." *Journal of Public Economics*, Vol. 25 (1984), pp. 1–12.
- BOLTON, P. AND FARRELL, J. "Decentralization, Duplication, and Delay." *Journal of Political Economy*, Vol. 98 (1990), pp. 803–826.
- BULOW, J. AND KLEMPERER, P. "Rational Frenzies and Crashes." *Journal of Political Economy*, Vol. 102 (1994), pp. 1–23.
- AND ———. "The Generalized War of Attrition." *American Economic Review*, Vol. 89 (1999), pp. 175–189.
- CABRAL, L.M.B. "Experience Advantages and Entry Dynamics." *Journal of Economic Theory*, Vol. 59 (1993), pp. 403–416.
- CRAMTON, P.C. "Strategic Delay in Bargaining with Two-Sided Uncertainty." *Review of Economic Studies*, Vol. 59 (1992), pp. 205–225.
- DENECKERE, R. AND PECK, J. "Demand Uncertainty, Endogenous Timing, and Costly Waiting: Jumping the Gun in Competitive Models." Mimeo, Department of Economics, The Ohio State University, 1999.
- DIXIT, A.K. AND SHAPIRO, C. "Entry Dynamics with Mixed Strategies." In L.G. Thomas III, ed., *The Economics of Strategic Planning: Essays in Honor of Joel Dean*. Lexington, Mass.: Lexington Books, 1986.
- FUDENBERG, D. AND TIROLE, J. "Preemption and Rent Equalization in the Adoption of New Technology." *Review of Economic Studies*, Vol. 52 (1985), pp. 383–401.
- HARSTAD, R.M. "Alternative Common-Value Auctions Procedures: Revenue Comparisons with Free Entry." *Journal of Political Economy*, Vol. 98 (1990), pp. 421–429.
- LEVIN, D. AND PECK, J. "To Grab for the Market or to Bide One's Time: A Dynamic Model of Entry." Mimeo, The Ohio State University, 2002.
- AND SMITH, J.L. "Equilibrium in Auctions with Entry." *American Economic Review*, Vol. 84 (1994), pp. 585–599.
- LI, H. AND ROSEN, S. "Unraveling in Matching Markets." *American Economic Review*, Vol. 88 (1998), pp. 371–387.
- MILGROM, P. "Rational Expectations, Information Acquisition, and Competitive Bidding." *Econometrica*, Vol. 49 (1981), pp. 921–943.
- REINGANUM, J.F. "On the Diffusion of New Technology: A Game Theoretic Approach." *Review of Economic Studies*, Vol. 48 (1981), pp. 395–406.
- ROTH, A.E. AND XING, X. "Jumping the Gun: Imperfections and Institutions Related to the Timing of Market Transactions." *American Economic Review*, Vol. 84 (1994), pp. 992–1044.
- SMITH, J.L. AND LEVIN, D. "Entry Coordination, in Auctions and Social Welfare: An Experimental Investigation." *International Journal of Game Theory*, Vol. 30 (2001), pp. 321–350.
- STOKEY, N.L. AND LUCAS, R.E., JR. *Recursive Methods in Economic Dynamics*. Cambridge, Mass.: Harvard University Press, 1989.
- VETTAS, N. "On Entry, Exit, and Coordination with Mixed Strategies." *European Economic Review*, Vol. 44 (2000a), pp. 1557–1576.
- . "Investment Dynamics in Markets with Endogenous Demand." *Journal of Industrial Economics*, Vol. 48 (2000b), pp. 189–203 and supplementary appendix.