Nonlinear Pricing with Competition

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Abstract

This paper studies how increased competition affects nonlinear pricing, in particular the number of contracts offered by firms. We present a model with both horizontally and vertically differentiated products, with the set of consumers served in the market being endogenously determined. Though firms are only able to sort consumers in the vertical dimension, horizontal differentiation affects screening in the vertical dimension. Using a two-phase optimal control technique, we characterize the symmetric equilibrium menu of contracts under different market structures. When the market structure moves from monopoly to duopoly, we show that each firm offers more contracts (serving more types of consumers) and quality distortions decrease. As the market structure becomes more competitive (when the number of firms increases further), the effect of increasing competition exhibits some non-monotonic features: when the initial competition is not too weak, a further increase in the number of firms will lead to more contracts being offered and a reduction in quality distortions; when the initial competition is weak, an increase in the number of firms will lead to fewer contracts being offered, though the effect on quality distortions is not uniform. Our predictions are largely consistent with some empirical studies.

Key words: Nonlinear pricing, product differentiation, contract menus, quality distortions

JEL: D40, D82, L10

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1 Introduction

As more Japanese car makers enter the US market, will GM or Ford offer more models targeting at different types of consumers? As more competitors enter the cellular phone market, will Verizon or Sprint offer more calling plans? A number of empirical studies suggest that as competition becomes more intense, each firm often offers more variety of goods or services. For example, in the face of increased competition, American Express introduced 12 to 15 new credit cards per year targeted at different customer segments (Forbes, July 1, 1996, “The Battle of Credit Cards”). Similar effects of competition on the variety of contracts/services is observed in the airline industry as well. Borenstein and Rose (1994) find that on routes with more competition, each airline offers more variety of air tickets. In the automotive industry, in response to the increased competition from foreign companies in 1980s, GM and Ford began to offer more variety of car models.

All these cases suggest that increased competition leads to more contracts offered by each individual firm. Since the work of Mussa and Rosen (1978) and Maskin and Riley (1984) on monopolistic nonlinear pricing, there is a growing literature on nonlinear pricing in competitive settings, see, for example, Spulber (1989), Champsaur and Rochet (1989), Wilson (1993), Stole (1995), Villas-Boas and Schmidt-Mohr (1999), Armstrong and Vickers (2001), and Rochet and Stole (1997, 2002). However, much remains to be done in understanding how increased competition affects firms’ nonlinear pricing strategies. In this paper, we focus on the effects of increased (horizontal) competition on the number of (vertical) contracts offered by each firm.

Specifically, we consider a market with both vertically and horizontally differentiated products. Consumers’ preferences over these products differ in two dimensions. In the horizontal dimension, consumers have different tastes over different brands; while in the vertical dimension consumers have different marginal utilities over quality. Thus each consumer is characterized by a horizontal type and a vertical type. Although neither type is observable to firms, in our model the single crossing property is only satisfied in the vertical dimension. As a result firms can only offer contracts to sort consumers with respect to their vertical types.1

Our base model includes both the monopoly and duopoly cases. In the duopoly case, there are two horizontally differentiated brands owned and operated by two separate firms. In the monopoly case, we assume that all the modeling elements are the same as in the duopoly case, except that the two brands are owned and operated by a single firm, the monopolist. This particular way of modeling

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1 For this reason our paper does not belong to the multi-dimensional screening literature (e.g., Laffont, Maskin and Rochet, 1987; McAfee and McMillan, 1988; Armstrong, 1996; and Rochet and Chone, 1998).
provides a well-controlled benchmark, with which the difference in market structures becomes the
only difference between the duopoly case and the monopoly case.

The key of our analysis comes from the interaction between horizontal differentiation and screening on the vertical dimension. Unlike in one dimensional screening models in which the individual rationality (IR) condition only binds for a single vertical type, in our model, the IR conditions bind for a continuum of vertical types due to the horizontal differentiation. Given the menus of contracts offered by the brands (firms), the IR conditions determine the market share for each brand (firm), which in turn affects the profitability of the contracts offered. Therefore, although horizontal differentiation does not have direct impact on the incentive compatibility (IC) conditions in the vertical dimension, it affects the IC conditions through the rent provisions to consumers.\(^2\) This interaction in turn affects the menu of contracts offered by each brand (firm). It is through this interaction that we identify the effect of increasing (horizontal) competition on the menu of contracts offered by each firm.

Our problem can be formulated as an optimal control problem, with consumers’ vertical types as the “time” variable, quality provision as the control variable, and rent provision as the state variable. Because consumers of high types enjoy higher informational rents, the market shares are increasing in vertical type. In fact, the space of consumers’ vertical types is partitioned into two ranges: the full coverage or competition range, and the partial coverage or local monopoly range. It turns out that when moving from the partial coverage range to the full coverage range, the objective function in our optimal control program fails to be differentiable in the state variable. The implication is that our problem cannot be tackled by the ordinary optimal control theory (e.g., Kamien and Schwartz, 1992). Following the work of Amit (1986) on a two-phase optimal control technique, we formulate a similar problem but allow for constraints on the optimal switching “time” and derive a set of necessary conditions which can be applied in all our cases.

Since firms possess equal production technologies, we focus on symmetric equilibria in which each firm offers the same menu of contracts. Applying the two-phase optimal control technique, we characterize the symmetric equilibrium menu of contracts for both monopoly and duopoly.\(^3\) In either

\(^2\)As is standard in the screening literature, any IC contract can be represented by a rent provision schedule, which governs the utilities of consumers in equilibrium.

\(^3\)The nonlinear pricing game should be formulated as a two-stage sequential game, with the consumers’ move following that of the firm(s). To simplify the analysis, we work with the reduced strategic game in which the consumers’ move is replaced by the correlated market shares (as functions of the menus of contracts offered by the firm(s)). With such simplification, it would be more consistent to call the equilibrium menu of contracts in the monopoly case the
case, the equilibrium menu of contracts is unique, and a positive measure of consumers are excluded from the market. Specifically, there is a participation threshold such that only the consumers with vertical types above this threshold are served by the market. Moreover, the equilibrium contracts in both cases exhibit perfect sorting. Thus, the equilibrium threshold can serve as a measure of the number of contracts offered by each firm: a lower threshold indicates more contracts targeting more (vertical) types of consumers, and a higher threshold indicates fewer contracts targeting fewer (vertical) types of consumers in equilibrium.

Despite the absence of a closed-form solution in the duopoly case, we are able to compare its equilibrium with the optimal menu of contracts in the monopoly benchmark. Specifically, we show that compared to the monopoly benchmark, more contracts are offered in the duopoly, and quality distortions decrease. This unambiguous ranking result is due to the interaction between horizontal competition and vertical screening. By moving from monopoly to duopoly, consumers in the full coverage range will necessarily enjoy more rent due to the competition between the two firms. The IC constraint (the screening condition) thus implies that either the participation threshold should be lowered (more contracts offered), or the quality provision schedule should be moved upward (less quality distortions).\(^4\) In other words, the competition in duopoly increases the rent provisions for higher type consumers, which relaxes the screening condition in the vertical dimension, leading to more contracts offered to consumers who were previously excluded, and a reduction in quality distortions.

