Investment Dynamics with Common and Private Values[1]

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[1]This material is based upon work supported by the NSF under Grant No. SES-0417352. Any opinions, findings and conclusions or recommendations expressed are those of the authors and do not necessarily reflect the views of the NSF. We thank the John Glenn Institute at Ohio State for their support. We are particularly grateful to Asen Ivanov for identifying problems in an earlier draft, and to an associate editor and two referees. This paper has benefitted from comments by seminar participants at ITAM, IUPUI, University of Michigan, The Ohio State University, The Second World Congress in Game Theory (Marseilles, France), Tel Aviv University, Vanderbilt University, and Washington University. E-mail: levin.36@osu.edu, peck.33@osu.edu.

Abstract

We study a dynamic investment game with two-dimensional signals, where each firm observes its continuously distributed idiosyncratic cost of investment and a discrete signal correlated with common investment returns. We demonstrate that the one-step property holds and provide an equilibrium existence/characterization result. "Reversals" are possible, where a large number of firms investing in a given round becomes bad news about investment returns. Welfare is compared to static and rigid-timing benchmarks, and computed for large economies. JEL classification: C73, D8

Keywords: endogenous timing, herding, reversal, multi-dimensional signals

1. Introduction

It is well known that efficient information aggregation may fail in models of social learning, including models of endogenous timing, in which firms have the option to delay their investment and learn from the choices of other firms. See, for example, Chamley and Gale [9]. However, existing models of social learning often make stark informational assumptions, so that a firm's investment perfectly reveals its signal. This paper attempts to relax these informational assumptions, by introducing a two-dimensional signal structure. This limits the inferences that can be drawn from other firms' choices. In our richer informational environment, the paper then reexamines the incentive to delay investment, and the implications for information aggregation and welfare.

Firms receive two signals, and then face a sequence of decisions about whether or not to invest. One signal is correlated with the aggregate state of the economy, which in our context is the unknown return on investment shared by all firms. We assume that this "common value" signal can take one of two values, and call a firm receiving the favorable signal a type-1 firm and a firm receiving the unfavorable signal a type-0 firm. The other signal is the cost of undertaking the investment, which is firm specific and independent of the costs faced by other firms. Observing the investment decisions of other firms could be used to improve inference about the aggregate state, but firms must disentangle whether another firm invests because it receives a favorable signal about investment returns or simply has a low cost.

Our equilibrium characterization overcomes new technical challenges to handle two dimensions, but the model is surprisingly well behaved. A symmetric equilibrium must involve cutoffs, such that, after any history, a type-1 (respectively, type-0) firm whose cost is below the type-1 (respectively, type-0) cutoff invests following that history. A potential source of difficulty is that, in identifying the firm that is indifferent between investing and waiting, the best waiting strategy could be quite complicated, investing in different rounds after different scenarios of market activity. With one-dimensional signals, it is easy to demonstrate the *one-step property*, which states that the marginal firm is indifferent between (i) investing right away and (ii) waiting, then investing in the following round or never investing. The intuition is that, if the marginal firm does not invest in round t, it becomes the most optimistic type remaining, so if it does not want to invest in round $t+1$, no firm will invest, and nothing can be learned by waiting further (See [9] or [8]). With two-dimensional signals, if the marginal type-1 firm does not invest in round t, perhaps the most optimistic type-0 firm will invest in round $t+1$, in which case valuable information could be learned by waiting further. We show that the one-step property holds in any equilibrium in which the investment-cost cutoffs are *aligned*.[2] We then characterize an equilibrium whose cutoffs are aligned, so the one-step property is satisfied. It turns

[2] That is, after any history in which the type-0 cutoff is interior, the type-1 cutoff is interior, and after any history in which the type-1 cutoff is interior and the previous round's type-0 cutoff is above the lower support of the cost distribution, the type-0 cutoff is interior.

out that the type-1 cutoff and the type-0 cutoff are related in a simple way, which allows us to reduce the two indifference equations to a single equation, for which we show that a solution exists. Our characterization can easily be converted into an algorithm for computing the equilibrium.

Introducing two-dimensional signals allows for a phenomenon that is completely new to this literature, which we call a "reversal," where more investment (generally good news) becomes bad news.[3] In round 0, a type-1 firm is far more likely to invest than a type-0 firm, so more investment in round 0 must be regarded as good news. After round 0, the lowest possible cost for a not-yet-invested type-0 firm is lower than the lowest possible cost for a not-yet-invested type-1 firm. After some histories, it is possible that investment is more likely to come from a type-0 firm than a type-1 firm, because the remaining type-0 firms have lower costs. If so, the higher the number of firms that invest, the *lower* the posterior probability each firm assigns to the high investment return. In such cases, more investment is bad news, and we have a reversal.

We derive the following welfare results. Ex ante expected total surplus is always higher in our equilibrium than in the static game in which firms must make a decision in round 0, and in particular, any firm that delays investment in our equilibrium strictly benefits from observing economic activity. Comparing our equilibrium to the "rigid timing" game, in which firms make a once-and-for-all decision in random and exogenously determined sequential order, the welfare comparison can go either way. As the number of firms approaches infinity, either the amount of investment

[3] For an application of reversals to dynamic English auctions, see [22].

remains small forever, or a positive fraction of firms invests, thereby revealing the investment return. We approximate the probability that the investment state is high and the economy "takes off," from which we compute the expected profit of each firm, as a function of their signals. When the degree of patience approaches infinity, the approximation is exact.

We consider the impact of investment subsidies on welfare. Because signals are two-dimensional, there are both positive and negative informational externalities to consider. When the number of firms is not too large, the beneficial effect that a subsidy encourages investment by type-1 firms might be outweighed by the harmful effect of encouraging investment by type-0 firms. For some parameters, the optimal subsidy is actually negative (a tax). For large economies, either (i) a positive fraction of firms invests in round 0 without a subsidy, revealing the investment state, so that (almost) the first-best is achieved,[4] or (ii) there exists a subsidy that induces a small amount of investment in round 0, but reveals the investment state and (almost) achieves the first-best. The optimal subsidy can either be permanent or temporary. A temporary (round 0 only) subsidy is particularly interesting, because it is narrowly targeted to type-1 firms with the lowest cost, and it achieves the first-best with a negligible total government subsidy payment.

In section 2, we present the model, provide some preliminary results, and provide a

[4] The qualifier, almost, is due to the fact that some firms invest in round 0, before the state is revealed, and those that invest in round 1 have their profits discounted. However, a planner that chooses which types invest in round 0, then learns the investment return and directs investment in round 1, cannot improve upon the allocation.

general existence/characterization result (Proposition 3). After histories allowing an interior solution, a simple relationship is derived, relating the type-0 and the type-1 investment cutoffs. In section 3, we provide an example of "reversals," in which more investment in round 0 is good news about investment returns, but more investment is bad news after some histories. Welfare issues are discussed. Section 4 considers asymptotic results, as the number of firms approaches infinity. Concluding remarks are offered in section 5. Proofs are in the Appendix.

Literature Review

The starting point for our model is the seminal paper by Chamley and Gale [9], who analyze models in which the investment cost for all firms is a fixed constant, so that the signal is one-dimensional. They find that there is a unique symmetric perfect Bayesian equilibrium in mixed strategies, and that the equilibrium is inefficient. Their results differ from ours in several respects. Their welfare results are stark: firms are no better off than in the static game, in which firms must invest without learning anything about other firms' information.[5] Also, reversals are impossible and a small subsidy always improves welfare. Equilibrium and optimal policy are studied for the large economy. The equilibrium involves a period of little investment, followed either by a crash or a surge, which is similar to our equilibrium. The main difference is that our model allows interesting parameter regions that induce full revelation and (almost) first-best investment. Also, we compute the probability of an investment

[5] This result would not hold for type-0 firms in their model, except that they assume that type-0 firms never have the option to invest.

surge in the good state, and the welfare of each of our heterogeneous firms. Optimal policy in our model is very different from [9] when the number of firms is small, for the reasons given above. For large economies, optimal policy is similar in the two models. The main distinctions are that we consider permanent subsidies as well as temporary ones, that we can evaluate the impact of a subsidy on the welfare of each of our heterogeneous firms, and that in our model the optimal temporary subsidy must be targeted to a small segment of the market.

Chamley [8] introduces a distribution of beliefs about the investment return, resulting from private information. Pure-strategy equilibrium is characterized, the possibility of multiple equilibrium is demonstrated, and the case of a large number of agents is studied. As in our model, firms that do not invest in the initial round strictly benefit from waiting, but in [8], reversals are impossible. For the large economy, the equilibrium is similar to ours, including the parameter ranges that induce first-best investment. However, welfare or policy are not considered in [8]. Another difference is technical: we identify the key condition on investment cutoffs, alignment, which allows the one-step property to hold, and solve two simultaneous indifference conditions.

In [17], any inefficiencies in these sorts of models are attributed to the fact that investment must be all or nothing, rather than varying with the strength of the signal. Their argument is that one can invert the investment function to recover the investor's type. However, with two-dimensional signals, the mapping from signals to investment choices would not be invertible, underscoring the importance of moving beyond a single dimension.[6]

Our model is related to some of the papers on herd behavior and information cascades. [3] and [2] consider a model in which each investor, in an exogenously determined sequence, faces an investment decision after privately observing a signal related to investment yields. In [3], cascades can arise in which everyone invests or in which no one invests. Since everyone must act when called upon, there is no bias towards delay and underinvestment. Similarly, in [2], cascades arise on one particular type of investment, but there is no inefficiency due to delay. In [16], a private value component is introduced to payoffs, and they find that agents eventually learn the true state of the world. Our model differs in the crucial timing dimension. Instead of facing an exogenously given single opportunity to invest, our firms endogenously choose when to invest, if at all. It is this ability to delay investment and free ride on the information provided by others that leads to lower and later investment. In [10], the timing of investment is endogenized, but in each period exactly one firm receives a signal, and in their equilibrium the exogenous sequence of firms receiving signals plays a prominent role.[7]

[6] We could allow firms in our model to choose any investment level between zero and one, without changing the results. Firms are risk neutral and can invest at most once, so any firm that invests would invest to the maximal extent possible. However, with risk averse firms, we conjecture that equilibrium would involve interior investment choices that do not fully reveal the investor's type. See also [10].

[7] In [7], they consider a model in which firms receive signals in each period, and are allowed to suspend and restart investment. In [18], investors privately observe the successes and failures of their investments, and update their beliefs about the unknown probability of success in order to decide whether or not to continue the investment. Investment returns depend on the overall level of investment as well as the success parameter, and at some point a coordination avalanche occurs.

We want to distinguish our approach from the large literature on multiple equilibrium and Keynesian coordination failure or trading externalities, based on the importance of self-fulfilling expectations of the *actions* that other agents are taking.[8] To emphasize the role of information and inference, we entirely eliminate the effect of one firm's actions on another firm's payoffs. We imagine that firms are not directly competing with each other, so that profitability depends on the aggregate shock and not the expansion decisions of other firms.[9] The delay and possible collapse of investment arising in our model is reminiscent of the Keynesian story of self-fulfilling pessimism.[10]

2. The Model

There are $n \geq 2$ risk-neutral firms or potential investors, and each firm privately observes a signal correlated with the return on investment common to all investors. Letting Z denote the investment return and X_i denote the "common value" signal of firm i, we assume that $Z \in \{0,1\}$ and $X_i \in \{0,1\}$. We also assume that the unconditional expected return is 0.5, and that signals are independent, conditional

See also [24] and [1] for static games exhibiting this contagion effect.

[8] See [12], [5], [6], [23], [11], and [19].

