Auctions with Uncertain Numbers of Bidders*

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Abstract

We investigate bidders' and seller's responses to ambiguity about the number of bidders in the first price auction (FPA) and the second price auction (SPA) with independent private valuations. We model ambiguity aversion using the maxmin expected utility model. We find that bidders prefer the number of bidders to be revealed in the FPA, are indifferent between revealing and concealing in the SPA, and prefer the SPA to the FPA. If bidders are more pessimistic than the seller then the seller prefers to conceal the number of bidders in the FPA, and prefers the FPA to the SPA.

Key Words: Auctions, Uncertainty, Ambiguity, Number of Bidders

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1. Introduction

Economic research on auctions and competitive bidding began in 1961 with Vickrey's seminal work. Most of the early auction literature assumes that the number of bidders, n, is both fixed and common knowledge. Using this *fixed–n* paradigm simplifies the analysis and produced many celebrated results regarding revenue-ranking of different auctions forms and optimal auction design. (e.g., Vickrey [28], Milgrom and Weber [20] and Myerson [21]). However, in the 1980's several auction researchers recognized that bidders in sealed-bid auctions usually do not know the number of rival bidders at the time that they submit their bids. Nonetheless, such information is critical for developing optimal bidding strategies.[1] In online auctions sites such as eBay, where each day a very large number of auctions are conducted generating high revenues, bidders rarely know the number of active bidders. Extensions of the *fixed–n* approach to deal with this issue take two forms. The first form is to model the number of rivals as resulting from an exogenous stochastic process; and the second form is to endogenize the entry process. In this paper, we focus only on the first form.[2]

In the first form, within the *Independent-Private-Value* setting, research focuses on the following natural questions:

1. How does uncertainty about the number of actual bidders in the auction affect bidding behavior and bidders' ranking of the the first price auction (FPA) vs. the second price auction (SPA)?

2. Should a seller who knows the actual number of active bidders, reveal or conceal such information to the bidders?

3. How does uncertainty change the (expected) revenue ranking of the FPA and the SPA away from the Vickrey equivalence result?

In the existing literature, the answers to these questions are considered by assuming that bidders share a common prior distribution of the number of active bidders in the auction and that they are risk averse (we label this approach as *pure risk approach*). However, here we propose a complementary, yet different, approach. In this approach, motivated by Ellsberg's [3] urn experiment, we assume that bidders may not have a prior distribution of the number of rivals, but they are nevertheless averse to the ambiguity generated by lack of information about the number of rivals in the auction. In Ellsberg's

[1] In the last spectrum auction in Great Britain that raised about $35 billion the issue of the number of competing companies was so central that the auction form was designed mainly with that in mind.

[2] A few examples for the first approach are in Matthews [15], McAfee and McMillan [17], Harstad, Kagel and Levin [7]. A few examples for the second approach are in Levin and Smith [13], McAfee and McMillan [18], Samuelson [25].

experiment, a decision maker is offered two urns, one that has 50 black and 50 red balls, and one that has 100 black and red balls in unknown proportions. Faced with these two urns, most decision makers bet on drawing either color from the first urn, rather than on drawing the same color from the second urn. It is easy to show that such behavior is inconsistent with the expected utility model. Intuitively, decision makers do not like betting on the second urn because they do not have enough information or, put differently, there is too much ambiguity. Being averse to ambiguity, they prefer to bet on the first urn. Ellsberg and many subsequent studies have[3] demonstrated that ambiguity aversion is common and incompatible with the standard expected utility theory.

In this paper, we investigate how bidders and seller respond to ambiguity about the number of bidders in the FPA and the SPA. Following Gilboa and Schmeidler [5] we model ambiguity aversion using the maxmin expected utility (MMEU) model[4]. In this model bidders have a set of probability measures regarding the number of bidders in the auction. They compute their expected utility in the auction as the minimum expected utility over the set of priors (we label this approach the *MMEU approach*). The MMEU model provides a natural framework to study uncertainty about the number of bidders. For comparison with the risk approach, we consider the case of pure uncertainty; that is we assume that bidders have linear utility functions. This assumption allows us to distinguish risk aversion from ambiguity aversion.

To frame our study, we consider two well known puzzles in auction theory. First, as observed by Rothkopf, Teisberg, and Kahn [23] sealed bid second-price auctions seldom occur in practice even though they have many nice properties. Several reasons have been provided for this phenomenon in the literature, here we provide another potential explanation of this puzzle. Second, a seller can always institute a procedure called *contingent bidding*, that resolves the uncertainty about the number of bidders even when she does not know that number at the time of the bidding. Under the contingent bidding procedure, suggested by Harstad, Kagel and Levin [7] each participating bidder submits a list of bids, one for every possible number of active bidders. The seller (e.g., the government) is precommitted to consider only the contingent bid corresponding to the actual number of submissions. Thus, each bidder effectively behaves as if he knows the number of rivals. However, in practice, we do not observe contingent bidding. Again, our results can potentially explain this puzzle.

For comparison and contrast, we present our main results under the MMEU approach along with the results from the pure risk approach from Matthews [15].

First, in the pure risk approach risk-averse bidders bid their valuations in the SPA

[3] See, for example, Camerer and Weber [1] for a survey.

[4] Gilboa and Schmeidler provide an axiomatic justification for the MMEU model in the Anscombe-Aumann framework which relies on using objective randomizations. In a setting like auctions, where objective randomizations may not be available, an axiomatic justification is provided by Casadesus-Masanell, Klibanoff and Ozdenoren [2].

but increase their bids relative to risk-neutral bidders in the FPA. Thus, bids are higher in the FPA but there is more variance in the SPA. As a result, an actual bidder: 1) prefers the SPA to the FPA if decreasing absolute risk aversion (DARA) holds, 2) is indifferent between the two if constant absolute risk aversion (CARA) holds, with risk neutrality being a special case, and 3) prefers the FPA to the SPA if increasing absolute risk aversion (IARA) holds. Under the MMEU approach, bidders still bid their valuations in the SPA but bid more aggressively in the FPA, as we move from a single prior to multiple priors, if priors used in equilibrium becomes more pessimistic for all bidders. However, even if moving from a single to multiple priors does not result in higher bidding in the FPA, bidders always prefer the SPA to the FPA and this preference is strict as long as, in equilibrium, all types of bidders do not use the same prior.

Second, in the pure risk approach, an actual bidder in the FPA who knows his valuation but not the actual number of bidders, n: 1) prefers n to be revealed if DARA holds, 2) is indifferent as to whether n is revealed if CARA holds and 3) prefers n to be concealed if IARA holds. In particular, a risk-neutral bidder is indifferent between the two policies in the pure risk framework. In contrast, in the MMEU approach, a bidder in the FPA with a linear utility function always prefers the revealing policy and this preference is sometimes strict. We interpret this result to indicate the robustness of bidder preference for the revealing policy in the FPA.

Third, in the FPA (it does not matter in the SPA), the seller in the risk approach prefers, given either CARA or DARA, the concealing policy as it yields a greater expected price in the FPA than does the revealing policy. Similarly, in the MMEU approach seller prefers the concealing policy in the FPA as long as the priors used in equilibrium under the concealing policy are more pessimistic then the seller's prior.

Finally, seller prefers the FPA to the SPA, given CARA or DARA, under the pure risk approach. In the MMEU approach, the seller prefers the FPA to the SPA as long the prior used in equilibrium of the FPA is more pessimistic than the seller's prior.

In section 3 we discuss two extensions of our model. In section 3.1 we introuduce ambiguity about valuations in addition to ambiguity about number of bidders. In section 3.2 we consider another common way to model ambiguity, Choquet expected utility model, and discuss how our results change under this alternative model of preferences.

There is a small but growing literature on auction theory with non-expected utility starting with Karni and Safra ([9], [10], [11]) and Karni [8]. Closer to this paper is the work by Lo [14], who studies first and second price auctions using MMEU preferences but focuses on the ambiguity about the other bidders' valuations whereas ours is on ambiguity about the number of bidders[5].