Our comparison between monopoly and duopoly also has implications about which market structure offers more contracts over any given quality interval. Our finding is that equally dense contracts are offered over the lower quality range in both market structures, but contracts offered over the higher quality range become denser moving from monopoly to duopoly.\(^5\) Based on a very different model, Johnson and Myatt (2003) show that an incumbent may respond to entry by either expanding (fighting brand) or contracting (pruning) the product line (the quality range).\(^6\) Thus our result is quite different from that of Johnson and Myatt: in their model, introducing competition only has an

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\(^4\) In equilibrium, both occur so each firm in the duopoly offers more contracts and quality distortions decrease.

\(^5\) If the number of contracts offered over a quality interval increases, we say that the contracts offered over that quality interval become denser.

\(^6\) Johnson and Myatt (2003) consider quantity competition between an incumbent and an entrant in a market with vertically differentiated products. In Johnson and Myatt (2005), they extend similar analysis to the case with multiple firms.
effect on the lower end of the product line, while in our model, moving from monopoly to duopoly has the most effect on the higher end of the quality range as more (or denser) contracts are offered in this range.

We also study how the degree of horizontal differentiation affects the equilibrium menu of contracts. It turns out that the effects under the two market structures are quite different. Under monopoly, as the two brands become less differentiated, each brand offers fewer contracts, and quality distortions become larger. The effects in the duopoly case are subtle. When the degree of horizontal differentiation (captured by parameter $k$ in our model) is smaller than some cutoff value, a decrease in $k$ results in more contracts offered and smaller quality distortions; when $k$ is larger than the cutoff value, a decrease in $k$ results in fewer contracts offered, though the effect on quality distortions is not uniform.\footnote{Roughly speaking, it results in larger quality distortions for lower types and smaller quality distortions for higher types.} Again these results are driven by the interplay between the horizontal differentiation and screening on the vertical dimension. In the monopoly case, a decrease in $k$ implies that if the rent provisions remain constant, the market shares in the partial coverage and full coverage ranges both expand, which creates room for the monopolist to reduce the rent provisions to the consumers. This can be achieved by raising the participation threshold and increasing quality distortions. In the duopoly, the degree of horizontal differentiation can also be interpreted in terms of the intensity of competition between the two firms: the lower the $k$, the more intense the competition. Now as $k$ decreases, the effect on the equilibrium rent provisions over the partial coverage range is still the same as that in the monopoly counterpart. However, the effect on the full coverage range is different. As $k$ decreases, the competition between the two firms on the competition range becomes more intense, which implies that the rent provisions for the consumers served over this range should be higher. Again by the screening condition, this can be achieved by lowering the participation threshold and decreasing quality distortions. Thus, in duopoly, the effects over the two ranges are exactly the opposite, with the net effect depending on which effect dominates. When the initial competition is not too weak, the effect over the competition range dominates, and when the initial competition is weak, the effect over the local monopoly range dominates.

We extend our analysis of the duopoly model to any finite $n$-firm case, and demonstrate that the analysis can be translated into that of the duopoly model by normalizing the degree of horizontal differentiation with the number of firms. In terms of this normalized degree of horizontal differentiation, we show that an increase in the number of firms is equivalent to a decrease in $k$ in the duopoly.
model. Hence all our results obtained in the duopoly case apply to the $n$-firm case. We thus conclude that when the initial competition level is not too low ($n$ is large), an increase in the number of firms results in more contracts offered by each firm and smaller quality distortions; while when the initial competition level is low ($n$ is small), an increase in the number of firms results in fewer contracts offered by each firm, and larger quality distortions for low types and smaller quality distortions for high types.

Our results are largely consistent with some existing empirical studies. In particular, the non-monotonic relationship between the number of firms and the number of contracts offered is consistent with a recent empirical study by Seim and Viard (2004), who test the effect of entry on the tariff choices (the number of contracts) of incumbent firms in the US cellular industry. We postpone a discussion of their study to the end of section 6.

As in our approach, a number of papers also study nonlinear pricing in competitive settings with both horizontally and vertically differentiated products (e.g., Gilbert and Matutes, 1993; Stole, 1995; Verboven, 1999; Villas-Boas and Schmidt-Mohr, 1999; and Ellison, 2005). However, these papers assume that all consumer types in the vertical dimension are served in the market. This full market coverage assumption does greatly simplify their analysis, but precludes the effect of competition on the number (measure) of contracts offered on vertical dimension, which is central to our analysis. Except for Stole (1995), in all the other papers mentioned above firms are only able to produce two exogenously given qualities.\footnote{In Ellison (2005), there are only two types of vertically differentiated consumers. Stole (1995) assumes that either the consumers’ horizontal types or the consumers’ vertical types are observable. Thus the impacts of horizontal differentiation on screening in the vertical dimension in his model are very different from those in our model.}

Our paper is most closely related to Rochet and Stole (1997, 2002), who consider both vertically and horizontally differentiated products with both quality and consumer types being continuous. Their framework is presumably more general, as they allow for general distributions about consumers’ types. However, there is a key difference between their model and ours. That is, in their settings, the lowest vertical type of consumers being served in the market is exogenously given, which also precludes the effect of increased competition on the consumer coverage in the vertical dimension, which is the focus of this current paper.

By focusing on equilibrium characterizations, Rochet and Stole (1997, 2002) show that in monopoly case, either bunching occurs at a lower interval or perfect sorting occurs with efficient quality provision for the lowest (vertical) type. However, in our analysis we show that bunching never occurs, and
the quality distortion is maximum for the lowest type. We demonstrate that this sharp difference is exactly due to the difference in our modelings: by endogenizing the set of consumers covered in the vertical dimension, we end up with different boundary conditions, which lead to a unique solution with perfect sorting in our case. Their result from the duopoly analysis is also quite different from ours. For example, in the case of full consumer coverage (in both horizontal and vertical dimensions), they show that each firm offers a cost-plus-fee pricing schedule, with each (vertical) type consumer receiving the efficient quality (a similar result obtained in Armstrong and Vickers (2001)), while in our case, efficient quality provision only occurs for the highest (vertical) type.

The paper is organized as follows. Section 2 introduces the base model with two brands. Section 3 lays down necessary preliminaries for our analysis. Section 4 derives the optimal symmetric menu of contracts under monopoly. Section 5 characterizes the symmetric equilibrium in the duopoly model, and investigates how the equilibrium menu of contracts changes as the market structure moves from monopoly to duopoly. We extend our analysis to the arbitrary n-firm case in Section 6, where we investigate the effects of further increased competition on the menu of contracts offered by each firm. Section 7 concludes.

2 The Model

To capture the effect of intensified competition on nonlinear pricing we consider a model in which consumers’ preferences differ both in the horizontal and vertical dimensions. Our basic model studies the two-brand case under both the duopoly and monopoly market structures. In Section 6 we extend our analysis to any finite n-brand (firm) case.

Under duopoly, two firms own two distinct brands, brand 1 and brand 2, respectively, and each firm (brand) offers a variety of vertically differentiated products. Specifically, each firm (brand) offers goods of different qualities, which are indexed by $q$, $q \in R^+$. Quality $q$ is both observable and contractible.