[9] The investment decision in our model is related to the decision to enter a market. See [13], [14], [26], and in particular, papers that incorporate private information by [4] and [21]. The present paper introduces common values, which allows us to interpret signals as information about demand. Another major difference is that a firm's revenues do not depend on the number of entrants.

[10] Keynes [20] argues (page 210) that pessimism causes consumers to reduce their demand, a sort of inaction. The reduction in consumption demand is not combined with an order for future consumption. Thus, firms could be deterred from investing, justifying the pessimism. Similarly, in our model, when a firm with a strong signal does not invest, the fact that the firm might be willing to invest in the future is not revealed to the market.

on Z. The accuracy of the signal is given by the parameter, $\alpha \in [\frac{1}{2}, 1]$:

$$pr(Z = 0 \mid X_i = 0) = pr(Z = 1 \mid X_i = 1) = \alpha.$$

When we have $\alpha = \frac{1}{2}$, common-value signals have no information content at all, and when we have $\alpha = 1$, a common-value signal fully reveals the aggregate state. Thus, the parameter α effectively captures the informativeness of the common-value signal, X_i. We call a firm that has received the high signal, $X_i = 1$, a type-1 firm and a firm that has received the low signal, $X_i = 0$, a type-0 firm.

Each firm i also privately observes a second signal, representing the idiosyncratic cost of undertaking the investment, c_i. We assume that c_i is independent of all other variables, and distributed according to the continuous and strictly increasing distribution function F, defined over the support, $[\underline{c}, \overline{c}]$. Assume that we have $0 \leq \underline{c} < 1 \leq \overline{c}$. The structure of signals is common knowledge.

Impatience is measured by the discount factor, $\delta < 1$. If firm i has cost c_i and the state is Z, its profits are zero if it does not invest, and $\delta^t(Z - c_i)$ if it invests in round t. We now describe the game. First, each firm observes its signals, (X_i, c_i). In each round, starting with round 0, each firm observes the history of play, and firms not yet invested simultaneously decide whether to invest. More formally, for $t = 0, 1, ...$, denote the action of firm i in round t as $e_i^t \in \{0, 1\}$, where the action, 0, represents no change in status (either not yet invested or invested in a previous round) and the action, 1, represents investing in that round. We assume that once a firm invests it remains invested. Let k^t denote the number of firms who invest in round t, $k^t = \sum_{i=1}^{n} e_i^t$, and denote the history of length t as $h^{t-1} = (k^0, k^1, ..., k^{t-1})$. We will sometimes denote the history, h^t, as (h^{t-1}, k^t). Let h denote the set of histories of any length, including the null history observed in round 0. A strategy for firm i is a mapping from signal realizations and histories into a decision of whether to invest, satisfying the restriction that a firm can change its investment status at most once.

Our solution concept is symmetric Bayesian Nash equilibrium. The following lemma is standard in the literature, and greatly simplifies the analysis, by showing that equilibrium is characterized by cutoff investment costs, such that any firm with investment cost below the cutoff will invest, if it has not already done so.

Lemma 1: *Suppose that F is continuous over the nondegenerate support, $[\underline{c}, \overline{c}]$. Then any symmetric Bayesian Nash equilibrium has the interval property. For any history, h^{t-1}, that arises with positive probability in the equilibrium, there are functions, $\beta_0(h^{t-1})$ and $\beta_1(h^{t-1})$, such that a type-0 firm (not previously invested) invests in round t if and only if $c_i \leq \beta_0(h^{t-1})$ holds, and a type-1 firm (not previously invested) invests in round t if and only if $c_i \leq \beta_1(h^{t-1})$ holds.*

Before characterizing the equilibrium, we introduce some notation. Central to the analysis is the probability of the high state ($Z = 1$), conditional on whether the firm is type-1 or type-0, and conditional on the investment history. The inferences that a firm draws from the investment history depend on the investment-cost cutoffs of the other firms. It will be convenient to let $H(h^{t-1})$ denote the ratio of the probability of h^{t-1} in the low state to the probability of h^{t-1} in the high state,[11]

$$H(h^{t-1}) = \frac{pr(h^{t-1} \mid Z=0)}{pr(h^{t-1} \mid Z=1)}.$$

Let $P_0(h^{t-1})$ denote the probability that we have $Z=1$, given the history, h^{t-1}, for a type-0 firm that has not yet invested. Let $P_1(h^{t-1})$ denote the probability that we have $Z=1$, given the history, h^{t-1}, for a type-1 firm that has not yet invested. Using Bayes' rule, we have

$$P_0(h^{t-1}) = \frac{1}{1+\frac{H(h^{t-1})\alpha}{(1-\alpha)}} \quad \text{and} \tag{2.1}$$

$$P_1(h^{t-1}) = \frac{1}{1+\frac{H(h^{t-1})(1-\alpha)}{\alpha}} \tag{2.2}$$

Define the probability that a type-0 (respectively, type-1) firm invests in round t, after the history h^{t-1}, by

$$q_0(h^{t-1}) = \frac{F(\beta_0(h^{t-1})) - F(\beta_0(h^{t-2}))}{1-F(\beta_0(h^{t-2}))}, \tag{2.3}$$

$$q_1(h^{t-1}) = \frac{F(\beta_1(h^{t-1})) - F(\beta_1(h^{t-2}))}{1-F(\beta_1(h^{t-2}))}. \tag{2.4}$$

From (2.3) and (2.4), we see that finding the investment cost cutoffs for round t is equivalent to finding the probabilities that a firm will invest in round t (having not

[11] These conditional probabilities depend on the investment cutoffs chosen in rounds 0 through $t-1$, and the fact that the firm considering this history has not yet invested.

yet invested). Define

$$
\begin{aligned}
Q_0(h^{t-1},k,q_0,q_1) &\equiv pr(k^t = k \mid h^{t-1}, Z=0, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1), &(2.5)\\
Q_1(h^{t-1},k,q_0,q_1) &\equiv pr(k^t = k \mid h^{t-1}, Z=1, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1), &(2.6)\\
Q(h^{t-1},k,q_0,q_1) &\equiv \frac{Q_0(h^{t-1},k,q_0,q_1)}{Q_1(h^{t-1},k,q_0,q_1)}. &(2.7)
\end{aligned}
$$

The ratio, $Q(h^{t-1}, k, q_0, q_1)$, represents the likelihood that a firm that has not yet invested observes k firms invest in round t, given that the state is low, relative to the likelihood of observing k firms invest in round t, given that the state is high. This likelihood ratio depends on the history, h^{t-1}, and depends on the probabilities that a type-1 firm and a type-0 firm invest after h^{t-1}.

Application of Bayes' rule yields the rule for updating beliefs when the mixing probabilities in round t are q_0 and q_1, given by

$$
\begin{aligned}
P_0(h^{t-1},k;q_0,q_1) &= \frac{1}{1+\frac{1-P_0(h^{t-1})}{P_0(h^{t-1})}Q(h^{t-1},k,q_0,q_1)} \quad \text{and} &(2.8)\\
P_1(h^{t-1},k;q_0,q_1) &= \frac{1}{1+\frac{1-P_1(h^{t-1})}{P_1(h^{t-1})}Q(h^{t-1},k,q_0,q_1)}. &(2.9)
\end{aligned}
$$

The following proposition establishes that more investment in round t implies a higher (lower) posterior probability that the state is high, if and only if a type-1 firm invests in round t with a higher (lower) probability than a type-0 firm.

Proposition 1: *If $q_1(h^{t-1}) > q_0(h^{t-1})$ holds, then $P_0(h^{t-1}, k; q_0, q_1)$ and $P_1(h^{t-1}, k; q_0, q_1)$*

are strictly increasing in k. If $q_1(h^{t-1}) < q_0(h^{t-1})$ holds, then $P_0(h^{t-1}, k; q_0, q_1)$ and $P_1(h^{t-1}, k; q_0, q_1)$ are strictly decreasing in k.

In equilibrium, the investment-cost cutoffs balance the benefits of investing and receiving profits earlier with the benefits of waiting and maintaining the option not to invest. To compute the option value of waiting, we develop notation for the expected profit that a type-1 firm with cost c_i saves in round t, by *not* investing in round $t+1$ when $k^t = k$ occurs, which is defined by

$$\begin{aligned}\theta_k(h^{t-1}, q_0, q_1, c_i) &= pr(Z = 0, k^t = k | h^{t-1}, X_i = 1, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)[c_i] - \\ pr(Z &= 1, k^t = k | h^{t-1}, X_i = 1, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)[1 - c_i].\end{aligned}$$

The first term above is the probability of the state being low and having k firms invest, multiplied by the resulting profit savings from not investing in period $t+1$. The second term above is the probability of the state being high and having k firms invest, multiplied by the resulting reduction in profits from not investing in period $t+1$. Note that $\theta_k(h^{t-1}, q_0, q_1, c_i)$ is positive when profits are negative, so that the option of not investing is valuable. Using Bayes' rule, this equation can be simplified to

$$\begin{aligned}\theta_k(h^{t-1}, q_0, q_1, c_i) &= (1 - P_1(h^{t-1}))Q_0(h^{t-1}, k, q_0, q_1)[c_i] - \\ &\quad P_1(h^{t-1})Q_1(h^{t-1}, k, q_0, q_1)[1 - c_i]. \qquad (2.10)\end{aligned}$$

For a type-0 firm, denote the corresponding profit savings as $\eta_k(h^{t-1}, q_1, q_2, c_i)$, which using Bayes' rule can be simplified to

$$\begin{aligned}\eta_k(h^{t-1}, q_0, q_1, c_i) &= (1 - P_0(h^{t-1}))Q_0(h^{t-1}, k, q_0, q_1)[c_i] - \\ &\quad P_0(h^{t-1})Q_1(h^{t-1}, k, q_0, q_1)[1 - c_i]. \qquad (2.11)\end{aligned}$$

In equilibrium, a firm whose investment cost equals the cutoff must be indifferent between investing immediately and waiting. However, the best continuation strategy involving waiting might be very complicated to compute, if the firm invests in different future rounds under different scenarios of market activity. Fortunately, in the equilibrium we characterize, a firm with the cutoff cost is indifferent between investing and waiting one more round and then making a final decision whether to invest. This one-step property is easy to demonstrate in other models with one-dimensional types. For our model, it requires the additional assumption that investment-cost cutoffs are aligned, as we explain below.

Definition 1: The investment-cost cutoff for a type-i firm after the history, h^{t-1}, is *interior* if $q_i(h^{t-1}) > 0$ holds. Investment cost cutoffs are *aligned* if, for every history, h^{t-1}, we have (i) $\beta_0(h^{t-1})$ is interior implies $\beta_1(h^{t-1})$ is interior, and (ii) $\beta_1(h^{t-1})$ is interior implies either $\beta_0(h^{t-1})$ is interior or $\beta_0(h^{t-1}) = \underline{c}$. An equilibrium satisfies the *one-step property* if for every history after which $\beta_i(h^{t-1})$ is interior, a type-i firm with cost $\beta_i(h^{t-1})$ is indifferent between investing in round t and investing in round

$t+1$ if and only if investment remains profitable.

In [9] and [8], signals are one-dimensional, and all equilibria must satisfy the one-step property. Suppose that, in a variation of our model, only type-1 agents had the opportunity to invest, essentially making it a one-dimensional model. Adapting the argument in [8], by continuity of payoffs, the cutoff type must be indifferent between investing in round t and waiting. If the cutoff type is not indifferent between investing in round t and making a final decision in round $t+1$, there must be a realization of k^t that is profitable, yet the firm does not invest. Since the cutoff type has a lower cost than any other remaining firm, then no other firm can be investing before the cutoff firm invests, so nothing can be learned by waiting, contradicting the supposition that waiting beyond round $t+1$ can be optimal.