The literature on ambiguity about the number of bidders is very sparse. The only

[5] Volij [29] and Ozdenoren [22] also look at uncertainty about other bidders' valuations and similar comments apply to these papers as well.

other work known to us in this area is that of Salo and Weber [24]. In their study, Salo and Weber use the Choquet expected utility model to examine bidding behavior in the FPA. They model ambiguity through a distortion of the uniform distribution over the number of actual bidders. They focus on bidding behavior, thus they do not provide results on bidder preferences for revealing or concealing policies, nor do they compare first and second price auctions. Moreover, our model subsumes the Salo and Weber model and thus provides a broader view of bidding behavior.

2. First and Second Price Auctions

2.1. Equilibrium Analysis

To study how bidder behavior varies across auction form, we compare first and second price auctions. In our model, we assume there is one indivisible good for sale. Following McAfee and McMillan [17] and Matthews [15], we consider first and second price auctions with a commonly known reservation price r. Bidders send their bids simultaneously. Bids lower than r are not accepted. We assume that the set of possible valuations of the bidders is $[0,1]$, with v_i denoting bidder i's valuation and $r<1$. Valuations of bidders are known privately. Furthermore, valuations are commonly known to be independent draws from the distribution $F(\cdot)$ on $[0,1]$ with positive density $f(\cdot)$.

In our model, there are at most m *potential* bidders indexed by $i=1,...,m$. We assume that the *actual* number of bidders participating in the auction is not known to the bidders. Following Gilboa and Schmeidler [5], we model bidders as maxmin expected utility (MMEU) maximizers. In this model, bidders have a set of priors regarding the number of actual bidders and their utility is the minimum expected utility as their prior varies over this set. Note that, this model captures bidder uncertainty aversion due to lack of information about the stochastic process that generates the actual number of bidders.

Formally let P be a closed and convex set of measures on the subsets of $\{1,...,m\}$. P represents a potential bidder's belief about the number of bidders in the auction, including himself, if he becomes an actual bidder. That is, for any $p\in P$ and $k\leq m$, and for any active bidder, $p(k)$ denotes the probability that there are k active bidders in the auction[6]. The probability that a potential bidder becomes an actual bidder does

[6] Alternatively, bidders may have beliefs over subsets of a fixed set of bidders where each subset indicates *identities* of the bidders participating in the auction. Such a model can be reduced to the current one where bidders have beliefs regarding only the *number* of participating bidders. We choose the reduced model for two reasons. First, as long as attention is restricted to symmetric equilibrium, information relevant for bidding is the number of participating bidders not their identities. Second, it seems unrealistic to assume that bidders' beliefs can be defined over a fixed set of bidders when bidders

not depend on the bidder's valuation, that is, P is assumed to be independent of bidder valuations[7] and to be common for all bidders.

In the second price auction, the bidder with the highest bid receives the object and pays the maximum of the second highest bid and the reservation price to the seller. It is easy to see that in the second price auction the usual equilibrium, where each bidder bids his true valuation as long as his valuation is over the reservation price, is the unique (dominant strategy) equilibrium even when bidders have ambiguity about the number of bidders.

In the first price auction, the bidder with the highest bid receives the object and pays this bid to the seller. Suppose the number of active bidders is $n \in \{1, ..., m\}$ and without loss of generality enumarate them as $1, ..., n$. Denote bidder j's bid by $b_j \in [0, \infty)$. Consider the payoff of participating bidder i. Let z_n^i be the highest competing bid for bidder i, i.e., $z_1^i = 0$ and $z_n^i = \max \{b_j \,| j \neq i, j \in \{1, ..., n\}\}$ for $n \geq 2$. Also, let t be the number of bidders tied with bidder i including himself, i.e., $t = \# \{j \,| b_j = b_i, j \in \{1, ..., n\}\}$. Denote bidder i's payoff by $\pi_i(v_i, b_i, z_n^i)$. In the first price auction, bidder i's payoff is given by,

$$\pi_i(v_i, b_i, z_n^i) = \begin{cases} \frac{1}{t}(v_i - b_i) & \text{if } b_i \geq z_n^i \text{ and } b_i \geq r \\ 0 & \text{if } b_i < z_n \text{ or } b_i < r \end{cases}. \tag{1}$$

Denote the strategy of bidder j by $s_j : [0, 1] \to [0, \infty)$. That is, if bidder j is a participating bidder with valuation v_j, he bids $s_j(v_j)$. Thus, $z_1^i = 0$ and $z_n^i = \max \{s_j(v_j) \,| j \neq i, j \in \{1, ..., n\}\}$ for $n \geq 2$. Let $H(\cdot \,| n)$ be the distribution function of z_n^i. For any $p \in P$, let $H^p(\cdot) = \sum_{n=1}^m p(n) H(\cdot \,| n)$.

Let $U_i\left(v_i, b_i, \{s_j\}_{j \neq i}\right)$ denote bidder i's utility in the first price auction if his valuation is v_i, his bid is b_i and the strategies of the other bidders are given by $\{s_j\}_{j \neq i}$. Since bidders have MMEU preferences, (ignoring ties) we can express bidder i's utility as,

$$U_i(v_i, b_i, \{s_j\}_{j \neq i}) = \begin{cases} \min_{p \in P} \left[(v_i - b_i) H^p(b_i)\right] & \text{if } b_i \geq r \\ 0 & \text{if } b_i < r \end{cases}$$

The vector of bidding strategies, $\{s_i\}_{i=1}^m$, is an equilibrium if for all i and $v_i \in [0, 1]$,

$$U_i(v_i, s_i(v_i), \{s_j\}_{j \neq i}) \geq U_i(v_i, b_i, \{s_j\}_{j \neq i}) \text{ for all } b_i \in [0, \infty).$$

Next, we look for a symmetric equilibrium of the first price auction. Suppose that bidder i knows that all bidders are using symmetric, strictly increasing and differentiable

may not even know who may be participating in the auction.

[7] Alternatively, first potential bidders learn their private valuations. Later, nature determines the actual bidders through a stochastic process that does not depend on the valuations of the potential bidders.

bidding functions, $s(\cdot)$. In this case, for any $b \in [0, \infty)$,

$$H^p(b) = \sum_{n=1}^{m} p(n) H(b|n) = \sum_{n=1}^{m} p(n) F^{n-1}\left(s^{-1}(b)\right). \tag{2}$$

Using (2), rewrite bidder i's utility as,

$$U_i(v_i, b_i, \{s, ..., s\}) = \begin{cases} (v_i - b_i) \min_{p \in P} \sum_{n=1}^{m} p(n) F^{n-1}\left(s^{-1}(b_i)\right) & \text{if } b_i \geq r \\ 0 & \text{if } b_i < r \end{cases}. \tag{3}$$

Letting $G^{\min}(v) = \min_{p \in P} \sum_{n=1}^{m} p(n) F^{n-1}(v)$, we rewrite (3) as,

$$U_i(v_i, b_i, \{s, ..., s\}) = \begin{cases} (v_i - b_i) G^{\min}\left(s^{-1}(b_i)\right) & \text{if } b_i \geq r \\ 0 & \text{if } b_i < r \end{cases}. \tag{4}$$

Note that (4) implies that the utility of any bidder in this auction is equal to the utility of an ambiguity neutral bidder in the traditional first price auction with two bidders where each bidder's valuation is drawn from the probability distribution $G^{\min}$. This observation leads to our first proposition.

Proposition 1. *In the first price auction the unique symmetric equilibrium bidding function is given by,*

$$s^{\min}(v) = v - \frac{\int_r^v G^{\min}(t)\, dt}{G^{\min}(v)} \text{ for all } v \in [r, 1].$$

The proof is quite standard and is omitted. As we shall see later our interesting results are obtained when the MMEU model "bites", namely when different types use different p's. Thus, in equilibrium, as is usual, strategies must be best responses to others strategies, but also beliefs must be consistent in the sense that, for any type, given the equilibrium strategies of bidders, the set of p's that minimizes a bidder's expected utility must also be the one that minimizes $G^{\min}$. One might wonder how all that collapses to a standard proof. How come we don't have to worry about beliefs? The answer is that in a symmetric monotonic equilibrium one wins when her signal is the largest of the signals of the n actual bidders. As a result the p's that minimize utility can be, and is defined independently of strategies and the correct minimizer is just those p's that minimize the probability that a certain type is the highest. This observation brings good and bad news. The good news is that our method is likely to work in applications of MMEU to games when the focus is on symmetric and monotonic equilibrium. But it also suggests (and we conjecture) that without symmetry solving for equilibrium may be harder than usual. The following example helps to illustrate this discussion and proposition 1.