There are a continuum of consumers in the market, whose preferences differ on two dimensions: the “taste” dimension over the brands and the “quality” dimension over the quality level. We model

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9 For the duopoly case, Rochet and Stole (2002) focus on either competitive regime or monopoly regime (in terms of consumer coverage in horizontal dimension). The mixed regime with both regimes present is analyzed in Rochet and Stole (1997).

10 Throughout “quality” should be interpreted as a summary measure for a variety of product characteristics, such as the safety, reliability, and durability etc.
the taste dimension as the horizontal “location” of a consumer on a unit circle representing the ideal brand for that consumer.\textsuperscript{11} As depicted in Figure 1 below, the locations of brand 1 and brand 2 evenly split the unit circle, and the distance from a consumer’s location to brand $i$’s location is proportionate to the loss in this consumer’s utility from consuming a product of this brand that is less than ideal. Let $d_i$ be the distance between a consumer’s location and brand $i$’s location, then $d_i$ is this consumer’s horizontal type, $i = 1, 2$. Because $d_1 + d_2 = 1/2$, either $d_1$ or $d_2$ alone fully captures a consumer’s preference over two brands.

![Figure 1: A Two-Brand Base Model](image)

Consumers’ varying preferences over the quality dimension are captured by $\theta$, $\theta \in [0, 1]$, which we call a consumer’s vertical type. Consumers of different vertical types have different marginal utilities over an increase of quality. A consumer is thus characterized by a two-dimensional type $(d_i, \theta)$ (either $i = 1$ or $i = 2$). Neither $\theta$ or $d_i$ is observable to either firm. We assume that consumers are continuously and uniformly located along the unit circle, and the vertical types of consumers at each location are distributed uniformly over the unit interval: $\theta \sim U[0, 1]$. A consumer’s horizontal location and vertical type are independent.

Each consumer demands at most one unit of a good. If a type-$(d_i, \theta)$ consumer purchases one unit of the brand-$i$ product with quality $q$ at price $t$, her utility is given by

$$u(q, t, d_i, \theta) = \theta q - t - kd_i$$

\textsuperscript{11}For two brands, it would be sufficient to use a unit interval. We work with a unit circle since doing so will make it easier to extend our model to the arbitrary $n$-brand or $n$-firm case later.
where $k, k > 0$, can be interpreted as the per unit “transportation” cost, which is a measure of the degree of horizontal differentiation between the two brands. The smaller the $k$, the less horizontally differentiated the two brands are.\footnote{If $k = 0$, our model collapses to the one with vertical differentiations only.} If a consumer purchases no product, her reservation utility is normalized to be 0.

We assume that the two brands (firms) have the same production technology. Specifically, to produce a unit of quality $q$ product a firm incurs a cost $c(q) = q^2/2$.\footnote{The quadratic functional form assumed here is not crucial. What is needed in our analysis is that the cost function should be convex.} Thus, each firm (brand) has a per-customer profit function given by

$$\pi(t, q) = t - q^2/2.$$ (2)

Each firm offers a menu of contracts, which is a collection of all the quality and price pairs. Given the menus of contracts offered by both firms (brands), consumers decide whether to make a purchase, and if they do, which brand to choose and which contract to accept. It is well known that in the environment of competitive nonlinear pricing, it is no longer without loss of generality to restrict attention to direct contracts.\footnote{As demonstrated in a series of examples in Martimort and Stole (1997) and Peck (1997), equilibrium outcomes in indirect mechanisms may not be supported when sellers are restricted to using direct mechanisms where buyers report only their private types. Moreover, as demonstrated by Martimort and Stole (1997), an equilibrium in such direct mechanisms may not be robust to the possibility that sellers might deviate to more complicated mechanisms. The reason for such failures, as pointed out by McAfee (1993) and Katz (1991), is that in competition with nonlinear pricing the offers made by other firms may also be private information of the consumers when they make their purchase decisions, which means that this private information can also potentially be used when firms set up their revelation mechanisms.} To sidestep this problem, as in Rochet and Stole (2002) we restrict attention to deterministic contracts.\footnote{See Rochet and Stole (2002) for a discussion on the restrictions resulted from focusing on deterministic contracts. More general approaches in restoring the “without loss of generality” implication of the revelation principle in the environment of competitive nonlinear pricing have been proposed and developed by, for example, Epstein and Peters (1999), Peters (2001), and Page and Monteiro (2003).} Since the preferences of a consumer with vertical type $\theta$ over the available price-quality pairs conditional on purchasing from a firm (brand) are independent of her horizontal type $d_i$, it is without loss of generality to consider direct contracts of the form \(\{q(\theta), t(\theta)\}_{\theta \in [0,1]}\) in what follows. More specifically, a menu of (direct) contracts is a pair of quality and pricing schedules \((q(\cdot), t(\cdot))\), mapping each reported vertical type to a pair of quality provision
and price. For brevity of exposition, from now on we will often refer to vertical types as the types, especially when there is no confusion in the context.

Our solution concept is Bertrand-Nash equilibrium: given the other firm’s menu of contracts, each firm maximizes its expected total profit by choosing its menu of contracts.

This basically completes a description of the duopoly model. For the monopoly model, our main goal is to lay down a benchmark with which we can identify the effect of competition on the menu of contracts. As such in the monopoly model we need to control for all but the market structure. We thus assume that in the monopoly case, all the modeling elements are the same as in the duopoly model, except that the two brands are now owned and operated by the same firm, which is the monopolist.\(^{16}\) The monopolist’s objective is to maximize the total profits from the two brands by choosing the menu of contracts for each brand.

3 Preliminaries

As an analytical benchmark, given (1) and (2), the first-best (efficient) quality provision is \(q^*(\theta) = \theta\). That is, in the efficient menu of contracts the quality provision should be such that the marginal cost of production \((q)\) equals the marginal benefit of consumption \((\theta)\). We can thus define \(\theta - q(\theta)\) as the quality distortion for type \(\theta\) given quality schedule \(q(\cdot)\).

**Incentive Compatible Contracts**

Let \(U_i(\hat{\theta}, \theta, d_i)\) be the utility obtained by a consumer of type \((\theta, d_i)\) who reports \(\hat{\theta}\) and purchases a unit of brand \(i\) product. Then

\[
U_i(\hat{\theta}, \theta, d_i) = \theta q_i(\hat{\theta}) - t_i(\hat{\theta}) - k d_i
\]

(3)

Incentive compatibility requires

\[
\forall(\theta, \hat{\theta}) \in [0, 1]^2, \ U_i(\theta, \theta, d_i) \geq U_i(\hat{\theta}, \theta, d_i) \text{ for } i = 1, 2
\]

(4)

Since (3) satisfies the single crossing property in \((\theta, q_i)\), we can show the following “constraint simplification” lemma.

\(^{16}\)Collusive duopoly would be an alternative interpretation of our monopoly model. A similar modeling technique is employed by Levin, Peck, and Ye (2005) to analyze the effect of introducing competition on firms’ quality disclosure strategies.
Lemma 1 The IC conditions (4) are satisfied if and only if the following two conditions hold:

1. \[ U_i(\theta, \theta, d_i) = \int_{\theta_i^*}^{\theta} q(\tau) d\tau - k d_i \] for all \( \theta \geq \theta_i^* \) and \( i = 1, 2 \), where \( \theta_i^* \in [0, 1) \).