In our two-dimensional model, this simple argument for the one-step property breaks down. Suppose that in equilibrium, a type-1 firm with investment cost of $\beta_1(h^{t-1})$ is indifferent between investing and waiting, but only because there is a realization of k^t that is profitable, yet this firm does not invest in round $t+1$. Since this firm has a lower cost than any other remaining type-1 firm, then no other type-1 firm can be investing before this firm invests. However, it is not necessarily true that nothing can be learned from waiting, because it is possible that a type-0 firm could invest in round $t+1$. None the less, Proposition 2 shows that a weaker version of the one-step property must hold, which turns out to be sufficient for our equilibrium characterization in Proposition 3.

Proposition 2: *Any equilibrium in which the investment cost cutoffs are aligned satisfies the one-step property.*

The following proposition characterizes an equilibrium that satisfies the one-step property. Therefore, interior investment cost cutoffs satisfy the indifference conditions,

$$\left(P_1(h^{t-1}) - \beta_1(h^{t-1})\right)\left(\frac{1-\delta}{\delta}\right) = \sum_{k=0}^{n-1} \max[0, \theta_k(h^{t-1}, q_0, q_1, \beta_1(h^{t-1}))] \text{ and } \quad (2.12)$$

$$\left(P_0(h^{t-1}) - \beta_0(h^{t-1})\right)\left(\frac{1-\delta}{\delta}\right) = \sum_{k=0}^{n-1} \max[0, \eta_k(h^{t-1}, q_0, q_1, \beta_0(h^{t-1}))]. \quad (2.13)$$

Conditions (2.12) and (2.13) can be derived by expressing the profits from waiting until round $t+1$ as the sum of two terms: the profits from always investing in round $t+1$, and the additional profits due to the option not to invest when k^t is unfavorable. We demonstrate that interior investment cost cutoffs are related according to

$$\frac{\beta_0(h^{t-1})}{1-\beta_0(h^{t-1})} = \frac{\beta_1(h^{t-1})}{1-\beta_1(h^{t-1})}\frac{(1-\alpha)^2}{\alpha^2}. \quad (2.14)$$

Therefore, it will be convenient to write an analogue of (2.12) which imposes this dependence of $\beta_0(h^{t-1})$, $q_0(h^{t-1})$, and $q_1(h^{t-1})$ on $\beta_1(h^{t-1})$, based on (2.3), (2.4), and

(2.14). Thus, we define the function,

$$b_0(\beta_1) = \max[\underline{c}, \frac{1}{1 + \left(\frac{1-\beta_1}{\beta_1}\right)\left(\frac{(1-\alpha)^2}{\alpha^2}\right)}], \tag{2.15}$$

and the function $q_0(\beta_1)$ is defined by substituting (2.15) into (2.3).

Proposition 3: *Under our maintained assumptions, there exists an equilibrium in which investment cost cutoffs are aligned, characterized as follows. After a history, h^{t-1}, in which investment is unprofitable for a type-1 firm with cost $\beta_1(h^{t-2})$, investment ceases. If investment is profitable for a type-1 firm with cost $\beta_1(h^{t-2})$, then $\beta_1(h^{t-1})$ is a solution to*

$$\left(P_1(h^{t-1}) - \beta_1\right)\left(\frac{1-\delta}{\delta}\right) = \sum_{k=0}^{n-1} \max[0, \theta_k(h^{t-1}, q_0(\beta_1), q_1(\beta_1), \beta_1)], \tag{2.16}$$

and $\beta_0(h^{t-1}) = b_0(\beta_1(h^{t-1}))$ holds. If investment is profitable for a type-1 firm with cost $\beta_1(h^{t-2})$, and $\beta_0(h^{t-2}) > \underline{c}$ holds, then $\beta_1(h^{t-1})$ and $\beta_0(h^{t-1})$ are both interior, and are related according to (2.14).

Proposition 3 characterizes an equilibrium.[12] A careful reading of the proof in-

[12] Refinements are not needed, because beliefs off the equilibrium path play no role in the analysis. If, after some history, h^{t-1}, $k^t = 0$ is off the equilibrium path, then it must be the case that all remaining firms are investing with probability one. After a deviation by firm i not to invest, the beliefs of other firms are irrelevant, since they have invested and have nothing more to do. If, after some history, h^{t-1}, $k^t = k > 0$ is off the equilibrium path, then it must be the case that no firm is investing. After a deviation by firm i to invest, its payoff is determined independent of the future play of the game, so the beliefs of other firms are irrelevant to the decision to deviate.

dicates that, when both cutoffs are interior and equation (2.14) is imposed, then conditions (2.12) and (2.13) reduce to the same equation. After any history, we can compute the equilibrium cutoffs by solving (2.16) for β_1. Then the equilibrium cutoffs are given by $\beta_1(h^{t-1}) = \beta_1$ and $\beta_0(h^{t-1}) = b_0(\beta_1)$. If at the solution for β_1, we have $b_0(\beta_1) > \underline{c}$, then both cutoffs are interior, and (2.14) is satisfied. If at the solution for β_1, we have $b_0(\beta_1) = \underline{c}$, then no type-0 firm invests after this history. The equilibrium cutoffs characterized in Proposition 3 are aligned. Following any history in which both cutoffs are interior, all histories that are profitable for the lowest cost uninvested type-1 firm are profitable for the lowest cost uninvested type-0 firm, and vice versa. Both cutoffs then shift together, until investment becomes unprofitable and ceases. Depending on parameters, there can be an initial phase of the equilibrium, in which a type 0 firm with cost $\underline{c}$ strictly prefers to delay investment. Proposition 3 provides a way to compute the equilibrium cutoffs numerically, history by history, for as many histories as one wishes.[13]

[13]Due to our assumption, $\overline{c} \geq 1$, investment cannot be profitable for all firms. Otherwise, there could be multiple equilibria, based on the boundary case in which all firms strictly prefer to invest rather than wait, because all other firms are investing. We conjecture that all symmetric equilibria are characterized by Proposition 3. Even if this conjecture is correct, multiple equilibria would arise if there were more than one solution to (2.16) after some histories.

3. Applying the Equilibrium Characterization: Reversals and Welfare

In this section, we apply Proposition 3 to derive some properties of the equilibrium. We first demonstrate that our characterization is consistent with the phenomenon of *reversals*, where after some histories, more investment during the current round is bad news about the investment return. Reversals arise because of the two-dimensional nature of signals. For example, if costs were deterministic as in [9], a firm that invests is revealed to have the good signal, which leads to higher posteriors about the investment return. However, in our model, after some histories, all uninvested type-1 firms have very high costs, while some uninvested type-0 firms could have much lower costs. It is not surprising, then, that an investor could be more likely to be a type-0 firm than a type-1 firm, so expected investment returns are decreasing in the number of investors (see Proposition 1). The second topic in this section is welfare. In evaluating the welfare of firms, one benchmark for comparison is the *static game*, in which firms decide whether to invest in round 0 or never invest. This benchmark is important, because any increase in welfare of the equilibrium allocation over the allocation from the static game represents the social benefit of learning. Another potentially interesting benchmark is the outcome of the "*rigid-timing*" game, in which firms face a once-and-for-all decision whether to invest in a randomly-determined sequence, observing the decisions of firms ahead of it in the queue. This rigid timing model, with exogenous timing of the investment decision, is assumed in much of the herding literature, including [3] and [2]. Besides being important in the literature, this benchmark allows us to measure the net advantage of being able to choose when to invest. Flexibility allows firms with the most favorable information to move first, providing benefits to the rest of the market. On the other hand, flexibility allows a firm for whom investment is profitable to delay, creating a negative externality for the rest of the market.

3.1. Reversals

We define a *reversal* to be an equilibrium in which after some histories, $P_0(h^{t-1}, k; q_0, q_1)$ and $P_1(h^{t-1}, k; q_0, q_1)$ are strictly decreasing in k, so more investment during the current round is actually bad news, reversing the situation in round 0. The following example of a reversal has three firms, and if exactly one firm invests in round 1, expected revenues increase. However, after one firm invests in round 0, type-0 firms are more likely than type-1 firms to invest in round 1, so investment in round 1 is bad news.

Example 1: Consider the following example, with $\alpha = .875$, $\delta = .9$, $n = 3$, and the distribution function given by,

$$F(c_i) = \frac{c_i}{3} \qquad \text{for } c_i \leq 3$$

The distribution function is uniform over the interval, $[0, 3]$.[14]

Table 1 presents our computations of the equilibrium cutoffs for the first few rounds, based on the equilibrium characterization given in Proposition 3.

	cutoff for investment		probability of investing	
history	$\beta_0(h^{t-1})$	$\beta_1(h^{t-1})$	$q_0(h^{t-1})$	$q_1(h^{t-1})$
null	.0891683	.827497	.0297228	.275832
(0)	no more investment		0	0
(1)	.303929	.955347	.0737797	.0588495
(2)	.701055	.991373	.210210	.0754319
$(1, 0)$	.315526	.957605	.00110429	.00430157
$(1, 1)$	no more investment		0	0

Table 1

From Table 1, we see why reversals can occur in equilibrium. More investment in round 0 is good news, because firms that invest are more likely to be type 1.[15] If no one invests in round 0, this is bad news, and there is no further investment. If one firm invests in round 0, this is good news, and expected revenue from investing increases for both types. Notice, however, that after one firm invests in round 0, a type-0 firm is more likely to invest than a type-1 firm. Therefore, after the history $(1, 1)$, the firm that invested in round 1 is much more likely to be a type-0 firm. While

[14] Example 1 can easily be altered to have $\overline{c} = 1$. Example 1 is equivalent to an example in which we have $F(c_i) = \frac{c_i}{3}$ for $c_i \leq 0.9$, and $F(c_i) = 7c_i - 6$ for $0.9 < c_i \leq 1$.

[15] We see that, for a type-i firm ($i = 0, 1$), we have $q_i(2) > q_i(1) > q_i(0)$ and $P_i(2) > P_i(1) > P_i(0)$.

the investment in round 0 is good news, the investment in round 1 is bad news, and investment ceases. On the other hand, if no one invests in round 1, this is good news, and we have a reversal. Thus, in round 2, we could see more investment after $(1, 0)$, but not after $(1, 1)$. Notice that, if two firms invest in round 0, a type-0 firm is also more likely to invest in round 1 than a type-1 firm. However, since there is only one remaining uninvested firm in round 1, obviously that firm does not learn anything from market activity in round 1.

In Example 1, following a history of the form, $(1, 0, 0, ...)$, there is always a positive probability of investment by a second firm, although this probability quickly approaches zero.[16] Each round in which there is no investment leads firms to slightly increase their posterior probability of the good state, because the posterior probability of the other uninvested firm being type 1 increases. This scenario is reminiscent of a war of attrition, although here there is no strategic interaction between firms.[17] If a second firm ever invests, this is bad news for the remaining firm, and investment ceases.

[16] A referee points out that, while it is true that the expected investment return is higher after the history $(1, 0)$ than after the history $(1, 1)$, the expected aggregate investment over all rounds is higher after the history $(1, 1)$.

[17] Reversals, in which more investment switches from being good news to being bad news, is reminiscent of the phenomenon documented by [25], in which timing games can switch from a "preemptive explosion" to a war of attrition.