Example 1. *Let $m = 4$ and P be the convex hull of p_1, p_2 and p_3 where:*

$$p_1(1) = 0.1, \quad p_1(2) = 0.1, \quad p_1(3) = 0.3, \quad p_1(4) = 0.5,$$
$$p_2(1) = 0 \quad p_2(2) = 0.5 \quad p_2(3) = 0 \quad p_2(4) = 0.5,$$
$$p_3(1) = 0.5, \quad p_3(2) = 0.5, \quad p_3(3) = 0, \quad p_3(4) = 0.$$

In words, p_1 puts probability 0.1 on the event that there is only one, probability 0.1 on the event that there are two, probability 0.3 on the event that there are three and probability 0.5 on the event that there are four active bidders. Probability measures p_2 and p_3 can be interpreted similarly. We assume that F is the uniform distribution over $[0,1]$. Let's compute $G^{\min}$ in this case.

$$\begin{aligned} G^{\min}(v) &= \min_{p \in P} \sum_{n=1}^{m} p(n) v^{n-1} \\ &= \min \left\{ 0.1 + 0.1v + 0.3v^2 + 0.5v^3, 0.5v + 0.5v^3, 0.5 + 0.5v \right\} \\ &= \begin{cases} 0.5v + 0.5v^3 & \text{if } v \leq 1/3 \\ 0.1 + 0.1v + 0.3v^2 + 0.5v^3 & \text{if } v > 1/3 \end{cases}. \end{aligned}$$

Therefore, the (conjectured) equilibrium bidding functions are given by:

$$\begin{aligned} s^{\min}(v) &= v - \frac{\int_0^v G^{\min}(t)\,dt}{G^{\min}(v)} \qquad (5) \\ &= \begin{cases} v - \frac{\int_0^v \left(0.5t + 0.5t^3\right)dt}{0.5v + 0.5v^3} & \text{if } v \leq 1/3 \\ v - \frac{\int_0^{1/3}\left(0.5t+0.5t^3\right)dt + \int_{1/3}^{v}\left(0.1+0.1t+0.3t^2+0.5t^3\right)dt}{0.1+0.1v+0.3v^2+0.5v^3} & \text{if } v > 1/3 \end{cases}. \end{aligned}$$

For $s^{\min}$ to be the symmetric equilibrium bidding function it must be a best response for a bidder to use $s^{\min}$ if all the other bidders use $s^{\min}$. Next we show that this is indeed the case. First consider a bidder who bids an amount b. Note that the bidder's utility (computed using MMEU) is given by:

$$\min_{p \in P} \sum_{n=1}^{m} p(n)(v-b) F^{n-1}\left(\left(s^{\min}\right)^{-1}(b)\right). \qquad (6)$$

The optimal bid b should maximize the expression above. The conjectured equilibrium bidding function is strictly increasing and therefore invertible. Let $b = s^{\min}(\hat{v})$. Thus $\left(s^{\min}\right)^{-1}(b) = \hat{v}$. Substituting $\hat{v}$ for $\left(s^{\min}\right)^{-1}(b)$ in equation (6) and plugging in $s^{\min}(\hat{v})$

from equation (5), we obtain,

$$
\begin{aligned}
&\min_{p\in P}\sum_{n=1}^{m} p(n)\left(v-s^{\min}(\hat{v})\right)F^{n-1}(\hat{v})\\
=&\left\{\begin{array}{c}\left(v-s^{\min}(\hat{v})\right)\left(0.5\hat{v}+0.5\hat{v}^{3}\right)\quad \text{if } \hat{v}\leq 1/3\\ \left(v-s^{\min}(\hat{v})\right)\left(0.1+0.1\hat{v}+0.3\hat{v}^{2}+0.5\hat{v}^{3}\right)\quad \text{if } \hat{v}>1/3\end{array}\right.\\
=&\left\{\begin{array}{c}(v-\hat{v})\left(0.5\hat{v}+0.5\hat{v}^{3}\right)+\int_{0}^{\hat{v}}\left(0.5t+0.5t^{3}\right)dt\quad \text{if } \hat{v}\leq 1/3\\ (v-\hat{v})\left(0.1+0.1\hat{v}+0.3\hat{v}^{2}+0.5\hat{v}^{3}\right)+\int_{0}^{1/3}\left(0.5t+0.5t^{3}\right)dt\\ +\int_{1/3}^{\hat{v}}\left(0.1+0.1t+0.3t^{2}+0.5t^{3}\right)dt\ \text{if } \hat{v}>1/3\end{array}\right. .
\end{aligned}
\tag{7}
$$

To see that this expression is maximized at $\hat{v}=v$, first suppose that $v\leq 1/3$. In this case the first part of the function is maximized at v. The second part of the function has a derivative of $(v-\hat{v})\left(0.1+0.6\hat{v}+1.5\hat{v}^{2}\right)\leq 0$, since $\hat{v}>1/3\geq v$ in this range. So the second part is maximized at $\hat{v}=1/3$, at which point the first part and the second part of the function have the same value. Therefore this function is maximized overall at $\hat{v}=v$. Now suppose that $v>1/3$. In this case the second part of the function is maximized at $\hat{v}=v$. The first part of the function has a derivative of $(v-\hat{v})\left(0.5+1.5\hat{v}^{2}\right)\geq 0$ since $\hat{v}\leq 1/3<v$ in this range. So the first part is maximized at $\hat{v}=1/3$. Again at this point the two parts of the function have the same value. So the function is maximized overall at $\hat{v}=v$. This completes the argument that bidding $s^{\min}(v)$ is a best response for type v.

Note that in example 1 a bidder with valuation less than 1/3 computes his utility using p_2 where as a bidder with valuation greater than 1/3 computes his utility using p_1. Intuitively, in this example, a bidder with a "low" valuation can win only if he does not have much competition. So low valuation bidders regard p_2 as the worst case. On the other hand, a bidder with a high valuation is fairly confident about winning against a medium number of actual bidders. Thus, for this bidder the worst case is given by p_1. Note that the distribution, $G^{\min}$, with respect to which the equilibrium bidding strategies are computed does not itself belong to P. This observation is important in obtaining our strict ranking results in what follows.

2.2. Comparative Statics of Bidding Behavior

In order to look at the effect of varying ambiguity aversion in the MMEU framework, first we need the following definition. We say that, in the MMEU framework, a bidder is *more ambiguity averse* than another if the set of priors of the former is larger than the latter.[8]

With MMEU, in the SPA, comparative statics on bidding behavior with respect to ambiguity aversion is trivial since it is a weakly dominant strategy to bid one's valuation no matter what the set of priors is. In the rest of this subsection, we focus on the bidding behavior in the FPA. In the following we define $P_v \subseteq P$ to be the set of priors that attain the minimum (when computing for $G^{\min}$) for a bidder with valuation v, that is,

$$P_v = \arg\min_{p\in P} \sum_{n=1}^{m} p(n) F^{n-1}(v). \tag{8}$$

Suppose $P \subset \hat{P}$. According to our definition a bidder with the set of priors $\hat{P}$ is more ambiguity averse then a bidder with the set of priors P. We assume that for all $v \in [r,1]$, there exist $p_v \in P_v$ and $\hat{p}_v \in \hat{P}_v$ such that,

$$\hat{p}_v(i)\, p_v(j) \geq \hat{p}_v(j)\, p_v(i) \text{ for } i > j. \tag{9}$$

This condition is very similar to the monotone likelihood ratio condition (see for example Milgrom [19]). Basically, it requires that for all bidders there is a minimizing distribution in $\hat{P}$ that is more pessimistic about the number of bidders than some minimizing distribution in P. Now, we are ready to state the next proposition.

Proposition 2. *Suppose $P \subset \hat{P}$, and assume that condition (9) holds. Suppose auctions A and $\hat{A}$ are first price auctions (with common reservation price r) and where bidders' beliefs about the number of bidders are given by P and $\hat{P}$ respectively. Then bids in auction $\hat{A}$ are at least as high as bids in auction A for all valuations. Moreover, bids in $\hat{A}$ are strictly higher for a positive measure of valuations if $P_v \cap \hat{P}_v = \emptyset$ for some $v \in [r,1]$.*

Note that, as is well known, condition (9) implies that $\hat{p}_v$ first order stochastically dominates p_v but the converse is not true. Moreover, first order stochastic domination is not enough for bidders to increase their bids. Also, to prove proposition 2, we only need to use condition (9). Even though this makes the proposition applicable outside the MMEU model, we restrict ourselves to the MMEU interpretation, since our main motivation is to study ambiguity aversion with respect to the number of bidders.