2. \( q_i(\theta) \) is increasing in \( \theta \)

where \( \theta_i^* \) is the lowest type that purchases from brand \( i \).

Lemma 1 is a standard result in the one-dimensional screening literature. This also applies to our model because consumers’ utility functions are separable in \( q \) and \( d_i \). As a result, if \( (q_i(\cdot), t_i(\cdot)) \) is incentive compatible for all the types at location \( d \), then it is also incentive compatible for all the types at any other location \( d' \neq d \). Here \( \theta_i^* \) can be regarded as a separate choice variable for brand \( i \): any consumer whose type is below \( \theta_i^* \) is excluded from the market for brand \( i \). Alternatively, one can interpret that brand \( i \) offers a null contract \( (q_i = 0 \text{ and } t_i = 0) \) to all consumers whose types are below \( \theta_i^* \).

Define

\[
y_i(\theta) = \int_{\theta_i^*}^{\theta} q_i(\tau) d\tau, \quad i = 1, 2.
\]

Then by Lemma 1 \( y_i(\theta) \) is the rent provision to the type \( (\theta, 0) \) consumer specified by the menu of IC contracts offered by brand \( i \). Equation (5) states that the rent provision for such a consumer can be represented by an integration over the quality provision schedule from the lowest type served in the market up to this consumer’s type.

The equilibrium utility enjoyed by a consumer of type \( (\theta, d_i) \) can now be written as \( y_i(\theta) - k d_i \). Moreover, the quality and the price specified in the original contract can be recovered from \( y_i(\theta) \) as follows:

\[
q_i(\theta) = y_i'(\theta) \text{ and } t_i(\theta) = \theta q_i(\theta) - y_i(\theta).
\]

Thus any menu of IC contracts can be characterized by rent provision schedules \( (y_i(\cdot), i = 1, 2) \). For this reason we will work with \( y_i(\cdot), i = 1, 2 \), to derive the equilibrium menu of contracts. Note that by definition, \( y_i(\theta) \) is continuous in \( \theta \).

Individual Rationality and Market Shares

Given rent provision schedules \( \{y_i(\theta)\}, i = 1, 2 \), each consumer decides whether to make a purchase, and if they do, what product (brand and quality) to purchase.
If a consumer of type \((\theta, d_i)\) chooses to purchase a product from brand \(i\), then the following two conditions must hold:

\[
\begin{align*}
    y_i(\theta) - kd_i & \geq 0 \\
    y_i(\theta) - kd_i & \geq y_{-i}(\theta) - k(1/2 - d_i)
\end{align*}
\]

The first condition states that the consumer obtains a positive utility from participation, and the second condition states that the consumer prefers brand \(i\) to the other brand. These two conditions jointly imply

\[
d_i \leq \min \left\{ \frac{y_i(\theta)}{k}, \frac{1}{4} + \frac{1}{2k} (y_i(\theta) - y_{-i}(\theta)) \right\} =: s_i(\theta)
\]

(6)

\(2s_i(\theta)\) is the total measure of type-\(\theta\) consumers who purchase brand \(i\) products, hence it is the market share of brand \(i\) from type-\(\theta\) consumers. Integrating \(2s_i(\theta)\) over \([\theta_i^*, 1]\), we obtain brand \(i\)'s market coverage, which is the total measure of consumers who purchase brand \(i\) products. Figure 2 below illustrates one half of the market coverage for each brand (the other half not shown is symmetric).

![Figure 2: An Illustration of Market Shares and Market Coverage](image)

From Figure 2, we can see that there is a cutoff type \(\hat{\theta}\) above which the market is fully covered (consumers are served regardless of their horizontal locations), and below which the market is not fully covered. Under duopoly the full coverage range \([\hat{\theta}, 1]\) can also be called the competition range since the two firms are competing for customers over this range, and the partial coverage range \([\theta_i^*, \hat{\theta}]\) can also be called the local monopoly range because over this range each firm enjoys local monopoly
power. Note that $\hat{\theta}$ is endogenously determined by the following condition:

$$y_1(\hat{\theta}) + y_2(\hat{\theta}) = \frac{k}{2}$$  \hspace{1cm} (7)

By offering a rent provision schedule $y_i(\theta)$, $\theta \in [\theta_i^*, 1]$, brand $i$’s expected profit from a type-$\theta$ consumer is given by $t_i(\theta) - c(\theta) = \theta q_i(\theta) - y_i(\theta) - \frac{1}{2}q_i^2(\theta)$. Hence given $y_{-i}(\cdot)$, brand $i$’s total expected profit is twice the following:

$$\int_{\theta_i^*}^{1} \left[ \theta q_i(\theta) - y_i(\theta) - \frac{1}{2}q_i^2(\theta) \right] s_i(\theta) d\theta$$  \hspace{1cm} (8)

The profit maximization problem of (8) can be formulated as an optimal control problem, with $\theta$ being the “time,” $q_i$ being the control variable, and $y_i$ being the state variable. However, the standard optimal control theory requires that the integrand of the objective function be differentiable in the state variable (e.g., Kamien and Schwartz, 1992), which fails in our problem: $s_i(\theta)$, and hence the integrand in (8) is not differentiable with respect to $y_i$ at $\hat{\theta}$.

It turns out that we can get around this nondifferentiability problem by using a special optimal control technique. In the partial coverage range, brand $i$’s market share is $y_i(\theta)/k$, while the market share in the full coverage range is $\left[ \frac{1}{4} + \frac{1}{2k} (y_i(\theta) - y_{-i}(\theta)) \right]$. Thus, (8) can be rewritten as the sum of two integrations:

$$\int_{\theta_i^*}^{\hat{\theta}} \left[ \theta q_i(\theta) - y_i(\theta) - \frac{1}{2}q_i^2(\theta) \right] \frac{y_i(\theta)}{k} d\theta$$

$$+ \int_{\theta}^{1} \left[ \theta q_i(\theta) - y_i(\theta) - \frac{1}{2}q_i^2(\theta) \right] \cdot \left[ \frac{1}{4} + \frac{1}{2k} (y_i(\theta) - y_{-i}(\theta)) \right] d\theta$$  \hspace{1cm} (9)

The problem of maximizing (9) subject to the transition equation $y'_i(\theta) = q_i(\theta)$ and the corresponding endpoint conditions can be viewed as an optimal control problem with two potential phases. What makes it different from ordinary one-phase optimal control is that now we also need to solve for the optimal switching “time” $\hat{\theta}$, at which the first phase switches to the second phase.

A closely related problem is analyzed by Amit (1986), who considers a petroleum recovery process that has two potential phases with different technologies yielding different extraction rates. Unlike in Amit’s model where the state variable is free at the switching time, in our problem $y_i(\hat{\theta})$ is constrained to satisfy (7). As a consequence, the set of necessary conditions for optimality derived in Amit cannot be directly applied to our model. In the next subsection we formulate a two-phase optimal control problem allowing for constraints on the switching point, and derive necessary conditions that can be directly applied to all our cases.
Rochet and Stole (1997) encounter the same optimal “switching” problem in their duopoly analysis of the “mixed regime” (where both the competition and local monopoly regimes are present). While they sidestep the problem by correctly applying smooth pasting (i.e., the continuity in both the state and control variables), their approach is not formally justified. In this sense our paper is the first to present a formal framework in tackling a general two-phase optimal control problem in the nonlinear pricing environment.