3.2. Welfare

Corollary of Proposition 3: *For the equilibrium characterized in Proposition 3, expected profits, conditional on* (X_i, c_i)*, are weakly higher than expected profits in the static game. If there is a positive probability of investment in round 0, then expected profits are strictly higher than in the static game for type-1 firms with* $c_i > \beta_1$*, and for type-0 firms with* $c_i > \beta_0$*.*

Remark 1: In [9], all firms have the same cost, and only firms with the favorable signal (type-1 firms in our terminology) have an opportunity to invest. Thus, all firms deciding whether to invest are identical. They find that there is a unique symmetric mixed-strategy equilibrium, and that ex ante welfare is the same as in the static game, with only one round of investing. This strong inefficiency result disappears when we have heterogeneity, including [8], with one-dimensional and heterogeneous types. Our Corollary indicates that all firms that do not invest in round 0 receive strictly higher profits, in the equilibrium to the flexible-timing game characterized in Proposition 3, than they would receive in the static game. These firms benefit from the ability to learn from market activity.

Remark 2: The ex ante welfare comparison between the flexible-timing game and the rigid-timing game is ambiguous. An advantage of the flexible-timing game is that firms with the most favorable signals invest first, so that firms who benefit most from observing market activity have an opportunity to delay their decision to invest. A disadvantage of the flexible-timing game is that some type-1 firms, for whom investment is profitable, delay their investment decision and thereby delay their favorable information from being observed by the market. In Example 2 below, welfare is higher in the flexible-timing game. If we were to change the parameters, however, welfare could easily be higher in the rigid-timing game. For example, consider a game with two firms, $n = 2$, a discount factor close to one, and investment costs uniformly distributed over $[\alpha - \varepsilon, \alpha]$, for small positive ε. In the flexible-timing game, the probability of investment in round 0 is close to zero, so welfare is close to zero. In the rigid-timing game, the first firm to act will invest if and only if it is of type 1, so the second firm will receive significant expected profits if it is also a type-1 firm.

Example 2: Consider the following example, with parameters, $n = 2$, $\alpha = .75$, and $\delta = 1$, and a cost distribution that is uniform on $[0, 1]$. Table 2 shows the unique symmetric equilibrium of our "flexible timing" game. For the purpose of comparison, Table 2 also shows the unique equilibrium of the rigid timing game usually studied in the herding literature.

	flexible timing game	rigid timing game
β_0	0.17913	0.25
β_1	0.66261	0.75
$\beta_0(1)$	0.37576	0.35714
$\beta_1(1)$	0.84417	0.83333
$\beta_0(0)$	no further investment	0.16667
$\beta_1(0)$	no further investment	0.64286
profit (ex ante)	0.16054	0.15848

Table 2: $n = 2,\ \ \delta = 1,\ \ \alpha = .75,\ \ c_i \sim U[0, 1]$

The fact that β_0 and β_1 are lower for the flexible timing game than for the rigid timing game illustrates the incentive to delay investment, due to the option value of not investing in round 1 if the other firm did not invest in round 0. A type-0 firm with cost $c_i \in (0.17913, 0.25)$ would receive positive profit by investing in round 0, but profit is higher by waiting until round 1. Similar reasoning applies to a type-1 firm with cost, $c_i \in (0.66261, 0.75)$. The cutoffs for investing in round 1, after observing the other firm invest in round 0, are higher for the flexible timing game than the rigid timing game. The reason is that there is a stronger inference that the other firm is a type-1 firm in the flexible timing game than in the rigid timing game, because β_1/β_0 is higher. Indeed, our simple example illustrates the role heterogeneous costs play in diluting the information gathered from another firm's investment. Suppose a firm observes its rival invest in round 0, and could infer that the rival is type-1. Then the hypothetical cutoffs for investment in round 1 would be $\widetilde{\beta}_0(1) = .50$ and $\widetilde{\beta}_1(1) = .90$. The actual values are significantly lower, reflecting the fact that investment by the rival is a noisy indicator that the rival is a type-1 firm. While the rival's investment is surely good news about the aggregate state, a firm must take into account the possibility that the rival is a type-0 firm with low investment cost.

Choosing the limiting discount factor, $\delta = 1$, allows a clean comparison of welfare in the two games, since forcing one of the firms to delay investment in the rigid timing game is not itself a source of inefficiency. Rather, any inefficiency that arises when a type-1 firm delays investment is due to the fact that the other firm cannot benefit from that information. Ex ante profit is higher for the flexible timing game, so the gains from endogenous sorting outweigh the loss due to strategic delay for this example. To put these profit values into perspective, ex ante profit in the static game (where no learning is possible) is 0.15625, and ex ante profit would be 0.25 if signals were perfectly accurate ($\alpha = 1$). The market clearly benefits from the opportunity to learn.

4. The Model with Many Firms

In the limit, as $n \to \infty$, the law of large numbers implies that the aggregate state could be known with certainty if the firms were to pool their information. Because we have a large market, there will be many firms with costs near the lower bound, so if all the type-1 firms with cost near $\underline{c}$ invest, the state will be revealed. Therefore, $\underline{c}$ plays an important role in characterizing the equilibrium. Indeed, the equilibrium of the large economy model shares many of the same features as [9] and [8], the latter being a one-dimensional model where information is heterogeneous but costs are not. However, there are several new results in this section. Proposition 4 basically extends the large economy results of [8] to a two-dimensional model. There are no surprises, although now there is a parameter region in which we have investment from a positive measure of type-0 and type-1 firms. Moreover, we derive expressions for the expected payoff of firms, as a function of the realized signals. [8] does not attempt such a welfare computation, and the computation is trivial in [9], where all type-1 firms are identical and type-0 firms do not have an opportunity to invest.

Proposition 4: *Fix α, δ, $\underline{c}$, $\overline{c}$, and the continuous and strictly increasing distribution function, F. Consider a sequence of economies, indexed by n, and let (β_0^n, β_1^n) be equilibrium investment cutoffs in round 0 for the economy with n firms. Consider a convergent subsequence, where $(\beta_0^n, \beta_1^n) \to (\beta_0^*, \beta_1^*)$. Then we have the following exhaustive possibilities:*

(1) Parameters satisfy $\underline{c} > \alpha$ [Region 1 in Figure 4.1], and cutoffs satisfy $\beta_0^* = \beta_1^* = \underline{c}$,

(2) Parameters satisfy $\frac{\alpha(1-\delta)}{1-\alpha\delta} < \underline{c} < \alpha$ [Region 2 in Figure 4.1], and cutoffs satisfy $\beta_0^* = \beta_1^* = \underline{c}$,

(3) Parameters satisfy $\frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta} < \underline{c} < \frac{\alpha(1-\delta)}{1-\alpha\delta}$ [Region 3 in Figure 4.1], and

cutoffs satisfy $\beta_0^* = \underline{c}$ *and* $\beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}$,

(4) Parameters satisfy $\underline{c} < \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta}$ *[Region 4 in Figure 4.1], and cutoffs satisfy* $\beta_0^* = \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta}$ *and* $\beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}$.

We now discuss the equilibrium cutoffs characterized in Proposition 4, leaving the more complicated part (2) for last. In region 1, for all firms, the cost exceeds the expected return, so no one would be willing to be the first to invest. Since no one invests in round 0, no further inference is made, and investment never occurs. This equilibrium is inefficient, because when investment returns are high, $Z = 1$, investment is profitable for all firms (ex post), yet no investment takes place. We also have no investment in the rigid timing game. For equilibria corresponding to part (3), type-0 firms do not invest in round 0, and type-1 firms invest with probability $F(\beta_1^n)$. By the law of large numbers, the limiting fraction of firms that invest in round 0 is $\alpha F(\beta_1^*)$ if the state is high, $Z = 1$, and the limiting fraction of firms that invest in round 0 is $(1-\alpha)F(\beta_1^*)$ if the state is low, $Z = 0$. Thus, activity in round 0 reveals the state, thereby justifying the hypothesized investment behavior in round 0. The outcome is nearly first-best efficient.[18] For equilibria corresponding to part (4), the

[18]The only departures from first-best efficiency are that (i) some firms might invest in round 0 in the low state, and (ii) some firms might delay their investment until round 1 in the high state. However, if it takes one round for a fully informed planner to communicate the state to the firms, it is impossible to improve on the equilibrium. By contrast, the rigid timing game outcome is inefficient even ignoring delays (assuming $\underline{c} > 0$), because a finite sequence of noninvestment can stop all investment forever.

limiting equilibrium cutoffs for round 0 are

$$\beta_0^* = \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta} \quad \text{and} \quad \beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}.$$

By the law of large numbers, the limiting fraction of firms that invest in round 0 is $\alpha F(\beta_1^*) + (1-\alpha)F(\beta_0^*)$ if the state is high, $Z = 1$, and the limiting fraction of firms that invest in round 0 is $(1-\alpha)F(\beta_1^*) + \alpha F(\beta_0^*)$ if the state is low, $Z = 0$. Thus, activity in round 0 reveals the state, thereby justifying the hypothesized investment behavior in round 0. Again, the equilibrium is nearly first-best efficient, while the rigid-timing game can exhibit inefficient herding.

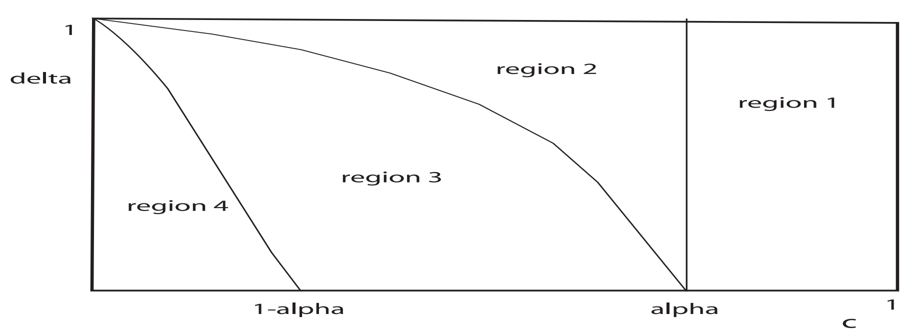

Figure 4.1: Parameter Regions

The more difficult and interesting case occurs when we consider equilibria described in part (2) of Proposition 4. Limiting investment cutoffs satisfy $\beta_0^* = \beta_1^* = \underline{c}$, so the probability that any particular firm invests in round 0 approaches zero. However, if firms are certain that there will be no investment, a type-1 firm with cost close to $\underline{c}$ should invest in round 0. This issue has been treated rigorously in the literature, for the case of one-dimensional signals, and the same logic applies here as well. For sufficiently large n, type-0 firms strictly prefer not to invest in round 0, $\beta_0^n = \underline{c}$. The cutoff for investment by type-1 firms is approaching $\underline{c}$, so an individual firm invests with a vanishing probability, so the expected aggregate number of firms who invest is positive and finite. Thus, observing only one or two firms invest in round 0 could be good news, for all values of n.

When δ approaches one and n approaches infinity, the limiting probability that no firm ever invests, given that we are in the high state, $Z = 1$, can be computed as $Q_1(\emptyset, 0, q_0, q_1) = \left(\frac{\alpha(1-\underline{c})}{(1-\alpha)\underline{c}}\right)^{-\alpha/(2\alpha-1)}$.The limiting probability that no firm invests in the low state can be computed as $Q_0(\emptyset, 0, q_0, q_1) = \left(\frac{(1-\underline{c})(1-\alpha)}{\alpha\underline{c}}\right)^{(1-\alpha)/(2\alpha-1)}$. (Derivations are given in the Appendix.) These probabilities of no investment are significant. For example, if we have $\delta \to 1$, $n \to \infty$, $\underline{c} = 1/2$, and $\alpha = 2/3$, the probability of no investment in the good and bad states, respectively, are 1/4 and 1/2.

From Proposition 4, we see that behavior in round 0 pins down the equilibrium payoffs of all firms when we are in region 1, region 3, or region 4. In region 1, payoffs are zero, because no one invests. In regions 3 and 4, those who wait will learn the investment state, so payoffs are $(1 - c_i)$ in the high state and zero in the low state.