[8] This definition is not only quite intuitive but also has a rigorous foundation within the MMEU framework provided by a number of existing notions in the literature. For example, Ghirardato and Marinacci [6], Epstein [4], and Schmeidler [26] have each proposed conditions on preferences that they consider to reflect uncertainty aversion. Given MMEU, Ghirardato and Marinacci and Schmeidler approaches yield exactly the definition above. Epstein's approach does as well, under the additional assumption that if there were an environment with no uncertainty, decision maker would be an expected utility maximizer.

2.3. Comparing Revealing and Concealing Policies

2.3.1. Bidders' Perspective

Assume that the seller knows the number of active bidders, or as in Harstad, Kagel and Levin [7] can employ contingent bidding that would allow bidders to resolve such uncertainty. Thus, the seller has two available policies to consider: revealing the number of bidders or concealing it. We start by evaluating the two policies from the bidders' interim perspective.

Proposition 3. *Consider an active bidder who knows his valuation but not the number of active bidders. In the SPA the bidder is indifferent between the revealing and the concealing policies. In the FPA the bidder weakly prefers the revealing policy, and this preference is strict for a positive measure of valuations if and only if* $\cap_{v\in[r,1]}P_v = \emptyset$.

Remember that in the SPA, equilibrium bidding behavior is the same with or without ambiguity about the number of bidders. From this it immediately follows that bidders are indifferent between the revealing and the concealing policies in the SPA. The proof of the statement about the FPA is in the appendix. To see that bidders strictly prefer the revealing policy only if $\cap_{v\in[r,1]}P_v = \emptyset$, note that if this condition is violated then there exists some $p \in P_v$ for all $v \in [r, 1]$. In other words, one can find a common prior according to which *all* bidders compute their bids. This situation is then essentially the same as one with only risk and with risk neutral bidders. For such an auction environment McAfee and McMillan [17] and Matthews [15] have already shown that bidders are indifferent between the revealing and concealing policies. Sufficiency of the condition is also proved in the appendix.

Note that proposition 3 takes an interim point of view, but it implies that ex ante (i.e., before bidders learn their types), bidders strictly prefer the revealing policy if and only if $\cap_{v\in[r,1]}P_v = \emptyset$.

In example 2, we use the set up in example 1 to illustrate proposition 3.

Example 2. *Suppose that the number of active bidders is commonly known by all bidders. Given the specifications in example 1, the equilibrium bidding function when there are* n *active bidders is given by,*

$$s^n(v) = v - \frac{\int_0^v t^{n-1}dt}{v^{n-1}} \text{ for all } r \le v \le 1.$$

Thus, the maxmin expected utility of a bidder under the revealing policy is given by,

$$\begin{aligned} & \min_{p\in P}\sum_{n=1}^{4}\left[\left(v-s^{n}\left(v\right)\right)F^{n-1}\left(v\right)p\left(n\right)\right] \\ = \; & \min_{p\in P}\sum_{n=1}^{4}\left[\left(\int_{0}^{v}t^{n-1}dt\right)p\left(n\right)\right]=0.25v^{2}+0.125v^{4}. \end{aligned} \tag{10}$$

The maxmin expected utility of a bidder under the concealing policy can be computed by evaluating equation (7) in example 1 at $\hat{v}=v$:

$$\begin{aligned} & \min_{p\in P}\sum_{n=1}^{4}\left[\left(v-s^{\min}\left(v\right)\right)F^{n-1}\left(v\right)p\left(n\right)\right] \\ = \; & \left\{\begin{array}{c} 0.25v^{2}+0.125v^{4} \quad \textit{if } v\leq 1/3 \\ -0.0148+0.1v+0.05v^{2}+0.1v^{3}+0.125v^{4} \ \textit{if } v>1/3 \end{array}\right. . \end{aligned} \tag{11}$$

Inspecting equations (10) and (11) shows that all types of bidders weakly prefer to know the number of bidders (the revealing policy) and for those bidders with valuations larger than 1/3 this preference is strict.

2.3.2. Seller's Perspective

Next, we turn to the comparison of revealing and concealing policies from the seller's perspective. It is immediate that for the SPA the seller is indifferent between the two policies. In the following, we focus on the seller's preference between the revealing and concealing policies in the FPA. In the analysis, we assume that the seller maximizes expected revenue with respect to the distribution p^t, and that

$$p_v\left(i\right)p^t\left(j\right)\geq p_v\left(j\right)p^t\left(i\right) \text{ for } i>j \tag{12}$$

for all $v\in[r,1]$ whenever p_v exists. Condition (12) requires that the priors used in equilibrium of the FPA by the bidders (under the concealing policy) are more pessimistic than the seller's prior.

We use proposition 2, to see whether the seller prefers the revealing or concealing policy. To answer this question let's compare the following three cases: (i) active bidders do not know how many other bidders participate in the auction and they do not know the true distribution on the number of bidders, (ii) active bidders do not know how many other bidders participate in the auction but they do know the true distribution on the number of bidders, (iii) active bidders know the actual number of bidders. With linear utility functions, McAfee and McMillan show that the expected selling price is higher in case (ii) then it is in case (iii). By proposition 2 we know that bidders bid higher in case (i) than they bid in case (ii). So the selling price is even higher in case (i). Moreover again by proposition 2 the expected selling price is strictly higher if $p_v \neq p^t$ for some $v \in [r, 1]$. In this case the seller strictly prefers the concealing policy. We summarize this argument in the next proposition.

Proposition 4. *Under condition (12), in the first price auction, the seller prefers the concealing policy to the revealing policy. This preference is strict if $p_v \neq p^t$ for some $v \in [r, 1]$.*

Notice that proposition 4 indicates tension between bidder and seller preferences. Bidders' prefer to know how many bidders participate in the FPA. On the other hand, the seller prefers to conceal this information. In the risk approach bidders prefer the revealing policy if they have DARA utility functions. Our result that ambiguity aversion also leads to bidder preference for revealing makes this finding more robust. Similarly our seller result is also robust since seller prefers the concealing policy in either case.

Next, we give an example where the seller strictly prefers the concealing policy, and the bidders strictly prefer the revealing policy.

Example 3. *Consider P as in example 1. Let's assume that the seller's prior is given by p_3, i.e. $p^t = p_3$. As we show in example 1 bidders whose valuations are less than $1/3$ have beliefs p_2, the others have beliefs p_1. In example 2 we show that bidders weakly prefer the revealing policy, and those with valuations more than $1/3$ strictly prefer the revealing policy. Moreover, condition (12) is satisfied, so the seller strictly prefers the concealing policy.*

2.4. Comparing First and Second Price Auctions

Our final proposition combines the previous propositions to determine buyer and seller preferences over the FPA and the SPA, when bidders do not know the number of rivals.

Proposition 5. *Bidders prefer the second price auction to the first price auction and under condition (12) the seller prefers the first price auction to the second price auction.*

Proof. By the Revenue Equivalence Theorem when risk neutral bidders know the number of rivals, they are indifferent between the first and the second price auction. In the first price auction, they prefer to know the number of active bidders as outlined in proposition 3. In the second price auction, they are indifferent between the revealing and concealing policies. Therefore, a bidder prefers the second to the first price auction when the number of rivals is unknown.

Furthermore, we know that, in a second price auction, it is a weakly dominant strategy for the bidders to bid their valuations. We also know by the Revenue Equivalence

Theorem that, in the expected utility case with risk neutral bidders, when bidders and seller know the number of bidders, the seller is indifferent between a first and second price auction. In the first price auction with ambiguous number of bidders, under condition (12), active bidders bid uniformly higher relative to the case when they know the true distribution p^t. Therefore, the seller actually prefers the first price auction. ■

3. Discussion

3.1. Ambiguity about valuations

Auctions where bidders have ambiguity about other bidders valuations (rather than the number of bidders) has been studied by Lo [14], Salo and Weber [24], Volij [29] and Ozdenoren [22]. In the previous section we studied ambiguity regarding the number of bidders. Isolating such ambiguity from ambiguity regarding bidders' valuations is useful: It simplifies the analysis, and allows a clean comparison with the existing "risk" literature regarding uncertainty in the number of rival bidders (which assumes a single prior regarding valuations).