A Two-Phase Optimal Control Procedure

Consider a slightly more general problem below.

\[
\max \int_{\theta_0}^{\theta_1} F_1(\theta, y(\theta), q(\theta)) \, d\theta + \int_{\theta_1}^{\theta_2} F_2(\theta, y(\theta), q(\theta)) \, d\theta 
\]

subject to:

\[
y'(\theta) = \begin{cases} f_1(\theta, y(\theta), q(\theta)) & : \theta_0 \leq \theta < \theta_1 \\ f_2(\theta, y(\theta), q(\theta)) & : \theta_1 < \theta \leq \theta_2 \end{cases}
\]

17 They implicitly assume the continuity of both the Hamiltonian and co-state at the switching point (the Erdmann-Weierstrass necessary conditions). However, as can be seen in our analysis, when the state variable is constrained at the switching point \( \hat{\theta} \), the transversality condition at \( \hat{\theta} \) does not necessarily imply the continuity in both the Hamiltonian and the co-state. Although smooth pasting turns out to be indeed the case in our analysis, it can be verified, following the two-phase optimal control technique introduced below, that when the state is not free at \( \hat{\theta} \), the co-state may be discontinuous at \( \hat{\theta} \). (Analogously we can show that when the switching point \( \hat{\theta} \) is not a choice variable, the Hamiltonian may be discontinuous.)
phases as follows:

\[
H = \begin{cases}
  H_1 = F_1 + \lambda f_1 & : \theta_0 \leq \theta < \theta_1 \\
  H_2 = F_2 + \lambda f_2 & : \theta_1 < \theta \leq \theta_2
\end{cases}
\]

**Lemma 2** The necessary conditions for the maximization problem (10)-(14) are:

\[
\frac{\partial H_1}{\partial q} = 0 \quad \theta_0 \leq \theta < \theta_1 \quad (15)
\]

\[
\frac{\partial H_2}{\partial q} = 0 \quad \theta_1 < \theta \leq \theta_2 \quad (16)
\]

\[
\frac{\partial H_1}{\partial y} = -\lambda' \quad \theta_0 \leq \theta < \theta_1 \quad (17)
\]

\[
\frac{\partial H_2}{\partial y} = -\lambda' \quad \theta_1 < \theta \leq \theta_2 \quad (18)
\]

\[
[H_1 - \lambda R'](\theta_1^{-}) = [H_2 - \lambda R'](\theta_1^{+}) \quad \text{if} \quad \theta_0 < \theta_1 < \theta_2 \quad (19)
\]

\[
[H_1 - \lambda R'](\theta_1^{-}) \leq [H_2 - \lambda R'](\theta_1^{+}) \quad \text{if} \quad \theta_0 = \theta_1 < \theta_2 \quad (20)
\]

\[
[H_1 - \lambda R'](\theta_1^{-}) \geq [H_2 - \lambda R'](\theta_1^{+}) \quad \text{if} \quad \theta_0 < \theta_1 = \theta_2 \quad (21)
\]

\[
H_1(\theta_0) = 0 \quad (22)
\]

\[
\lambda(\theta_2) = 0 \quad (23)
\]

**Proof.** See Appendix. ■

Conditions (15)-(18) and (22)-(23) are the standard necessary conditions for optimality. Conditions (19)-(21) govern the optimal switching “time” between the two phases. Let \( \hat{H}_i = H_i - \lambda R' \) be the pseudo-Hamiltonian modified from the original Hamiltonian (due to the restrictions on the switching “time” \( \theta_1 \)). Then (19) says that if there is a type \( \theta_1 \) at which the pseudo-Hamiltonians of the two phases are equal, then it is optimal to start with phase I and then switch to phase II at \( \theta_1 \). However, if no such \( \theta_1 \) can be found, then it is optimal to either skip phase I and start with phase II right away, or stay with phase I and never enter phase II, depending on whether \( \hat{H}_1(\cdot) \) is bigger or smaller than \( \hat{H}_2(\cdot) \).

Lemma 2 can be of some independent interest. It can be applied to similar two-phase optimal control problems where the state variable at the switching point has to satisfy some restrictions. Lemma 2 can be easily extended to problems with multiple controls and multiple states, in which case \( q, y, \) and \( R \) should each be interpreted as a vector of functions.
4 Monopoly

Under monopoly, the two brands are owned by a single firm. The monopolist’s objective is to maximize the joint profits from the two brands. Since consumers are uniformly distributed along the horizontal dimension and the two brands’ production technologies are symmetric, we focus on the symmetric solution in which each brand offers the same menu of contracts and the resulting market shares are symmetric.\(^{18}\) We can thus drop the subscripts to write \(y_i(\theta) = y(\theta), i = 1, 2\). Simplifying (6), the market share becomes

\[
s_i(\theta) = s(\theta) = \min \left\{ \frac{y(\theta)}{k^*}, \frac{1}{4} \right\}
\]

By Lemma 1, the monopolist’s problem can be formulated as the following program:

\[
\begin{align*}
\max & \quad \int_{\theta^*}^{1} \left[ \theta q(\theta) - y(\theta) - \frac{1}{2} q^2(\theta) \right] s(\theta) d\theta \\
\text{s.t.} & \quad y'(\theta) = q(\theta) \\
& \quad q'(\theta) \geq 0 \\
& \quad y(\theta^*) = 0
\end{align*}
\]

where \(\theta^*\) is the lowest type of consumers who are served in the market, that is, \(y(\theta^*) = 0\).\(^{19}\) Since \(q(\theta) > 0\) for \(\theta > \theta^*\), \(y(\theta)\) must be strictly increasing. This implies that there is a unique \(\hat{\theta}\) such that \(y(\theta)/k \leq 1/4\) for \(\theta \leq \hat{\theta}\) and \(y(\theta)/k > 1/4\) for \(\theta > \hat{\theta}\). The above program can thus be rewritten as a two-phase optimal control problem:

\[
\begin{align*}
\max & \quad \int_{\theta^*}^{1} \left[ \theta q(\theta) - y(\theta) - \frac{1}{2} q^2(\theta) \right] \frac{y(\theta)}{k} d\theta + \int_{\hat{\theta}}^{1} \left[ \theta q(\theta) - y(\theta) - \frac{1}{2} q^2(\theta) \right] \frac{1}{4} d\theta \\
\text{s.t.} & \quad y'(\theta) = q(\theta) \\
& \quad q'(\theta) \geq 0 \\
& \quad y(\hat{\theta}) = k/4 \\
& \quad y(\theta^*) = 0
\end{align*}
\]

\(^{18}\)We focus on the symmetric solution here for ease of comparison with the duopoly case, where we will focus on symmetric equilibrium in which each firm offers the same menu of contracts. While a formal proof is not attempted here, we conjecture that symmetric solution is optimal for the monopolist.