Thus, the overall payoff is $(1 - c_i)$ multiplied by the probability of the high state, α for type-1 firms and $(1 - \alpha)$ for type-0 firms. To understand the payoffs when we are in region 2, the key idea is that, either investment collapses after a negligible fraction of firms invest, or a positive measure of type-1 agents invest, thereby revealing the state. Thus, a firm that waits until the previous round's investment cutoff is at least $\underline{c} + \varepsilon$ will invest if and only if the economy "takes off," where the state is revealed to be high. By continuity, a type-1 firm with cost $\underline{c}$, that invests in round 0, must receive essentially the same payoff as a type-1 firm with cost $\underline{c} + \varepsilon$, that waits. When we have $\delta = 1$, this observation allows us to compute exactly the probability of the economy taking off, which allows an exact computation of payoffs for all firms, as a function of their signals. When we have $\delta < 1$, we can bound the payoffs, between the payoff the firm would receive if it always waited until *after* the takeoff occurs, and the payoff the firm would receive if it could somehow observe the state and decide whether to invest *during* the round in which the takeoff occurs.

Proposition 5: *Fix α, δ, $\underline{c}$, $\overline{c}$, and the continuous and strictly increasing distribution function, F. Consider a sequence of economies, indexed by n, and let $W^n(0, c_i)$ and $W^n(1, c_i)$ be equilibrium profits, conditional on being a type-0 or type-1 firm with investment cost $c_i \leq \overline{c}$, for the economy with n firms. Consider a convergent subsequence, where $(W^n(0, c_i), W^n(1, c_i)) \to (W^*(0, c_i), W^*(1, c_i))$. Then we have:*

(1) In region 1, $W^*(0, c_i) = W^*(1, c_i) = 0$,

(2) In region 2,

$$\delta\left(\frac{\alpha-\underline{c}}{1-\underline{c}}\right)\left(\frac{1-\alpha}{\alpha}\right)(1-c_i) \leq W^*(0,c_i) \leq \frac{1}{\delta}\left(\frac{\alpha-\underline{c}}{1-\underline{c}}\right)\left(\frac{1-\alpha}{\alpha}\right)(1-c_i),$$
$$\delta\left(\frac{\alpha-\underline{c}}{1-\underline{c}}\right)(1-c_i) \leq W^*(1,c_i) \leq \frac{1}{\delta}\left(\frac{\alpha-\underline{c}}{1-\underline{c}}\right)(1-c_i),$$

(3) In region 3,

$$\begin{aligned}
W^*(0,c_i) &= \delta(1-\alpha)(1-c_i),\\
W^*(1,c_i) &= \delta\alpha(1-c_i) \quad \text{for } c_i \geq \frac{\alpha(1-\delta)}{1-\alpha\delta},\\
W^*(1,c_i) &= \alpha - c_i \quad \text{for } c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta},
\end{aligned}$$

(4) In region 4,

$$\begin{aligned}
W^*(0,c_i) &= \delta(1-\alpha)(1-c_i), \quad \text{for } c_i \geq \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta},\\
W^*(0,c_i) &= 1-\alpha-c_i, \quad \text{for } c_i < \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta},\\
W^*(1,c_i) &= \delta\alpha(1-c_i) \quad \text{for } c_i \geq \frac{\alpha(1-\delta)}{1-\alpha\delta},\\
W^*(1,c_i) &= \alpha - c_i \quad \text{for } c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}.
\end{aligned}$$

Suppose that we have $\delta \simeq 1$. Proposition 5 establishes that the probability of the economy taking off (to an outside observer with no signal) is $\frac{\alpha-\underline{c}}{\alpha(1-\underline{c})}$. There is no chance of overinvestment, where a positive fraction of firms invest in the low state. Consider what happens to welfare as we vary $\underline{c}$. For $\underline{c} = 0$, we have $W^*(0, c_i) = (1-\alpha)(1-c_i)$ and $W^*(1, c_i) = \alpha(1-c_i)$, which implies that we achieve the first-best. Firms receive the profits they would receive if they acted with full knowledge of the state. For small $\underline{c}$, the outcome is nearly first-best efficient. The chance of investment collapse in the high state (the fraction investing in the high state is zero) is small. As $\underline{c}$ rises, the chance of investment collapse rises, but if the fraction investing becomes positive, all firms will know that the state is high and invest.

Back to the case of general δ, notice that the characterization in Proposition 5 does not assume anything about the distribution, F, except that it is continuous and strictly increasing! The only feature of F that affects equilibrium profits is the lower support, $\underline{c}$. If we start in region 3 or 4, and consider any firm with cost $c_i > \underline{c}$, then changing the distribution in any way that lowers $\underline{c}$ will have no effect on the firm's expected profits. However, if we start in region 2, then lowering $\underline{c}$ will increase the firm's expected profits. If we start in region 1, then lowering $\underline{c}$ enough to move the economy into another region will increase the firm's expected profits; if we remain in region 1, then there is no effect.

Let us summarize the efficiency properties of large markets. Suppose all firms would delay investment if waiting allows firms to learn the state (region 1 or region 2). Then equilibrium is inefficient, free riding limits information flow and leads to underinvestment. However, equilibrium yields higher welfare than the static game in region 2. When some firms receive favorable enough signals that they would not delay investment, even if waiting would allow them to learn the state (region 3 or region 4), then the market aggregates information efficiently. Even though a single firm has a blunt instrument for conveying information, invest or not, a market with a large number of firms can be highly informative.

We now briefly consider policy implications. For the case of two firms, the effect of an investment subsidy on welfare is ambiguous. For $c_i \sim U[0,1]$, $\alpha = 0.75$, $\delta = 0.9$, it turns out that a small investment subsidy is welfare diminishing! With two-dimensional types, the beneficial effect of encouraging investment by type-1 firms is offset by the harmful effect of encouraging investment by type-0 firms. For arbitrarily large economies, if the economy is in region 3 or region 4 without the subsidy, we have full revelation of the state after round 0, and an efficient selection of which firms should invest in round 0. Therefore, the optimal subsidy is zero. If the economy is in region 1 or region 2 without the subsidy, then a permanent subsidy that shifts the distribution of net costs to the boundary of region 2 and region 3 leads to full revelation after round 0. Also, the fraction of firms that invest in round 0 is zero, so there is no distortion created by inducing firms to invest in round zero rather than waiting to learn the state.[19]

[19] It is also possible to design a temporary subsidy only received by those who invest in round 0, yielding full information revelation and a total subsidy payment made by the government (on a per capita basis) that is arbitrarily close to zero. For details, see the working paper.

5. Conclusions

One might argue that firms could credibly announce their signals to the market, thereby avoiding the inefficiencies arising in our model due to limited information flow. We feel that our results, and the literatures on herding and coordination failures in general, should not be dismissed based on this argument. It is shown in [15] that, when communication costs are zero, it is impossible to implement the fully revealing equilibrium with a static mechanism or the extensive mechanisms they consider. Their analysis applies to our context as well. In reality, it is costly to send and to receive communication, especially in large markets. In a model that includes a communication decision as well as an investment decision, there may be an incentive for firms to free-ride by not incurring the communication costs. The resulting equilibrium with strategic communication delay might closely resemble our equilibrium with strategic investment delay. Furthermore, as the size of the market increases, communication costs become large.

Our assumption of conditional independence implies that, for large economies, the state could be known for certain if firms pool their information. Future work will extend the model to information structures in which the state cannot be known with certainty. One idea is to replicate the economy, so that there are n classes of firms, with r identical firms in each class receiving identical signals. As r approaches infinity, the aggregate information possessed by all firms remains constant. A special feature of the replication economy is that firms know that there is a tie for who has the lowest cost, leading to complicated mixed-strategy equilibria, in spite of the continuous investment cost distribution. We are currently exploring this and other information structures.

A macroeconomic interpretation of the model is that the economy is in recession, but the investment climate might have improved. In equilibrium, firms with favorable signals might delay their investment, and there is a positive probability that no one invests, even if the investment climate has improved and the recession should be over. Although this is the implication of the theory, notice that firms with low investment costs are almost indifferent between investing in round 0 and waiting, and that common knowledge of rationality might be a strong assumption in practice. Some type-1 firms might instead see profitable opportunities and invest, as in Keynes' notion of *animal spirits*. The fascinating point here is that this urge to invest can actually improve the informativeness of markets, thereby improving economic efficiency! This phenomenon would be interesting to test experimentally.

6. Appendix: Proofs

Proof of Lemma 1. A type-1 firm that invests in round t receives expected profit, $pr(Z = 1 \mid h^{t-1}, X_i = 1) - c_i$. Suppose the firm does not invest, and instead chooses continuation strategy, s_i. The equilibrium strategies of the other firms and the history, h^{t-1}, determine the expected profit from the continuation strategy, which can be written as $R_1(h^{t-1}, s_i) - \varphi_1(h^{t-1}, s_i)c_i$, where $R_1(h^{t-1}, s_i)$ denotes expected discounted revenue of a type-1 firm and $\varphi_1(h^{t-1}, s_i)c_i$ denotes expected discounted investment cost of a type-1 firm, given h^{t-1} and s_i. If, given the history, h^{t-1}, a type-1 firm with cost c_i invests in round t, it follows that

$$pr(Z = 1 \mid h^{t-1}, X_i = 1) - R_1(h^{t-1}, s_i) \geq [1 - \varphi_1(h^{t-1}, s_i)]c_i \tag{6.1}$$

holds for all continuation strategies, s_i. From (6.1), and the fact that $\varphi_1(h^{t-1}, s_i) < 1$ holds, it follows that (6.1) holds as a strict inequality for all $c_i' < c_i$ and all continuation strategies, s_i. Therefore, if a type-1 firm with cost c_i invests in round t, a type-1 firm with a lower cost also invests in round t, unless it has already invested. An identical argument applies to type-0 firms. Because F is continuous and the support is nondegenerate, the probability that a firm's cost is exactly $\beta_0(h^{t-1})$ or $\beta_1(h^{t-1})$ is zero, so assuming that firms with these cutoff costs invest is without loss of generality. This establishes the interval property of equilibrium. ■

Proof of Proposition 1. We will derive and simplify expressions for the numerator and denominator of $Q(h^{t-1}, k, q_0, q_1)$. These expressions are complicated by the fact that, given the state, we do not know the number of type-1 firms who have not yet invested at the beginning of round t. Let $\overline{\alpha}_0$ denote the probability that a firm is of type-0, given that the state is low, given the history, h^{t-1}, and given that the firm has not invested by the beginning of round t. Similarly, let $\overline{\alpha}_1$ denote the probability that a firm is of type-1, given that the state is high, given the history, h^{t-1}, and given that the firm has not invested by the beginning of round t. From Bayes' rule, we have

$$\overline{\alpha}_0 = pr(X_j = 0 \mid h^{t-1}, Z = 0, j \text{ not invested}) = \frac{1}{1 + \frac{1-\alpha}{\alpha}\left[\frac{1-F(\beta_1(h^{t-2})}{1-F(\beta_0(h^{t-2})}\right]}, \tag{6.2}$$

$$\overline{\alpha}_1 = pr(X_j = 1 \mid h^{t-1}, Z = 1, j \text{ not invested}) = \frac{1}{1 + \frac{1-\alpha}{\alpha}\left[\frac{1-F(\beta_0(h^{t-2})}{1-F(\beta_1(h^{t-2})}\right]}. \tag{6.3}$$