Moreover, introducing ambiguity only about the number of bidders may be appropriate when bidding is done in an established market that has just been opened to a larger pool of bidders. Such a case happened in Department of Defense auctions of surplus military equipment about ten years ago. Because the market had long been established, there was no uncertainty regarding the distribution of valuations. However, when the market was opened to internet bidding, the pool of bidders definitely changed and thus there may well have been uncertainty regarding the distribution of bidder participation[9].

There is another, perhaps more important, reason for treating the two kinds of ambiguity separately. The main results of this paper are robust under a formulation of the problem in which the bidders view the two sources of ambiguity as independent. The notion of independence that we use is in the spirit of the notion described in Gilboa and Schmeidler [5]. We relegate the formal analysis to subsection 4.3 of the appendix. We show that in this setting propositions 1- 4 hold essentially without modification. In proposition 5 we show that, when there is ambiguity about the number of bidders participating in the auction, bidders prefer the SPA to the FPA and under condition (12) the seller prefers the FPA to the SPA. When there is also ambiguity about valuations, this result holds under two additional assumptions. First, we assume that the seller is ambiguity neutral[10]. Second we assume that bidders are more pessimistic about the distribution of valuations than the seller is. Proof is in subsection 4.3.

[9] See Katzman, Mixon and Austin [12] for more information on the Department of Defense auctions. We thank an anonymous referee for pointing out this example.

[10] This assumption simplifies the analysis carried out in the appendix, but it can be relaxed.

3.2. Choquet Expected Utility

An alternative approach to MMEU in modelling ambiguity aversion is the Choquet expected utility (CEU) model as developed by Schmeidler [26]. In the CEU model expected utility is computed with respect to a nonadditive probability (or capacity) using the Choquet integral.

CEU preferences satisfy a property called comonotonic independence. An important implication of comonotonic independence is that comonotonic acts[11] are evaluated by taking the expectation with respect to the same distribution (while the utility of acts that are not comonotonic can be evaluated via different distribution functions). This property of the CEU model has an important implication in our framework. In equilibrium of the FPA (both with revealing and concealing policies) bidders face payoff profiles that are non-increasing in the number of bidders, and such profiles are comonotonic. Thus, bidders with CEU preferences use the same single distribution to evaluate all these acts, and therefore they must be indifferent between the revealing and concealing policies and between first- and second-price auctions in the CEU framework.[12]

In contrast, bidders with MMEU preferences satisfy a weaker property called C-independence. C-independence implies that acts whose utility profiles are affine transformations of one another are evaluated by the same distribution.[13] Therefore, in general acts that are monotone decreasing (or increasing)[14] may be evaluated with different distributions, allowing for interesting preferences in the FPA under which ambiguity aversion is not irrelevant.

Specifically, we have shown that when MMEU preferences bite (do not degenerate to the risk case) bidders strictly prefer the revealing policy to the concealing policy. This, seems to be a more natural result that is in the spirit of risk analysis (at least in the DARA case) while CEU is silent. It also shows that C-independence assumption may be more flexible than comonotonic independence assumption in applications where agents payoffs are monotonic in the underlying variable of interest, such as the FPA and the SPA where bidders payoffs are non-decreasing both in other bidders' valuations and number of bidders.

[11] Two acts are comonotonic if one act gives a more desirable outcome in one state compared to another state, then so does the other one.

[12] Formal definitions and the proof of this statement are relegated to the appendix.

[13] On the other hand, MMEU approach has an additional axiom, called Uncertainty Aversion (see Gilboa and Schmeidler[5]) not in the CEU model. Thus, in general, CEU preferences are not a subset of MMEU preferences.

[14] Note that two monotone decreasing (or increasing) acts are always comonotonic, but they may not be affine transformations of one another.

4. Appendix

4.1. Proof of Proposition 2

Let $s^{\min}$ and $\hat{s}^{\min}$ be the equilibrium bid functions in auctions A and $\hat{A}$ respectively. We first show that $\hat{s}^{\min}(v) \geq s^{\min}(v)$ for almost all $v \in [r, 1]$.

Let $G^{\min}(v) = \min_{p \in P} \sum_{n=1}^{m} p(n) F^{n-1}(v)$ and $\hat{G}^{\min}(v) = \min_{p \in \hat{P}} \sum_{n=1}^{m} p(n) F^{n-1}(v)$ for $v \in [0, 1]$. Both $G^{\min}$ and $\hat{G}^{\min}$ are differentiable almost everywhere. Moreover, if $G^{\min}$ is differentiable at some v, then its derivative is given by,

$$G^{\min\prime}(v) = \sum_{n=1}^{m} (n-1) p_v(n) F^{n-2}(v) F'(x)$$

where $p_v \in P_v$. Similarly,

$$\hat{G}^{\min\prime} = \sum_{n=1}^{m} (n-1) \hat{p}_v(n) F^{n-2}(v) F'(x)$$

where $\hat{p}_v \in \hat{P}_v$.

Next we show that,

$$\frac{\hat{G}^{\min\prime}(v)}{\hat{G}^{\min}(v)} \geq \frac{G^{\min\prime}(v)}{G^{\min}(v)} \text{ for almost all } v \in [r, 1]. \tag{13}$$

Fix some $v \in [r, 1]$. Suppose that p_v and $\hat{p}_v$ satisfy condition (9). Inequality in (13) holds iff,

$$\begin{aligned}
& \frac{\sum_{n=1}^{m} (n-1) \hat{p}_v(n) F^{n-2}(v) F'(x)}{\sum_{n=1}^{m} \hat{p}_v(n) F^{n-1}(v)} \geq \frac{\sum_{n=1}^{m} (n-1) p_v(n) F^{n-2}(v) F'(v)}{\sum_{n=1}^{m} p_v(n) F^{n-1}(v)} \\
\Leftrightarrow \quad & \left(\sum_{n=1}^{m} (n-1) \hat{p}_v(n) F^{n-1}(v) \right) \left(\sum_{n=1}^{m} p_v(n) F^{n-1}(v) \right) \\
\geq \quad & \left(\sum_{n=1}^{m} (n-1) p_v(n) F^{n-1}(v) \right) \left(\sum_{n=1}^{m} \hat{p}_v(n) F^{n-1}(v) \right) \\
\Leftrightarrow \quad & \sum_{n=1}^{m} \sum_{j=1}^{m} (n-1) \hat{p}_v(n) p_v(j) F^{n+j-2}(v) \geq \sum_{n=1}^{m} \sum_{j=1}^{m} (j-1) p_v(j) \hat{p}_v(n) F^{n+j-2}(v) \\
\Leftrightarrow \quad & \sum_{n=1}^{m} \sum_{j=1}^{m} (n-j) \hat{p}_v(n) p_v(j) F^{n+j-2}(v) \geq 0 \qquad (14)
\end{aligned}$$

Note that inequality (14) holds under condition (9). Ozdenoren [22] shows that inequality (13) holds for almost all $v \in [r, 1]$ if and only if $\hat{s}^{\min}(v) \geq s^{\min}(v)$ for almost all $v \in [r, 1]$. This concludes the first part of the proof.

Next, suppose that $P_v \cap \hat{P}_v = \emptyset$ for some $v \in [r, 1]$. Then condition (9) is strict for this v and for some $n > j$. This in turn implies that inequality (13) is strict for this v. By Ozdenoren [22] $\hat{s}^{\min}$ is strictly larger than $s^{\min}$ for a positive measure subset of $[r, 1]$ if (13) is strict for some $v \in [r, 1]$.