\(^{19}\)If \(y(\theta^*) > 0\), then for some sufficiently small \(\epsilon\), it can be verified that some type-\((\theta^* - \epsilon)\) consumers would prefer accepting contract \(y(\theta^*)\) to staying out of the market, which contradicts the assumption that \(\theta^*\) is the lowest type being served.
As is standard in the literature, instead of solving the original program above, we will solve the relaxed program by dropping the monotonicity constraint $q'(\theta) \geq 0$. Later on we will verify the monotonicity of $q(\theta)$ to justify the solution. To apply Lemma 2, we define

$$H = \begin{cases} 
H_1 = \left[ \theta q - y - \frac{1}{2} q^2 \right] \frac{y}{k} + \lambda q & : \theta^* \leq \theta < \hat{\theta} \\
H_2 = \left[ \theta q - y - \frac{1}{2} q^2 \right] \frac{1}{4} + \lambda q & : \hat{\theta} < \theta \leq 1
\end{cases}$$

For Phase I (partial coverage range), the optimality condition and the co-state equation are given by

$$\frac{\partial H_1}{\partial q} = (\theta - q) \frac{y}{k} + \lambda = 0 \quad (24)$$

$$\frac{\partial H_1}{\partial y} = -\lambda' = -\frac{1}{k} \left( 2y - \theta q + \frac{1}{2} q^2 \right) \quad (25)$$

Differentiating (24) with respect to $\theta$, and substituting $\lambda'$ from (25) into (24), we obtain the following differential equation.

$$3y - \frac{1}{2} y'^2 - yy'' = 0 \quad (26)$$

Combining with the lower endpoint condition $y(\theta^*) = 0$, it can be verified that the solution to (26) is given by: $^{20}$

$$y(\theta) = \frac{3}{4} (\theta - \theta^*)^2 \quad (27)$$

$$q(\theta) = \frac{3}{2} (\theta - \theta^*) \quad (28)$$

Similarly, in phase II (full coverage range) the optimal $y(\theta)$ is characterized by

$$\frac{\partial H_2}{\partial q} = (\theta - q) \frac{1}{4} + \lambda = 0 \quad (29)$$

$$\lambda' = -\frac{\partial H_2}{\partial y} = \frac{1}{4}$$

and the transversality condition at $\theta = 1$:

$$\lambda(1) = 0$$

The solution to this system is given by:

$$y(\theta) = \theta^2 - \theta + \beta \quad (30)$$

$$q(\theta) = 2\theta - 1 \quad (31)$$

$^{20}$As shown in Rochet and Stole (2002) (appendix, p. 304), if a convex solution to differential equation (26) exists for a given set of boundary conditions, it is unique.
where $\beta$ is a parameter yet to be determined. Note that in both phases $q(\theta)$ is increasing in $\theta$. Thus the solutions to the relaxed program are also the solutions to the original program. Moreover, since in both phases $q(\theta)$ is strictly increasing in $\theta$, the optimal menu of contracts exhibits perfect sorting. That is, among consumers served in the market, different types will select different contracts in equilibrium.

To determine $\hat{\theta}$, we have two conditions:

\begin{align*}
y(\hat{\theta}^-) &= \frac{3}{4}(\hat{\theta} - \theta^*)^2 = \frac{k}{4} \quad (32) \\
y(\hat{\theta}^+) &= \hat{\theta}^2 - \hat{\theta} + \beta = \frac{k}{4} \quad (33)
\end{align*}

In addition, we have a transversality condition about the optimal switching point $\hat{\theta}$. Since $R'(\theta) = \frac{k}{4}$, $R' = 0$. Thus (19) becomes:

\[ H_1(\hat{\theta}) = H_2(\hat{\theta}) \quad (34) \]

Solving (24) for $\lambda(\hat{\theta}^-)$, and (29) for $\lambda(\hat{\theta}^+)$, then substituting the expressions of $\lambda(\hat{\theta}^-)$ and $\lambda(\hat{\theta}^+)$, and $y(\hat{\theta}) = \frac{k}{4}$ into (34), we can verify that $q(\cdot)$ is continuous at $\hat{\theta}$.\(^{21}\) Thus from (28) and (31) we have

\[ \frac{3}{2}(\hat{\theta} - \theta^*) = 2\hat{\theta} - 1 \]

Combining this with (32) and (33), we have

\begin{align*}
\theta^*M &= \frac{1}{2} - \frac{1}{12}\sqrt{3k} \quad (35) \\
\hat{\theta}^M &= \frac{1}{2} + \frac{1}{4}\sqrt{3k} \quad (36) \\
\beta &= \frac{1}{4} + \frac{1}{16}k
\end{align*}

It is easily verified that (34) has an interior solution only when $k < \frac{4}{3}$. If $k \geq \frac{4}{3}$, $H_1 \geq H_2$ for all $\theta \leq 1$, thus we would have the corner solution $\hat{\theta} = 1$. That is, if $k \geq \frac{4}{3}$ phase II is never entered (no interaction between the two brands). In that case we can use the transversality condition $\lambda(1) = 0$ to pin down $\theta^*M = \frac{1}{3}$. The above analysis is summarized below.

**Proposition 1** In the monopoly model, the optimal symmetric menu of contracts is unique and exhibits perfect sorting. Specifically, for $k \in (0, \frac{4}{3})$,

\[ y(\theta) = \begin{cases} 
\frac{3}{4}(\theta - \theta^*M)^2 & : \theta^*M \leq \theta \leq \hat{\theta}^M \\
\theta^2 - \theta + \frac{1}{4}k & : \hat{\theta}^M < \theta \leq 1
\end{cases} \]

\(^{21}\)Thus the optimal solution at $\hat{\theta}$ satisfies smooth pasting, that is, both $y(\cdot)$ and $q(\cdot)$ are continuous at $\hat{\theta}$.  

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where $\theta^* M$ and $\hat{\theta}^M$ are given by (35) and (36), respectively. For $k \geq \frac{4}{3}$,

$$y(\theta) = \frac{3}{4} \left( \theta - \frac{1}{3} \right)^2, \quad \theta \in \left[ \frac{1}{3}, 1 \right].$$

The optimal menu of contracts exhibits several salient features. First, there is always a positive measure of types of consumers (regardless of horizontal location) who are excluded from the market ($\theta^* M > 0$). The underlying reason for the exclusion is the informational rent consideration. Offering contracts to all types may increase the firm’s profit from those types in $[0, \theta^* M)$. However, doing so necessarily increases the informational rent to all types above $\theta^* M$ due to the screening condition (5), which reduces the firm’s profit from those types. The optimal $\theta^* M$, which balances the above two opposing effects, should thus be strictly above zero.

Second, there is quality distortion for all but the highest type consumers, i.e., $q(\theta) < \theta$ for all $\theta \in (\theta^* M, 1)$. This is again driven by the informational rent consideration. Offering a higher quality provision up to some certain type, say $\theta'$, would enable the firm to charge higher prices for the consumers with types up to $\theta'$. However, doing so would necessarily increase the informational rent to all the consumers with types above $\theta'$ due to (5). By distorting quality downward, the firm optimally balances these two opposing effects.