The following probabilities are conditional on firm i not having invested before round t. Let $\overline{n}$ denote the number of firms that have not yet invested before round t, not including firm i. Of these firms, let κ denote the number of type-1 firms, and let k_1 denote the number of these type-1 firms that invest in round t. Then we can write:

$$pr(k^t = k \mid h^{t-1}, Z = 0) = \sum_{k_1=0}^{k} \sum_{\kappa=k_1}^{\overline{n}} \left[\begin{array}{c} \binom{\overline{n}}{\kappa}\overline{\alpha}_0^{\overline{n}-\kappa}(1-\overline{\alpha}_0)^{\kappa}\left[\binom{\kappa}{k_1}q_1^{k_1}(1-q_1)^{\kappa-k_1}\right] \\ \left[\binom{\overline{n}-\kappa}{k-k_1}q_0^{k-k_1}(1-q_0)^{\overline{n}-\kappa-k+k_1}\right] \end{array} \right] \tag{6.4}$$

and

$$pr(k^t = k \mid h^{t-1}, Z = 1) = \sum_{k_1=0}^{k} \sum_{\kappa=k_1}^{\overline{n}} \left[\begin{array}{c} \binom{\overline{n}}{\kappa}\overline{\alpha}_1^{\kappa}(1-\overline{\alpha}_1)^{\overline{n}-\kappa}\left[\binom{\kappa}{k_1}q_1^{k_1}(1-q_1)^{\kappa-k_1}\right] \\ \left[\binom{\overline{n}-\kappa}{k-k_1}q_0^{k-k_1}(1-q_0)^{\overline{n}-\kappa-k+k_1}\right] \end{array} \right]. \tag{6.5}$$

Equations (6.4) and (6.5) can be simplified as follows:

$$pr(k^t = k \mid h^{t-1}, Z = 0) = \binom{\overline{n}}{k} A^k (1-A)^{\overline{n}-k} \quad \text{and} \tag{6.6}$$

$$pr(k^t = k \mid h^{t-1}, Z = 1) = \binom{\overline{n}}{k} B^k (1-B)^{\overline{n}-k}, \tag{6.7}$$

where $A \equiv \overline{\alpha}_0 q_0 + (1-\overline{\alpha}_0)q_1$ and $B \equiv (1-\overline{\alpha}_1)q_0 + \overline{\alpha}_1 q_1$. From (2.7), (6.6), and (6.7), we have

$$Q(h^{t-1}, k, q_0, q_1) = \left(\frac{1-A}{1-B}\right)^{\overline{n}} \left(\frac{A(1-B)}{B(1-A)}\right)^k. \tag{6.8}$$

Equations (2.8) and (2.9) imply that $P_0(h^{t-1}, k; q_0, q_1)$ and $P_1(h^{t-1}, k; q_0, q_1)$ are increasing in k if and only if $Q(h^{t-1}, k, q_0, q_1)$ is decreasing in k. From (6.8), it follows that $Q(h^{t-1}, k, q_0, q_1)$ is decreasing in k if and only if $B > A$ holds, which is equivalent to the condition,

$$(\overline{\alpha}_0 + \overline{\alpha}_1 - 1)(q_1 - q_0) > 0. \tag{6.9}$$

Algebraic manipulation of equations (6.2) and (6.3) establishes that $\overline{\alpha}_0 + \overline{\alpha}_1 - 1$ must be positive, and the result follows. ■

Proof of Proposition 2.

Case 1: Suppose that the one-step property does not hold immediately following some history in which both cutoffs are interior. Then for some type, $i \in \{0, 1\}$, a type-i firm with cost $\beta_i(h^{t-1})$ has a continuation strategy, σ, that yields higher expected profits than making a final decision in round $t+1$, and denote the difference in expected profits by ε. Now consider the continuation strategy, σ^T, which is identical to σ, except that the firm makes a final decision whether to invest in round T if it has not yet invested (investing after all histories h^T that are subhistories of h^{t-1} and are profitable). Because of discounting, there exists T such that the difference in expected profits, of σ^T vs. making a final decision in round $t+1$, is at least $\varepsilon/2$.

Consider a history, $h^{T-2} = (h^{t-1}, k^t, ..., k^{T-2})$. Because cutoffs are aligned, there are two possibilities: (i) no firm will ever invest following this history, in which case investing when profitable is preferred to waiting until round T, or (ii) both cutoffs are interior, so a firm with cost $\beta_i(h^{T-2})$ is indifferent between investing in round $T-1$ and waiting, which implies that a firm with the lower cost $\beta_i(h^{t-1})$ prefers to invest in round $T-1$. Thus, the continuation strategy σ^{T-1} is weakly preferred to σ^T. Working backwards, we see that the continuation strategy σ^{t+1} is weakly preferred to σ^T, contradicting the previous conclusion that the expected profits of σ^T exceed the expected profits of σ^{t+1} by at least $\varepsilon/2$.

Case 2: Suppose that the one-step property does not hold immediately following some history in which only $\beta_1(h^{t-1})$ is interior. Then a type-1 firm with cost $\beta_1(h^{t-1})$ has a continuation strategy, σ, that yields higher expected profits than making a final decision in round $t+1$, and denote the difference in expected profits by ε. Cutoffs are aligned, so after all subsequent histories, h^{T-2}, either (i) no firm will ever invest following this history, in which case investing when profitable in round $T-1$ is preferred to waiting until round T, or (ii) $\beta_1(h^{T-2})$ is interior, so a firm with cost $\beta_1(h^{T-2})$ is indifferent between investing in round $T-1$ and waiting, which implies that a firm with the lower cost $\beta_1(h^{t-1})$ prefers to invest in round $T-1$. Therefore, we reach the same contradiction as we did in case 1. Because investment-cost cutoffs are aligned, these are the only two cases, so the one-step property must hold. ■

Proof of Proposition 3. To make the notation less cumbersome, we will drop the explicit functional dependence on h^{t-1}, q_0, and q_1. From (2.10) and (2.11), we can write the indifference conditions (2.12) and (2.13) as

$$
\begin{aligned}
(P_1 - \beta_1)\left(\frac{1-\delta}{\delta}\right) &= \sum_{k=0}^{n-1} \max[0, P_1(\beta_1 - 1)Q_1(k) + \beta_1(1-P_1)Q_0(k)] \text{ and} \\
(P_0 - \beta_0)\left(\frac{1-\delta}{\delta}\right) &= \sum_{k=0}^{n-1} \max[0, P_0(\beta_0 - 1)Q_1(k) + \beta_0(1-P_0)Q_0(k)] \ ,
\end{aligned}
$$

which equivalently can be expressed as

$$
\frac{(P_1 - \beta_1)}{\beta_1(1-P_1)}\left(\frac{1-\delta}{\delta}\right) = \sum_{k=0}^{n-1} \max[0, \frac{P_1(\beta_1 - 1)Q_1(k)}{\beta_1(1-P_1)} + Q_0(k)] \text{ and} \tag{6.10}
$$

$$
\frac{(P_0 - \beta_0)}{\beta_0(1-P_0)}\left(\frac{1-\delta}{\delta}\right) = \sum_{k=0}^{n-1} \max[0, \frac{P_0(\beta_0 - 1)Q_1(k)}{\beta_0(1-P_0)} + Q_0(k)] \ . \tag{6.11}
$$

Suppose equation (2.14) holds. From (2.1) and (2.2), equation (2.14) is equivalent to

$$
\frac{P_0}{1-P_0}\left(\frac{1-\beta_0}{\beta_0}\right) = \frac{P_1}{1-P_1}\left(\frac{1-\beta_1}{\beta_1}\right). \tag{6.12}
$$

Using (6.12), we can rewrite (6.11) as

$$
\frac{(P_0 - \beta_0)}{\beta_0(1-P_0)}\left(\frac{1-\delta}{\delta}\right) = \sum_{k=0}^{n-1} \max[0, \frac{P_1(\beta_1 - 1)Q_1(k)}{\beta_1(1-P_1)} + Q_0(k)] \ . \tag{6.13}
$$

We claim that (6.10) is satisfied if and only if (6.13) is satisfied. The right sides of the two equations are the same, so we must show that the left sides are also the same.

From (6.12), we can solve for β_0 in terms of β_1,

$$\beta_0 = \frac{P_0\beta_1(1-P_1)}{P_0\beta_1 + P_1 - P_0P_1 - P_1\beta_1},$$

which, when substituted into $\frac{(P_0-\beta_0)}{\beta_0(1-P_0)}$ and simplified, can be shown to equal $\frac{(P_1-\beta_1)}{\beta_1(1-P_1)}$, establishing the claim.

The equilibrium is constructed as follows, history by history. If investment is unprofitable for a type-1 firm with cost $\beta_1(h^{t-2})$, there is no investment and investment remains unprofitable thereafter. If investment is profitable for a type-1 firm with cost $\beta_1(h^{t-2})$, then $\beta_1(h^{t-1})$ is chosen to solve (2.16), and $\beta_0(h^{t-1})$ is determined by (2.15). We now show that a solution exists with $\beta_1 \in [\beta_1(h^{t-2}), P_1]$. First note that both sides are continuous in β_1. If $\beta_1 = P_1$ holds, then the left side of (2.16) is zero and the right side is nonnegative, so the difference is less than or equal to zero. A solution will exist if we can show that, when $\beta_1 = \beta_1(h^{t-2})$ holds, the difference is greater than or equal to zero. Let us consider the two possible cases, assuming this construction was used in all previous histories.

In case 1, the round $t-1$ cutoff for type-0 firms was $\underline{c}$, which implies

$$\frac{\beta_0(h^{t-2})}{1-\beta_0(h^{t-2})} \geq \frac{\beta_1(h^{t-2})}{1-\beta_1(h^{t-2})}\frac{(1-\alpha)^2}{\alpha^2} \tag{6.14}$$

and $b_0(\beta_1(h^{t-2})) = \underline{c}$. Therefore, if in round t we have $\beta_1 = \beta_1(h^{t-2})$, then $b_0(\beta_1) = \underline{c}$, which implies $q_0 = 0$ and $q_1 = 0$. This, in turn, implies that there is no investment in round t, so that the right side of (2.16) is zero. Since the left side of (2.16) is positive, a solution exists with $\beta_1 \in [\beta_1(h^{t-2}), P_1]$.

In case 2, both cutoffs in round $t-1$ were interior, which implies

$$\frac{\beta_0(h^{t-2})}{1-\beta_0(h^{t-2})} = \frac{\beta_1(h^{t-2})}{1-\beta_1(h^{t-2})}\frac{(1-\alpha)^2}{\alpha^2} \tag{6.15}$$

and $b_0(\beta_1(h^{t-2})) = \beta_0(h^{t-2})$. Therefore, if in round t we have $\beta_1 = \beta_1(h^{t-2})$, then $b_0(\beta_1) = \beta_0(h^{t-2})$, which implies $q_0 = 0$ and $q_1 = 0$. By the reasoning of case 1, a solution exists with $\beta_1 \in [\beta_1(h^{t-2}), P_1]$.

We now show that the cutoffs constructed from the above procedure constitute an equilibrium. Following the history, h^{t-1}, clearly a type-1 firm with cost $\beta_1(h^{t-1})$ is indifferent between investing in round t and making a final decision in round $t+1$. Suppose $b_0(\beta_1) > \underline{c}$ holds. Because we know that the indifference condition for a type-0 firm is equivalent to (6.11), and that we have

$$\frac{(P_0-\beta_0)}{\beta_0(1-P_0)} = \frac{(P_1-\beta_1)}{\beta_1(1-P_1)}$$

when (6.12) holds, it follows that a type-0 firm with cost $\beta_0(h^{t-1})$ is indifferent between investing in round t and making a final decision in round $t+1$. Suppose instead that $b_0(\beta_1) = \underline{c}$ holds. Since, in order for a type-0 firm to be indifferent, its investment cost would have to be less than (or possibly equal to) $\underline{c}$, it is optimal for all type-0 firms to wait.