4.2. Proof of Proposition 3

Suppose that the number of active bidders is commonly known by all bidders. The equilibrium bid function when there are n active bidders is given by,

$$s^n(v) = v - \frac{\int_r^v F^{n-1}(t)\,dt}{F^{n-1}(v)} \text{ for all } r \leq v \leq 1.$$

Therefore, the maxmin expected utility of a bidder under the revealing policy is given by,

$$\min_{p \in P} \sum_{n=1}^{m} \left[(v - s^n(v)) F^{n-1}(v) p(n) \right].$$

The equilibrium bid of a bidder under the concealing policy is given by,

$$s^{\min}(v) = v - \frac{\int_r^v G^{\min}(t)\,dt}{G^{\min}(v)}$$

where $G^{\min}(x) = \min_{p \in P} \sum_{n=1}^{m} p(n) F^{n-1}(x)$. Therefore, the maxmin expected utility of a bidder under the concealing policy is given by,

$$\min_{p \in P} \sum_{n=1}^{m} \left[\left(v - s^{\min}(v)\right) F^{n-1}(v) p(n) \right].$$

Now note that,

$$\min_{p \in P} \sum_{n=1}^{m} \left[(v - s^n(v)) F^{n-1}(v) p(n) \right] = \min_{p \in P} \sum_{n=1}^{m} \left(\int_r^v F^{n-1}(t)\,dt \right) p(n)$$

$$= \min_{p \in P} \int_r^v \left(\sum_{n=1}^{m} F^{n-1}(t) p(n) \right) dt \geq \int_r^v \min_{p \in P} \left(\sum_{n=1}^{m} F^{n-1}(t) p(n) \right) dt$$

$$= \int_r^v G^{\min}(t)\,dt = \frac{\int_r^v G^{\min}(t)\,dt}{G^{\min}(v)} \min_{p \in P} \sum_{n=1}^{m} \left[F^{n-1}(v) p(n) \right]$$

$$= \min_{p \in P} \sum_{n=1}^{m} \left(v - s^{\min}(v)\right) F^{n-1}(v)\, p(n),$$

where the first and last equalities are obtained by plugging the expressions for of s^n and $s^{\min}$ respectively, the inequality is obtained by concavity of min and the rest of steps are evident. Observe that we just showed that all types of bidders weakly prefer the revealing policy in the FPA.

Now suppose that $\cap_{v \in [r,1]} P_v = \emptyset$. This means that for any $\hat{p} \in P$ there exists $v \in [r, 1]$ such that $\min_{p \in P} \sum_{n=1}^{m} F^{n-1}(v)\, p(n) < \sum_{n=1}^{m} F^{n-1}(v)\, \hat{p}(n)$. By continuity of F we can extend this inequality to a positive measure set around v. Therefore,

$$\int_r^1 \min_{p \in P} \left(\sum_{n=1}^{m} F^{n-1}(t)\, p(n) \right) dt < \min_{p \in P} \int_r^1 \left(\sum_{n=1}^{m} F^{n-1}(t)\, p(n) \right) dt. \tag{15}$$

To see this suppose that r.h.s. of the inequality in equation (15) is minimized by $\hat{p} \in P$. Given $\hat{p} \in P$, we have already argued that there is a positive measure set of valuations such that the integrand on the l.h.s. is strictly smaller than the integrand on the r.h.s. and it is weakly smaller otherwise. Thus we obtain the strict inequality.

Equation (15) shows that a bidder with valuation 1 strictly prefers the revealing policy. By continuity there exists $\varepsilon > 0$ such that all bidders with valuations larger than $1 - \varepsilon$ strictly prefer the revealing policy, completing the proof.

4.3. Ambiguity about valuations: Formal analysis

Suppose that, as before, P is the set of measures on the number of bidders. Now we introduce another set, Δ, a closed convex set of probability measures on Borel subsets of $[0, 1]$. This set represents the ambiguity about the measure generating valuations. We assume that each measure $\delta \in \Delta$ has a differentiable distribution function $F_\delta \equiv \delta([0, v])$. We denote the set of distribution functions corresponding to the measures in Δ by Δ^D.

Next, we construct a set of measures that represents jointly the ambiguity about the number of bidders and the highest valuation among rivals. We pick a measure $p \in P$ on the number of bidders, and a measure $\delta \in \Delta$ on valuations. From these two measures, we generate a joint distribution on the product space of the number of bidders and the highest valuation among the rivals bidders given the number of bidders. The set of all distributions constructed this way is generally not a closed, convex set. Since the set of priors in the MMEU model needs to be closed and convex, we take the closed convex hull of this set. As we shall see below, enlarging the set this way does not really matter, since the minimum is always attained within the smaller set.

Since, in this construction, for each $p \in P$, we consider all possible $\delta \in \Delta$ and vice versa, the two kinds of ambiguity do not interact. In other words, ambiguity about the valuations does not vary with the ambiguity about the number of bidders. This is the sense that the two kinds of ambiguity are treated independently.

We define a state space, $\Omega = \{(n, v) : n \in \{1, ..., m\}, v \in [0, 1]\}$ where n stands for the number of bidders and v stands for the highest valuation among the remaining bidders. Now, we construct a set of probability measures on this space, that represents the ambiguity about the distribution of the number of bidders, as well as, the ambiguity about the distribution of valuations.

Let Σ denote Borel measurable subsets of $[0,1]$. Let $\mathcal{M}$ denote all subsets of $\{1, ..., m\}$. The set of events are $\mathcal{M} \otimes \Sigma = \{N \times V : N \in \mathcal{M}, V \in \Sigma\}$.

Let δ_n^1 denote the probability that highest valuation among $n \geq 1$ bidders is v, where valuations are drawn independently from the distribution F_δ. That is,

$$\delta_n^1(v) = nF_\delta(v)^{n-1} F_\delta'(v)$$

for $v \in [0,1]$. For $n = 0$, measure δ_0^1 puts point mass on 0, that is $\delta_0^1(0) = 1$ and $\delta_0^1(v) = 0$ for $v \in (0,1]$. The probability that $v \in V$ for any $V \in \Sigma$ is,

$$\delta_n^1(V) = \int_V nF_\delta(v)^{n-1} F_\delta'(v)\,dv. \tag{16}$$

For $p \in P$ and $\delta \in \Delta$ define a probability measure π on $\mathcal{M} \otimes \Sigma$ by,

$$\pi(N \times V) = \sum_{n \in N} p(n)\,\delta_{n-1}^1(\{v \,|(n, v) \in N \times V\}) \tag{17}$$

where $N \times V \in \mathcal{M} \otimes \Sigma$. It is easy to verify that π is a probability measure. Define Π to be the set of all probability measures on $\mathcal{M} \otimes \Sigma$ generated as in equation (17) from some $p \in P$ and $\delta \in \Delta$. Let $\mathrm{co}\,\Pi$ be the convex hull of Π and $\overline{\mathrm{co}}\Pi$ be the closed and convex hull of Π. Now, suppose that bidders' preferences are given by MMEU where the set of priors is given by $\overline{\mathrm{co}}\Pi$ and the utility is linear.

Now, consider bidder i, in the FPA, with valuation v_i who bids $b_i \geq r$. Suppose that bidder i knows that all bidders are using increasing bidding functions, $s(\cdot)$. The utility of bidder i is then given by:

$$\begin{aligned} U_i(v_i, b_i, s, ..., s) &= \min_{\pi \in \overline{\mathrm{co}}\Pi} (v_i - b_i)\,\pi\left(\{1, ..., m\}, \left[0, s^{-1}(b_i)\right]\right) \\ &= \min_{\pi \in \overline{\mathrm{co}}\Pi} (v_i - b_i) \sum_{n=1}^{m} \pi\left(\{n\}, \left[0, s^{-1}(b_i)\right]\right) \\ &= \inf_{\pi \in \mathrm{co}\,\Pi} (v_i - b_i) \sum_{n=1}^{m} \pi\left(\{n\}, \left[0, s^{-1}(b_i)\right]\right). \end{aligned} \tag{18}$$

Now fix $\pi \in \operatorname{co}\Pi$. There exists a positive integer k, $\pi_i \in \Pi$, $\lambda_i \geq 0$ for $i = 1, ..., k$ such that $\sum_{i=1}^{k} \lambda_i = 1$ and

$$\pi\left(\{n\}, [0,v]\right) = \sum_{i=1}^{k} \lambda_i \pi_i \left(\{n\}, [0,v]\right).$$