Finally, the optimal contracts exhibit perfect sorting. That is, different types of consumers choose different contracts. In the standard one-dimensional screening model without horizontal differentiation, a sufficient condition to guarantee perfect sorting is that the distribution of consumers’ types satisfies the monotone hazard rate property (MHRP). Our model differs from the standard screening model in that products are horizontally differentiated, and the effective “density” function in the vertical dimension would now be $s(\theta)$, the market share of type $\theta$ consumers.

Our results thus imply that bunching does not occur and the quality provision for the lowest type covered is always distorted downwards. These are in stark contrast with the results obtained by Rochet and Stole (1997, 2002), who show that either bunching occurs at a lower interval, or perfect sorting occurs with efficient quality provision for the lowest type. This sharp difference between our results and theirs first appears puzzling, given that the differential equation (26) is a special case of the Euler equation derived in Rochet and Stole (who consider more general distributions). The key to solve the puzzle is to observe the difference in boundary conditions. Note that in Rochet and Stole the ratio of the lowest type to the highest type $\gamma = \frac{\theta}{\bar{\theta}}$ is assumed to be greater than $1/2$. This implies that all the (vertical) types are covered and in effect, the lowest type covered is exogenously given. As a result, the state variable $y$ is free at the lowest type $\bar{\theta}$, which gives rise to the boundary
condition $\lambda(\theta) = 0$. This boundary condition in turn implies efficient quality provision at $\theta$ if the monotonicity constraint on $q$ is satisfied (perfect sorting case). Note also that sorting can become quite costly for the monopolist given the requirement of no quality distortion at $\theta$, which explains why bunching may occur at a lower interval starting from $\theta$. On the other hand, in our model the lowest possible type $\underline{\theta}$ is 0 ($\gamma = 0$), thus not all types will be covered and the lowest type covered, $\theta^*$, is endogenously determined. This leads to a different set of boundary conditions: $y(\theta^*) = 0$ and $H(\theta^*) = 0$. Combined with the differential equation (26), these conditions pin down a unique perfect sorting solution in which $q(\theta^*) = 0$. \(^{22}\)

Thus in a sense our analysis is complementary to that in Rochet and Stole: while they study the case with full coverage of vertical types ($\gamma$ is big), we analyze the case with endogenously determined coverage of vertical types ($\gamma$ is small). It is worth noting that two cases lead to dramatically different results. In fact our results are more in line with those in the classical Mussa and Rosen (1978) paper without horizontal differentiation: the quality provision for the lowest type covered is always distorted downward, and the quality schedule is fully separating if the distribution of types satisfies the regularity condition (MHRP). To better understand the link between our results and those of Rochet and Stole, let’s fix the upper bound of the vertical type, $\overline{\theta}$, and gradually raise $\theta$, starting from 0. When $\underline{\theta}$ is 0, our results apply: there is an endogenously determined lowest type covered $\theta^*$, with perfect sorting and $q(\theta^*) = 0$. This feature stays the same until $\underline{\theta}$ is raised just above $\theta^*$. When $\underline{\theta}$ is just above $\theta^*$, the case of Rochet and Stole applies since all the vertical types are covered. If the monotonicity constraint does not bind, the boundary condition at $\underline{\theta}$ requires efficient quality provision at $\underline{\theta}$. \(^{23}\) But continuity implies that the optimal solution should not change drastically at $\underline{\theta} = \theta^*$. Thus the monotonicity must fail, leading to bunching at the lower end near $\underline{\theta}$. Intuitively, when $\underline{\theta}$ is slightly above $\theta^*$ ($\gamma$ is relatively small), efficient quality provision at $\underline{\theta}$ is costly since it increases the informational rent for all higher types, the measure of which is big since $\gamma$ is relatively small. The optimality thus requires bunching. As $\underline{\theta}$ is further raised close to $\overline{\theta}$ ($\gamma$ becomes big enough), efficient quality provision at $\underline{\theta}$ becomes less costly since there are fewer higher types. As a result, the monotonicity constraint is more likely to be satisfied even with efficient quality provision at $\underline{\theta}$. Therefore, perfect sorting is more likely when $\gamma$ is big. This explains why in Rochet and Stole the solution involves perfect sorting when $\gamma$ is sufficiently large.

\(^{22}\) It can be easily verified that the quadratic functional form solution, which works in our case, does not satisfy the differential equation system in Rochet and Stole, simply because it violates their boundary conditions.

\(^{23}\) This would imply that $\lim_{\underline{\theta} \to \theta^+} q(\underline{\theta}) = \theta^*$ while $\lim_{\underline{\theta} \to \theta^-} q(\underline{\theta}) = 0$. 

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Given that the optimal contracts exhibit perfect sorting, there is a one-to-one mapping between the number of contracts offered and the (vertical) types of consumers served. Thus the number of contracts offered can be measured in terms of the (Lebesgue) measure of vertical types of consumers covered in the market. As a direct consequence, the lowest type served in the market, or the participation threshold, $\theta^*_{M}$, becomes a measure of the number of contracts offered. Specifically, as $\theta^*_{M}$ decreases, more contracts are offered in the sense that more (vertical) types of consumers are served. On the other hand, as $\theta^*_{M}$ increases, fewer contracts are offered in the sense that fewer (vertical) types of consumers are served. We are interested in how the degree of horizontal differentiation, which is parameterized by $k$, affects the number (measure) of contracts that the firm offers. Equation (35) shows that for $k \in (0, 4/3)$, $\theta^*_{M}$ is decreasing in $k$, and for $k \geq 4/3$, $\theta^*_{M} = 1/3$ is independent of $k$. Thus when two brands become more horizontally differentiated (a bigger $k$), the monopolist offers more contracts. From (28) and (31) it can also be seen that quality distortions become smaller in Phase I but are unaffected in Phase II. We summarize these results in the following proposition.

**Proposition 2** In the monopoly model, when two brands become more horizontally differentiated, the monopolist offers more contracts targeting more consumer types, and quality distortions become smaller in the partial coverage range and remain unaffected in the full coverage range.

To understand the intuition of this result, we first need to understand the effects on profit of increasing the rent provision. Raising rent provisions (hence the total rent) to consumers has two effects. The first is to reduce the firm’s profitability per consumer (which can be termed as the *marginal effect*), and the second is to attract more consumers (which can be termed as the *market share effect*). Thus profit maximization requires an optimal balance between these two opposing effects. Note that with asymmetric information, the firm cannot freely vary the rent provision for certain types of consumers without affecting the rent provisions to other types. That is, rent provisions can only be adjusted subject to the screening condition, (5), which implies that changing the rent provision for some type will affect the rent provisions for all the types above. Hence the optimal rent provision schedule reflects an optimal trade-off between the marginal effect and market share effect subject to the screening condition.