Finally, our construction ensures that cutoffs are aligned. The argument given in the proof of Proposition 2 establishes that no firm has a profitable deviation that involves not investing and possibly waiting for more than one period. ■

Proof of Proposition 4. Consider parameters in region 1. Even a type-1 firm with cost $\underline{c}$ would receive negative expected profits by investing, so no firm invests in round 0. Therefore, nothing is learned from past behavior, so there is no investment in subsequent rounds.

Consider parameters in region 2. Suppose we have $\beta_1^* > \beta_0^* \geq \underline{c}$. For sufficiently large n, the law of large numbers implies the following. If we have $Z = 1$, with probability arbitrarily close to one, the fraction of firms investing in round 0 is arbitrarily close to $\alpha F(\beta_1^*) + (1-\alpha)F(\beta_0^*)$. If we have $Z = 0$, with probability arbitrarily close to one, the fraction of firms investing in round 0 is arbitrarily close to $\alpha F(\beta_0^*) + (1-\alpha)F(\beta_1^*)$. Because these fractions are different, firms can infer the true state from round 0 activity, with probability arbitrarily close to one. A type-1 firm with cost c receives expected profits of $\alpha - c$ by investing in round 0, but would receive arbitrarily close to $\delta\alpha(1-c)$ by waiting until round 1. Because $\frac{\alpha(1-\delta)}{1-\alpha\delta} < \underline{c}$ holds, it follows that all type-1 firms should wait until round 1, contradicting $\beta_1^* > \underline{c}$. If we have $\beta_0^* > \beta_1^* \geq \underline{c}$, once again firms can infer the state from round 0 activity, and we reach the same contradiction.

Suppose we have $\beta_0^* = \beta_1^* > \underline{c}$. For all $\varepsilon > 0$, there exists N such that $n > N$

implies $\beta_1^n < \beta_1^* + \varepsilon$ and $\beta_0^n > \beta_0^* - \varepsilon$. Therefore, if firm i is type-1 with cost $c_i = \beta_1^* + \varepsilon$, it prefers not to invest in round 0, choosing instead some other strategy, s_i^n. Thus, we have

$$\alpha - \beta_1^* - \varepsilon < \alpha\pi(s_i^n, \beta_1^* + \varepsilon \mid Z = 1) + (1 - \alpha)\pi(s_i^n, \beta_1^* + \varepsilon \mid Z = 0),$$

which implies

$$\alpha - \beta_1^* - \varepsilon < \alpha\pi(s_i^n, \beta_1^* \mid Z = 1) + (1 - \alpha)\pi(s_i^n, \beta_1^* \mid Z = 0) + \varepsilon, \tag{6.16}$$

where $\pi(s, c \mid Z = z)$ denotes the discounted expected profits for a firm with investment cost c, playing the strategy s, given that the state is z.[20] If firm i is type-0 with cost $c_i = \beta_0^* - \varepsilon$, it prefers to invest in round 0, rather than choosing the strategy, s_i^n. Thus, we have

$$1 - \alpha - \beta_0^* + \varepsilon > (1 - \alpha)\pi(s_i^n, \beta_0^* - \varepsilon \mid Z = 1) + \alpha\pi(s_i^n, \beta_0^* - \varepsilon \mid Z = 0),$$

which implies

$$1 - \alpha - \beta_0^* + \varepsilon > (1 - \alpha)\pi(s_i^n, \beta_0^* \mid Z = 1) + \alpha\pi(s_i^n, \beta_0^* \mid Z = 0) - \varepsilon. \tag{6.17}$$

[20] Since there are no histories to observe and we are conditioning on the true state, firm i's common value signal, X_i, provides no additional information about expected revenues. Therefore, $\pi(s, c \mid Z = z)$ is independent of a firm's common value signal.

From (6.16), (6.17), and $\beta_0^* = \beta_1^*$, we have

$$0 < (2\alpha - 1)\left[\pi(s_i^n, \beta_1^* \mid Z = 1) - \pi(s_i^n, \beta_1^* \mid Z = 0) - 1\right] + 4\varepsilon. \tag{6.18}$$

Because $\pi(s_i^n, \beta_1^* \mid Z = 1) < \delta(1 - \beta_1^*)$ and $\pi(s_i^n, \beta_1^* \mid Z = 0) > -\delta\beta_1^*$ must hold, it follows that the term in brackets in (6.18) is negative. Therefore, for sufficiently small ε, we have a contradiction. The remaining cases are: (i) $c_i = \beta_1^* + \varepsilon$ (which is impossible), $\beta_1^* = \overline{c}$ (which is also impossible), or $\beta_0^* = \beta_1^* = \underline{c}$.

Consider parameters in region 3. Suppose we have $\beta_1^* \neq \beta_0^*$. By the law-of-large-numbers argument given above for region 2, for sufficiently large n, a firm not investing in round 0 will be able to infer the true state arbitrarily precisely. Based on the parameter range for region 3, all type-0 firms would rather learn the state in round 1 than invest in round 0. A type-1 firm with cost c_i would rather invest in round 0 than learn the state in round 1 if and only if we have $c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}$. Therefore, we must have $\beta_0^* = \underline{c}$ and $\beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}$.

Suppose we have $\beta_0^* = \beta_1^* = \underline{c}$. For a type-1 firm with cost, $c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}$, investing in round 0 strictly dominates any other strategy, so we cannot have $\beta_1^* = \underline{c}$.

Suppose we have $\overline{c} > \beta_0^* = \beta_1^* > \underline{c}$. By the identical argument given above for region 2, we reach a contradiction. The remaining possibility for region 3 is $\beta_0^* = \beta_1^* = \overline{c}$, which is impossible.

Consider parameters in region 4. Suppose we have $\beta_1^* \neq \beta_0^*$. By the law-of-large-numbers argument given above for region 2, for sufficiently large n, a firm not investing in round 0 will be able to infer the true state arbitrarily precisely. Based on the parameter range for region 4, a type-0 firm with cost c_i would rather invest in round 0 than learn the state in round 1 if and only if we have $c_i < \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta}$. A type-1 firm with cost c_i would rather invest in round 0 than learn the state in round 1 if and only if we have $c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}$. Therefore, we must have $\beta_0^* = \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta}$ and $\beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}$.

Suppose we have $\beta_0^* = \beta_1^* = \underline{c}$. For a type-1 firm with cost, $c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}$, investing in round 0 strictly dominates any other strategy, so we cannot have $\beta_1^* = \underline{c}$.

Suppose we have $\overline{c} > \beta_0^* = \beta_1^* > \underline{c}$. By the identical argument given above for region 2, we reach a contradiction. The remaining possibility for region 4 is $\beta_0^* = \beta_1^* = \overline{c}$, which is impossible. ■

Derivations of $Q_1(\emptyset, 0, q_0, q_1)$ and $Q_0(\emptyset, 0, q_0, q_1)$ when $n \to \infty$ and $\delta \to 1$. As $\delta \to 1$, the indifference equation for type-1 firms, (2.16), implies that the option not to invest in round 2 after $k^1 = 0$ is converging to zero for the marginal type-1 firm, so we have $\theta_0(\emptyset, q_0, q_1, \underline{c}) = 0$, which can be solved for $Q(\emptyset, 0, q_0, q_1)$, yielding

$$Q(\emptyset, 0, q_0, q_1) = \frac{(1-\underline{c})\alpha}{\underline{c}(1-\alpha)}. \tag{6.19}$$

Since (abbreviating the functional notation) Q is the ratio of the probability of zero investment in the low state to the high state, we also have

$$Q = \frac{(1-(1-\alpha)q_1)^n}{(1-\alpha q_1)^n},$$

which can be solved for q_1, yielding

$$q_1 = \frac{Q^{\frac{1}{n}} - 1}{\alpha Q^{\frac{1}{n}} - (1-\alpha)}. \tag{6.20}$$

Substituting (6.20) into $Q_1 = \lim_{n\to\infty}(1-\alpha q_1)^n$, we have

$$Q_1 = \lim_{n\to\infty}(1-\alpha\frac{Q^{\frac{1}{n}} - 1}{\alpha Q^{\frac{1}{n}} - (1-\alpha)})^n. \tag{6.21}$$

Taking logs of both sides of (6.21), and using l'Hopital's rule, yields

$$Q_1 = Q^{-\alpha/(2\alpha-1)}$$

so the result follows from (6.19). To derive $Q_0(\emptyset, 0, q_0, q_1)$, we substitute (6.20) into $Q_0 = \lim_{n\to\infty}(1-(1-\alpha)q_1)^n$, take logs of both sides, and use l'Hopital's rule.

Proof of Proposition 5. Part (1) is obvious. Suppose the parameters are in region 3, and consider a type-1 firm with investment cost below β_1^*. Then Proposition 3 implies that for sufficiently large n, this firm will invest in round 0, in which case $W^*(1, c_i) = \alpha - c_i$ holds. All other firms will not invest in round 0, for sufficiently large n, due to the assumption, $\overline{c} \geq 1$. If other firms invest in round 1 if and only if we have $k^0/n \geq F(\beta_1^*)/2$, then by the law of large numbers, the probability of investing in the low state converges to zero, and the probability of investing in the high state converges to one. Since the probability of the high state is α for a type-1 firm and $1-\alpha$ for a type-0 firm, part (3) of Proposition 8 follows. The same argument applies to region 4, except that the firms that do not invest in round 0 should invest in round 1 if and only if we have $k^0/n \geq [F(\beta_0^*) + (F(\beta_1^*)]/2$.

Suppose the parameters are in region 2. For the equilibrium of the economy with n firms, let $E^{t,\varepsilon}$ be the event that a type-1 firm with investment cost $\underline{c} + \varepsilon$ invests in round t, and define $T_1^n(1,\varepsilon) = \sum_{t=1}^{\infty} \delta^t pr(Z = 1 \text{ and } E^{t,\varepsilon} \mid X_i = 1)$, define $T_1^n(0,\varepsilon) = \sum_{t=1}^{\infty} \delta^t pr(Z = 0 \text{ and } E^{t,\varepsilon} \mid X_i = 1)$, define $T_0^n(1,\varepsilon) = \sum_{t=1}^{\infty} \delta^t pr(Z = 1$ and $E^{t,\varepsilon} \mid X_i = 0)$, and define $T_0^n(0,\varepsilon) = \sum_{t=1}^{\infty} \delta^t pr(Z = 0 \text{ and } E^{t,\varepsilon} \mid X_i = 0)$.