Thus,

$$\begin{aligned} \sum_{n=1}^{m} \pi\left(\{n\}, [0,v]\right) &= \sum_{n=1}^{m} \sum_{i=1}^{k} \lambda_i \pi_i \left(\{n\}, [0,v]\right) = \sum_{i=1}^{k} \lambda_i \sum_{n=1}^{m} \pi_i \left(\{n\}, [0,v]\right) \\ &\geq \min_{\pi_i, i=1,\ldots,k} \sum_{n=1}^{m} \pi_i \left(\{n\}, [0,v]\right) \geq \inf_{\pi \in \Pi} \sum_{n=1}^{m} \pi\left(\{n\}, [0,v]\right). \end{aligned}$$

Since the above inequality holds for all $\pi \in \operatorname{co}\Pi$,

$$\inf_{\pi \in \operatorname{co}\Pi} \sum_{n=1}^{m} \pi\left(\{n\}, [0,v]\right) \geq \inf_{\pi \in \Pi} \sum_{n=1}^{m} \pi\left(\{n\}, [0,v]\right).$$

On the other hand, since $\operatorname{co}\Pi \supseteq \Pi$, the reverse inequality is also true. Thus,

$$\inf_{\pi \in \operatorname{co}\Pi} \sum_{n=1}^{m} \pi\left(\{n\}, [0,v]\right) = \inf_{\pi \in \Pi} \sum_{n=1}^{m} \pi\left(\{n\}, [0,v]\right). \tag{19}$$

From (18) and (19) we conclude that,

$$U_i(v_i, b_i, s, ..., s) = \inf_{\pi \in \Pi} (v_i - b_i) \sum_{n=1}^{m} \pi\left(\{n\}, \left[0, s^{-1}(b_i)\right]\right). \tag{20}$$

Since $\pi \in \Pi$ if and only if there exists $p \in P$ and $\delta \in \Delta$ such that equation (17) holds, we can write equation (20) as,

$$\begin{aligned} U_i(v_i, b_i, s, ..., s) &= \min_{p \in P, \delta \in \Delta} (v_i - b_i) \sum_{n=1}^{m} p(n)\, \delta\left(\left[0, s^{-1}(b_i)\right]\right)^{n-1} \\ &= \min_{p \in P, F \in \Delta^D} (v_i - b_i) \sum_{n=1}^{m} p(n)\, F\left(s^{-1}(b_i)\right)^{n-1} \end{aligned} \tag{21}$$

where inf is replaced with min since both P and Δ are closed and bounded.

Let $\tilde{F}(v) = \min_{F \in \Delta^D} F(v)$. Now, note that for any $v \in [0,1]$ and $p \in P$,

$$\min_{F \in \Delta^D} \sum_{n=1}^{m} p(n)\, F(v)^{n-1} = \sum_{n=1}^{m} p(n)\, \tilde{F}(v)^{n-1}. \tag{22}$$

In particular $\tilde{F}$ is not a function of p. Using equation (22), rewrite equation (21) as,

$$U_i(v_i, b_i, s, ..., s) = \min_{p \in P} (v_i - b_i) \sum_{n=1}^{m} p(n) \tilde{F}\left(s^{-1}(b_i)\right)^{n-1} \tag{23}$$

for all $v_i \geq r$. Note, that equation (3) is the same as equation (23), if we replace F by $\tilde{F}$. Thus, we can replicate our earlier analysis, by replacing F by $\tilde{F}$ everywhere, to obtain the following proposition.

Proposition 6. *An actual bidder who knows his valuation but not the actual number of bidders weakly prefers the revealing policy, and this preference is strict for a positive measure of valuations if and only if* $\cap_{v \in [r,1]} P_v = \emptyset$. *Propositions 2 and 4 hold without modification. In particular, under condition (12), the seller prefers the concealing policy to the revealing policy even when there is ambiguity about the valuations.*

Proofs are omitted since they are identical to proofs of earlier results.

In proposition 5 we show that, when there is ambiguity about the number of bidders participating in the auction, bidders prefer the SPA to the FPA and under condition (12) the seller prefers the FPA to the SPA. When in addition there is ambiguity about valuations, this result holds under two additional assumptions. First, we assume that the seller is ambiguity neutral with distribution $F_t \in \Delta$ with positive density. The assumption that the seller is ambiguity neutral is not essential and is made just to simplify the analysis. Second we assume that $\tilde{F}$ has positive density and

$$\frac{\tilde{F}'(v)}{\tilde{F}(v)} \geq \frac{F_t'(v)}{F_t(v)} \text{ for all } v \in (0,1]. \tag{24}$$

This assumption means that bidders are more pessimistic about the distribution of valuations under $\tilde{F}$ than under the seller's belief F_t.

Proposition 7. *Under the previous two assumptions, bidders prefer the SPA to the FPA. Moreover, as before, under condition (12) the seller prefers the FPA to the SPA.*

Proof. To see this note that when the bidders and seller are both ambiguity neutral with beliefs represented by distribution F_t then both the bidders and the seller are indifferent between the FPA and the SPA. We know from Ozdenoren [22] that when bidders are ambiguity averse about the valuations of other bidders, they bid (weakly) more in the FPA if and only if (24) holds. On the other hand, in the SPA, for any bidder it is a weakly dominant strategy to bid his valuation. Therefore, bidders prefer the SPA to the FPA when there is ambiguity about valuations. When there is ambiguity about the number of bidders participating in the auction, bidders prefer the revealing policy in the FPA and they are indifferent between the revealing and concealing policies, and therefore in an environment where the number of bidders is concealed bidders preference for the SPA is even stronger. Seller prefers the FPA since from Ozdenoren [22] and proposition 2, it follows that when (24) and (12) both hold bidders bid more in the FPA. ■

4.4. Definitions and proofs omitted from section 3.2

We start this section by formally describing the CEU representation.

Definition 1. *A* capacity ρ *on* $\{1,...,m\}$ *is a set function that is monotone, i.e. if for* $A, B \subset \{1,...,m\}$, $A \subseteq B$ *then* $\rho(A) \leq \rho(B)$, *and satisfies* $\rho(\emptyset) = 0$ *and* $\rho(\{1,...,m\}) = 1$.

Suppose the decision maker's Von Neumann-Mongenstern utility function is given by $u : \mathbb{R} \to \mathbb{R}$ and prior on $\{1,...,m\}$ is given by a capacity ρ. According to CEU, the utility of the decision maker from a nonnegative act $\pi : \{1,...,m\} \to \mathbb{R}$ is,

$$U(\pi) = \int_0^\infty \rho(\{n \in \{1,...,m\} : u(\pi(n)) \geq \alpha\}) \, d\alpha. \tag{25}$$

Just like in the earlier analysis, using standard arguments we can show that it is dominant strategy for bidders with CEU preferences who do not know the number of bidders to bid their valuations in the SPA. Therefore, below we look at bidding behavior in the FPA.

Suppose the number of participating bidders is $n \geq 1$ and without loss of generality index them by $j = 1,...n$. Let $G_\rho(v) = \sum_{n=1}^m \left(F^{n-1}(v) - F^n(v)\right) \rho(\{i : i \leq n\})$. As before, suppose $s_j : [0,1] \to [0,\infty)$ is the strategy of bidder $j \in \{1,...,m\}$. Let, $z_n^i = \max\{s_j(v_j) \,|\, j \neq i, j = 1,...,n\}$ if $n \geq 2$ and 0 if $n = 1$. Let $H(\cdot\,|n)$ be the distribution function of z_n^i. Let $H_\rho(\cdot) = \sum_{n=1}^m (H(\cdot\,|n) - H(\cdot\,|n+1)) \rho(\{i : i \leq n\})$.

Let $U_i\left(v_i, b_i, \{s_j\}_{j \neq i}\right)$ denote bidder i's utility in the first price auction if his valuation is v_i, his bid is b_i and the strategies of the other bidders are given by $\{s_j\}_{j\neq i}$. Using the formula for Choquet expected utility given in (25), we can express bidder i's utility in the FPA as,

$$\begin{aligned} U_i(v_i, b_1, \{s_j\}_{j\neq i}) &= \begin{cases} \sum_{n=1}^m (v_i - b_i)(H(b_i\,|n) - H(b_i\,|n))\rho(\{i : i \leq n\}) & \text{if } b_i \geq r \\ 0 & \text{if } b_i < r \end{cases} \\ &= \begin{cases} (v_i - b_i) H_\rho(b_i) & \text{if } b_i \geq r \\ 0 & \text{if } b_i < r \end{cases}. \end{aligned}$$

Next, we look for a symmetric equilibrium of the FPA. Suppose that bidder i knows that all bidders are using symmetric, increasing and differentiable bidding functions, $s(\cdot)$. Now, $H_\rho(b_i) = G_\rho\left(s^{-1}(b_i)\right)$,and if bidder i bids $b_i \geq r$, he has utility,

$$U_i(v_i, b_i, s, \ldots, s) = (v_i - b_i)\, G_\rho\left(s^{-1}(b_i)\right).$$

Using arguments that are very similar to the ones in proposition 1, we can now show that in the FPA the unique symmetric equilibrium bidding function is given by,

$$s_\rho(v) = v - \frac{\int_r^v G_\rho(t)\, dt}{G_\rho(v)} \text{ for all } v \in [r, 1].$$

Proposition 8. *Bidders with CEU preferences are indifferent between the revealing and the concealing policies in the FPA and the SPA.*

Proof. This result follows immediately for the SPA, since bidders bid their valuations under both policies. Next we prove the result for the FPA. Define $\pi(n) = (v - s^n(v))\, F^{n-1}(v)$ for $n \in \{1, ..., m\}$ and $\pi(m+1) = 0$. Using the formula for Choquet expected utility given in (25), and noting that π is non-increasing, we can compute the utility of a bidder under the revealing policy as,

$$\begin{aligned}
& \int_0^\infty \rho(\{n \in \{1, ..., m\} : \pi(n) \geq \alpha\})\, d\alpha \\
= & \sum_{n=1}^{m} (\pi(n) - \pi(n+1))\, \rho(\{i : i \leq n\}) \\
= & \sum_{n=1}^{m} \pi(n)\, (\rho(\{i : i \leq n\}) - \rho(\{i : i \leq n-1\})) \\
= & \sum_{n=1}^{m} \left(\int_r^v F^{n-1}(t)\, dt\right) (\rho(\{i : i \leq n\}) - \rho(\{i : i \leq n-1\})). \qquad (26)
\end{aligned}$$

Define $\hat{\pi}(n) = (v - s_\rho(v)) F^{n-1}(v)$. Again using (25), noting that $\hat{\pi}$ is non-increasing, we can compute the utility of a bidder under the concealing policy as,

$$\begin{aligned}
& \int_0^\infty \rho(\{n \in \{1, ..., m\} : \hat{\pi}(n) \geq \alpha\}) \, d\alpha \\
= & \sum_{n=1}^{m} (\hat{\pi}(n) - \hat{\pi}(n+1)) \rho(\{i : i \leq n\}) \\
= & \sum_{n=1}^{m} \hat{\pi}(n) (\rho(\{i : i \leq n\}) - \rho(\{i : i \leq n-1\})) \\
= & \sum_{n=1}^{m} \left(\frac{\int_r^v G_\rho(t) \, dt}{G_\rho(v)} F^{n-1}(v) \right) (\rho(\{i : i \leq n\}) - \rho(\{i : i \leq n-1\})) \\
= & \int_r^v \sum_{n=1}^{m} \left(F^{n-1}(t) \left[\rho(\{i : i \leq n\}) - \rho(\{i : i \leq n-1\}) \right] \right) dt \\
= & \sum_{n=1}^{m} \left(\int_r^v F^{n-1}(t) \, dt \right) \left[\rho(\{i : i \leq n\}) - \rho(\{i : i \leq n-1\}) \right]. \qquad (27)
\end{aligned}$$

Comparing equations (26) and (27), we see that bidders are indifferent between the revealing and the concealing policies in the FPA. ■

Proposition 9. *Bidders with CEU preferences are indifferent between the FPA and the SPA.*

Proof. Risk neutral bidders who know the number of bidders are indifferent between the FPA and the SPA. By proposition 8 bidders are also indifferent between knowing or not knowing the number of bidders both in the FPA and in the SPA. Therefore bidders are indifferent between the FPA and the SPA even when they do not know the number of bidders. ■

References

[1] C. Camerer and M. Weber, Recent developments in modelling preference: uncertainty and ambiguity. J. Risk Uncertainty 5 (1992), pp. 325-70.

[2] R. Casadesus-Masanell, P. Klibanoff and E. Ozdenoren, Maxmin expected utility over Savage acts with a set of priors. J. Econ. Theory 92 (2000), pp. 35–65.

[3] D. Ellsberg, Risk, ambiguity, and the Savage axioms. Quart. J. Econ. 75 (1961), pp. 643-69.

[4] L. Epstein, A definition of uncertainty aversion. Rev. Econ. Stud. 66 (1999), pp. 579-608.

[5] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior. J. Math. Econ. 18 (1989), pp. 141-53.

[6] P. Ghirardato and M. Marinacci, Ambiguity made precise: a comparative foundation. J. Econ. Theory 102 (2002), pp. 251–289.

[7] R. M. Harstad, J. H. Kagel and D. Levin, Equilibrium bid function for auctions with uncertain number of bidders. Econ. Letters 33 (1990), pp. 35-40.

[8] E. Karni, On the equivalence between descending bid auctions and first price sealed bid auctions. Theory Dec. 25 (1988), pp. 211-217.

[9] E. Karni and Z. Safra, Vickrey auctions in the theory of expected utility with rank dependent probabilities. Econ. Letters, 20 (1986), pp. 15-18.

[10] E. Karni and Z. Safra, Ascending bid auctions with behaviorally consistent bidders. Annals of Operations Research, 19 (1989), pp. 435-446.

[11] E. Karni and Z. Safra, Dynamic consistency, revelation in auctions and the structure of preferences. Rev. Econ. Stud. 56 (1989), pp. 421-434.

[12] B. Katzman, S. Mixon and A. Austin, Optimal reserve prices in military surplus auctions. Kennesaw State University, mimeo, 2003.

[13] D. Levin and J. L. Smith, Equilibrium in auctions with entry. Amer. Econ. Rev. 84 (1994), pp. 585-99.

[14] K.C. Lo, Sealed bid auctions with uncertainty averse bidders. Econ. Theory, 12 (1998), pp. 1-20.

[15] S.A. Matthews, Comparing auctions for risk averse buyers: A buyer's point of view. Econometrica 55 (1987), pp. 633-46.

[16] E. Maskin and J. Riley, Uniqueness in sealed high bid auctions. Harvard University and UCLA Working Paper, 1996.

[17] R. P. McAfee and J. McMillan, Auctions with a stochastic number of bidders. J. Econ. Theory 43 (1987), pp. 1-19.

[18] R. P. McAfee and J. McMillan, Auctions with entry. Econ. Letters 23 (1987), pp. 343-47.

[19] P.R. Milgrom, Good news and bad news: Representation theorems and applications. Bell J. Econ. 12 (1981), pp. 380-391.

[20] P. R. Milgrom and R. J. Weber, A theory of auctions and competitive bidding. Econometrica 50 (1982), pp. 1089-1122.

[21] R. B. Myerson, Optimal auction design. Mathematics of Operations Research 6 (1981), pp. 58-73.

[22] E. Ozdenoren, Auctions and bargaining with a set of priors. University of Michigan, mimeo, 2001.

[23] M. H. Rothkopf, T. J. Teisberg and E. P. Kahn, Why are Vickrey auctions rare? J. Polit. Economy 98 (1990), pp. 94-109.

[24] A. Salo and M. Weber, Ambiguity aversion in first-price sealed bid auctions. J. Risk Uncertainty 11 (1995), pp. 123-37.

[25] W. F. Samuelson, Competitive bidding with entry costs. Econ. Letters 17 (1985), pp. 53-57.

[26] D. Schmeidler, Subjective probability and expected utility without additivity. Econometrica 57 (1989), pp. 571-87.

[27] J. L. Smith and D. Levin, Ranking auctions with risk averse bidders. J. Econ. Theory 68 (1996), pp. 549-61.

[28] W. Vickrey, Counterspeculation, auctions and competitive sealed tenders. J. Finance 16 (1961), pp. 8–37.

[29] O. Volij, Payoff equivalence in sealed bid auctions and the dual theory of choice under risk. Econ. Letters 76 (2002), pp. 231-237.

[30] R. Wilson, A bidding model of perfect competition. Rev. Econ. Stud. 44 (1997), pp. 511-18.