Clearly, $W^n(1, c_i)$ must be continuous and decreasing in c_i, because otherwise some type-1 firm would have a profitable deviation to imitate the strategy chosen by a type-1 firm with nearby investment cost. For all $\varepsilon_1 > 0$, there exists $\varepsilon_2 > 0$, such that $\varepsilon < \varepsilon_2$ implies

$$W^n(1, \underline{c}) - W^n(1, \underline{c} + \varepsilon) < \varepsilon_1. \tag{6.22}$$

We have $W^n(1, \underline{c}) = \alpha - \underline{c}$ (they invest in round 0) and $W^n(1, \underline{c} + \varepsilon) = T_1^n(1,\varepsilon)(1 - \underline{c} - \varepsilon) - T_1^n(0,\varepsilon)(\underline{c} + \varepsilon)$. Imposing $\varepsilon < \min[\varepsilon_1, \varepsilon_2]$, it follows from (6.22) that we have

$$\alpha - \underline{c} - T_1^n(1,\varepsilon)(1 - \underline{c}) + T_1^n(0,\varepsilon)(\underline{c}) < 2\varepsilon_1. \tag{6.23}$$

Claim: For all c_i and all $\varepsilon > 0$, there exists N such that $n > N$ implies

$$W^n(1, c_i) \geq \delta T_1^n(1, \varepsilon)(1 - c_i) - \varepsilon_1. \tag{6.24}$$

Proof of Claim: Consider a type-1 firm with investment cost c_i, which waits until the round after a type-1 firm with cost $\underline{c}+\varepsilon$ would invest. That is, we are considering histories such that $\beta_1^n(h^{t-1}) \geq \underline{c}+\varepsilon$. By the law of large numbers, for sufficiently large n, with probability arbitrarily close to one, the cumulated investment in state 1 is arbitrarily close to $\alpha F(\beta_1^n(h^{t-1}))+(1-\alpha)F(\beta_0^n(h^{t-1}))$, and the cumulated investment in state 0 is arbitrarily close to $(1-\alpha)F(\beta_1^n(h^{t-1})) + \alpha F(\beta_0^n(h^{t-1}))$. A type-1 firm with cost $\beta_1^n(h^{t-1})$ is indifferent between investing in round t and some continuation strategy, s, in which the probability of eventual investment is less than one. (This fact follows from $\delta < 1$ and our assumption, $\bar{c} \geq 1$, which guarantees that there are type-1 firms with costs above $\beta_1^n(h^{t-1})$ that do not invest.) However, a type-0 firm with the same cost, $\beta_1^n(h^{t-1})$, must strictly prefer the continuation strategy, s, because $P_1(h^{t-1}) > P_0(h^{t-1})$. Thus, we conclude that $\beta_0^n(h^{t-1}) < \beta_1^n(h^{t-1})$ must hold. By investing in round $t+1$ if and only if we have $\beta_1^n(h^{t-1}) \geq \underline{c}+\varepsilon$ and

$$\frac{\sum_{\tau=1}^{t} k^\tau}{n} \geq \frac{F(\beta_0^n(h^{t-1})) + F(\beta_1^n(h^{t-1}))}{2},$$

the probability of investing in the high state (when we have $\beta_1^n(h^{t-1}) \geq \underline{c} + \varepsilon$) is arbitrarily close to one, and the probability of investing in the low state (when we have $\beta_1^n(h^{t-1}) \geq \underline{c} + \varepsilon$) is arbitrarily close to zero. Thus, adopting this strategy yields expected profit arbitrarily close to $\delta T_1^n(1, \varepsilon)(1 - c_i)$, thereby establishing the Claim.

From (6.23), and the fact that $T_1^n(0, \varepsilon) \geq 0$, we have

$$T_1^n(1, \varepsilon) \geq \frac{\alpha - \underline{c}}{1 - \underline{c}} - \frac{2\varepsilon_1}{1 - \underline{c}}. \tag{6.25}$$

From (6.24) and (6.25), we have

$$W^n(1, c_i) \geq \delta \left[\frac{\alpha - \underline{c}}{1 - \underline{c}}\right](1 - c_i) - \delta \left[\frac{2\varepsilon_1}{1 - \underline{c}}\right](1 - c_i) - \varepsilon_1. \tag{6.26}$$

Letting $c_i = \underline{c}$ hold in (6.24), we have $\delta T_1^n(1, \varepsilon)(1 - \underline{c}) - \varepsilon_1 \leq \alpha - \underline{c}$, and using (6.23), this becomes $\delta T_1^n(1, \varepsilon)(1 - \underline{c}) - \varepsilon_1 < T_1^n(1, \varepsilon)(1 - \underline{c}) - T_1^n(0, \varepsilon)(\underline{c}) + 2\varepsilon_1$, implying

$$T_1^n(0, \varepsilon)(\underline{c}) < T_1^n(1, \varepsilon)(1 - \underline{c})(1 - \delta) + 3\varepsilon_1. \tag{6.27}$$

A type-1 firm with cost c_i cannot possibly do better than to decide whether or not to invest during the same round that firm $(1, \underline{c} + \varepsilon)$ invests, but with full knowledge of the state. Therefore, we have

$$W^n(1, c_i) \leq T_1^n(1, \varepsilon)(1 - c_i). \tag{6.28}$$

Since firm $(1, \underline{c})$ weakly prefers to invest in round 0, rather than mimic the strategy of firm $(1, \underline{c} + \varepsilon)$, we have $\alpha - \underline{c} \geq T_1^n(1, \varepsilon)(1 - \underline{c}) - T_1^n(0, \varepsilon)(\underline{c})$. Thus, we have

$$T_1^n(1, \varepsilon)(1 - \underline{c}) \leq \alpha - \underline{c} + T_1^n(0, \varepsilon)(\underline{c}). \tag{6.29}$$

Using (6.27) and (6.29), we have $T_1^n(1, \varepsilon)(1 - \underline{c}) \leq \alpha - \underline{c} + T_1^n(1, \varepsilon)(1 - \underline{c})(1 - \delta) + 3\varepsilon_1$, from which we have

$$T_1^n(1, \varepsilon) \leq \frac{\alpha - \underline{c}}{\delta(1 - \underline{c})} + \frac{3\varepsilon_1}{\delta(1 - \underline{c})}. \tag{6.30}$$

From (6.28) and (6.30), we conclude

$$W^n(1, c_i) \leq \left[\frac{\alpha - \underline{c}}{\delta(1 - \underline{c})}\right](1 - c_i) + \frac{3\varepsilon_1(1 - c_i)}{\delta(1 - \underline{c})}. \tag{6.31}$$

For type-0 firms, a simple calculation yields

$$\begin{aligned} T_0^n(1, \varepsilon) &= \frac{1 - \alpha}{\alpha} T_1^n(1, \varepsilon) \quad \text{and} \\ T_0^n(0, \varepsilon) &= \frac{\alpha}{1 - \alpha} T_1^n(0, \varepsilon). \end{aligned}$$

The law-of-large-numbers argument given in the Claim also establishes

$$\begin{aligned} W^n(0, c_i) &\geq \delta T_0^n(1, \varepsilon)(1 - c_i) - \varepsilon_1, \text{ implying} \\ W^n(0, c_i) &\geq \delta \frac{1 - \alpha}{\alpha} T_1^n(1, \varepsilon)(1 - c_i) - \varepsilon_1. \end{aligned} \tag{6.32}$$

From (6.25) and (6.32), it follows that

$$W^n(0,c_i) \geq \delta\left[\frac{1-\alpha}{\alpha}\right]\left[\frac{\alpha-\underline{c}}{1-\underline{c}}\right](1-c_i) - \delta\left[\frac{1-\alpha}{\alpha}\right]\left[\frac{2\varepsilon_1}{1-\underline{c}}\right](1-c_i) \tag{6.33}$$

holds. A type-0 firm with cost c_i cannot possibly do better than to decide whether or not to invest during the same round that firm $(1, \underline{c}+\varepsilon)$ invests, but with full knowledge of the state. Therefore, we have

$$W^n(0,c_i) \leq T_0^n(1,\varepsilon)(1-c_i) = .\frac{1-\alpha}{\alpha}T_1^n(1,\varepsilon)(1-c_i). \tag{6.34}$$

Combining (6.30) and (6.34), we have

$$W^n(0,c_i) \leq \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{\alpha-\underline{c}}{\delta(1-\underline{c})}\right](1-c_i) + \left[\frac{1-\alpha}{\alpha}\right]\frac{3\varepsilon_1(1-c_i)}{\delta(1-\underline{c})}. \tag{6.35}$$

Because inequalities (6.28), (6.31), (6.33), and (6.35) hold for all ε_1, the results for region 2 follow. ■

References

[1] S. Baliga and T. Sjostrom, Arms races and negotiations, *Review of Economic Studies* vol. 71(2) (2004), 351-369.

[2] A. V. Banerjee, A simple model of herd behavior, *Quarterly Journal of Economics* 107 (3) (1992), 797-817.

[3] S. Bikhchandani, D. Hirshleifer, and I. Welch, A theory of fads, fashion, custom, and cultural change as informational cascades, *Journal of Political Economy*, Vol. 100 (1992), 992-1026.

[4] P. Bolton and J. Farrell, Decentralization, duplication, and delay, *Journal of Political Economy*, Vol. 98 (1990), 803-826.

[5] J. Bryant, A simple rational expectations Keynes-type model, *Quarterly Journal of Economics* 98 (3) (1983), 525-528.

[6] J. Bryant, The paradox of thrift, liquidity preference and animal spirits, *Econometrica* 55(5) (1987), 1231-1235.

[7] A. Caplin and J. Leahy, Business as usual, market crashes, and wisdom after the fact, *American Economic Review* 84(3) (1994), 548-565.

[8] C. Chamley, Delays and equilibria with large and small information in social learning, *European Economic Review* 48 (2004), 477-501.

[9] C. Chamley and D. Gale, Information revelation and strategic delay in a model of investment, *Econometrica*, Volume 62, Issue 5 (1994), 1065-1085.

[10] V. V. Chari and P. J. Kehoe, Financial crises as herds: overturning the critiques, *Journal of Economic Theory* 119(1) (2004), 128-150.

[11] R. Cooper and A. John, Coordinating coordination failures in Keynesian models, *Quarterly Journal of Economics* 103 (3) (1988), 441-463.

[12] P. Diamond, Aggregate demand management in search equilibrium, *Journal of Political Economy*, Vol. 90 (1982), 881-894.

[13] A. Dixit and C. Shapiro, Entry dynamics with mixed strategies, In L. G. Thomas III, ed., *The Economics of Strategic Planning: Essays in Honor of Joel Dean*. Lexington: Lexington Books, 1986.

[14] D. Fudenberg and J. Tirole, Preemption and rent equalization in the adoption of new technology, *Review of Economic Studies* 52 (1985), 383-401.

[15] J. Glazer and A. Rubinstein, Motives and implementation: on the design of mechanisms to elicit opinions, *Journal of Economic Theory* 79(2) (1998), 157-173.

[16] J. K. Goeree, T. R. Palfrey, and B. W. Rogers, Social learning with private and common values, *Economic Theory* 28 (2006), 245-264.

[17] F. Gul and R. Lundholm, Endogenous timing and the clustering of agents' decisions, *Journal of Political Economy*, No. 5, Vol. 103 (1995), 1039-1066.

[18] T. D. Jeitschko and C. R. Taylor, Local discouragement and global collapse: A Theory of Coordination Avalanches, *American Economic Review* 91 (1) (2001), 208-224.

[19] L. E. Jones and R. E. Manuelli, The coordination problem and equilibrium theories of recessions, *American Economic Review* 82 (3) (1992), 451-471.

[20] J. M. Keynes, *The General Theory of Employment, Interest, and Money*, (Harcourt Brace Jovanovich: New York and London), First Harbinger Edition, 1964.

[21] D. Levin and J. Peck, To grab for the market or to bide one's time: a dynamic model of entry. *Rand Journal of Economics* 34(3) (2003), 536-556.

[22] D. Levin, J. Peck, and L. Ye, Bad news can be good news: early dropouts in an English auction with multi-dimensional signals, *Economics Letters* 95 (2007), 462-467.

[23] P. Milgrom and J. Roberts, Rationalizability, learning, and equilibrium in games with strategic complementarities, *Econometrica* 58(6) (1990), 1255-1277.

[24] S. Morris and H. S. Shin, Unique equilibrium in a model of self-fulfilling currency attacks, *American Economic Review* 88 (3) (1998), 587-597.

[25] A. Park and L. Smith, Caller number five: timing games that morph from one form to another, April 2003.

[26] N. Vettas, On entry, exit, and coordination with mixed strategies, *European Economic Review*, Vol. 44 (2000), 1557-1576.