In view of this insight, it is now straightforward to think through the intuition behind Proposition 2. As $k$ increases, by fixing the previous menu of contracts (holding $y(\cdot)$ fixed), $\hat{\theta}$ increases and $y(\theta)/k$ decreases, which implies that the market shares in both the full and partial coverage ranges shrink.
To counter this effect, the monopolist has an incentive to increase \( y(\theta) \) in an attempt to partially restore the loss of the market shares. By the screening condition (5), this can be achieved by either moving the schedule \( q(\cdot) \) upward or pushing \( \theta^{*M} \) downward, and both occur in equilibrium (due to equation (28)). On the other hand, as \( k \) decreases, holding \( y(\cdot) \) fixed leads to a decrease in \( \hat{\theta} \) and an increase in \( y(\theta)/k \), which results in an increase in the total market share. This creates room for the monopolist to reduce rent provisions so as to increase the profit per consumer. Again by the screening condition this can be achieved by raising \( \theta^{*M} \) or moving up the schedule \( q(\cdot) \) (both of which occur in equilibrium).

Hence Proposition 2 is driven by an interaction between horizontal differentiation and screening in the vertical dimension, which occurs through the rent provision schedule \( y(\theta) \). As can be seen, this insight will serve as a general intuition for most of the other results that we are going to present.

5 Duopoly

Unlike in the monopoly model where the monopolist’s objective is to maximize the joint profits from the two brands, in the duopoly model each firm’s objective is, given the other firm’s menu of contracts, to maximize its own profit by choosing a menu of contracts. As mentioned earlier, our solution concept is Bertrand-Nash equilibrium. Since both firms are symmetric in terms of their production technology and market positions, we focus on symmetric equilibrium, in which each firm offers the same menu of contracts, hence the same rent provision schedule \( y^{*}(\theta), \theta \in [\theta^{*D}, 1] \) (\( \theta^{*D} \) is the lowest type that is served in the market). Formally, the pair \((y^{*}(\theta), y^{*}(\theta))\) constitutes a Bertrand-Nash equilibrium if given \( y_{-i}(\theta) = y^{*}(\theta), \theta \in [\theta^{*}, 1] \), firm \( i \)’s best response is to choose \( y_{i}(\theta) = y^{*}(\theta), \theta \in [\theta^{*}, 1] \) as well.

Given the two firms’ rent provision schedules \( y_{1}(\theta) \) and \( y_{2}(\theta) \), the consumers’ type space is demarcated into two ranges: the competition range \((\theta > \hat{\theta})\), and the local monopoly range \((\theta < \hat{\theta})\). The switching point \( \hat{\theta} \) is determined by

\[
y_{i}(\hat{\theta}) = \frac{k}{2} - y_{-i}(\hat{\theta})
\]

Suppose \( y_{-i}(\theta) = y^{*}(\theta), \theta \in [\theta^{*}, 1] \), then firm \( i \)’s relaxed program (by ignoring the monotonicity of
\[ q_i \) is as follows:

\[
\max \int_{\hat{\theta}}^{\theta_i^*} \left[ \theta q_i(\theta) - y_i(\theta) - c(q_i(\theta)) \right] \frac{y_i(\theta)}{k} \, d\theta \\
+ \int_{\hat{\theta}}^{1} \left[ \theta q_i(\theta) - y_i(\theta) - c(q_i(\theta)) \right] \cdot \left[ \frac{1}{4} + \frac{1}{2k} (y_i(\theta) - y^*(\theta)) \right] \, d\theta
\]

s.t. \( y_i'(\theta) = q_i(\theta) \)

\( y_i(\theta_i^*) = 0, \ \theta_i^* \) free

\( y_i(\hat{\theta}) = \frac{k}{2} - y^*(\hat{\theta}), \ \hat{\theta} \) free

\( y_i(1) \) free

To apply Lemma 2, we define

\[
H = \begin{cases} 
H_1 = \left[ \theta q_i - y_i - \frac{1}{2} q_i^2 \right] \frac{y_i}{k} + \lambda q_i : \ \theta_i^* \leq \theta < \hat{\theta} \\
H_2 = \left[ \theta q_i - y_i - \frac{1}{2} q_i^2 \right] \cdot \left[ \frac{1}{4} + \frac{1}{2k} (y_i(\theta) - y^*(\theta)) \right] + \lambda q_i : \ \hat{\theta} < \theta \leq 1
\end{cases}
\]

For phase I (\( \theta < \hat{\theta} \)), we can follow exactly the same steps as in the monopoly model to obtain

\[
y^*(\theta) = \frac{3}{4} (\theta - \theta^*)^2
\]

\[
q^*(\theta) = \frac{3}{2} (\theta - \theta^*)
\]

For phase II (\( \theta > \hat{\theta} \)), the optimality condition and the co-state equation are given by

\[
\frac{\partial H_2}{\partial q_i} = (\theta - q_i) \left[ \frac{1}{4} + \frac{1}{2k} (y_i - y^*) \right] + \lambda = 0
\]

\[
\lambda' = -\frac{\partial H_2}{\partial y_i} = \left[ \frac{1}{4} + \frac{1}{2k} (y_i - y^*) \right] - \frac{1}{2k} \left[ \theta q_i - y_i - \frac{1}{2} q_i^2 \right]
\]

For \( y^*(\theta) \) to constitute a symmetric equilibrium, the above equations hold when evaluated at \( y_i = y^* \).

Thus we have

\[
0 = (\theta - q^*) \frac{1}{4} + \lambda \\
\lambda' = \frac{1}{4} - \frac{1}{2k} \left[ \theta q^* - y^* - \frac{1}{2} q^*^2 \right]
\]

After eliminating \( \lambda \) from the above equations we obtain the following differential equation:

\[
y'''' = 2 - \frac{2}{k} \left( \theta y'' - y^* - \frac{1}{2} y^*^2 \right)
\]
Letting $y_i = y_{-i} = y^*$ in (37), we have the endpoint condition at $\hat{\theta}$:

$$y^*(\hat{\theta}) = \frac{k}{4}$$

(42)

To derive the transversality condition for $\hat{\theta}$, first note that $R(\hat{\theta}) = \frac{k}{2} - y^*(\hat{\theta})$. This implies that

$$R'(\hat{\theta}^-) = -y''(\hat{\theta}^-) = -q^*(\hat{\theta}^-), \text{ and}$$

$$R'(\hat{\theta}^+) = -y''(\hat{\theta}^+) = -q^*(\hat{\theta}^+)$$

Substituting the above two equations and (42) into the transversality condition (19), and following the similar steps as in the monopoly case we can verify that

$$q^*(\hat{\theta}^-) = q^*(\hat{\theta}^+)$$

(43)

i.e., $q^*(\cdot)$ must be continuous at $\hat{\theta}$ in equilibrium. So again, smooth pasting is satisfied in equilibrium.

By the continuity of $y^*(\cdot)$, we can combine (39) and (42) to get

$$\hat{\theta} - \theta^* = \sqrt{k/3}$$

(44)

Plugging (44) into (40), we have

$$y''(\hat{\theta}) = q^*(\hat{\theta}) = \sqrt{3k}/2$$

(45)

Finally $\lambda(1) = 0$ implies that

$$y''(1) = q^*(1) = 1$$

(46)

Now the existence of a symmetric equilibrium boils down to the existence of a $\hat{\theta} \in (0,1]$ and a convex function $y^*(\cdot)$ defined over $[\hat{\theta},1]$,$^{24}$ which satisfy the differential equation (41) and three boundary conditions (42), (45) and (46). To save notation we drop the superscripts to rewrite the system as follows:

$$\begin{cases}
   y'' = 2 - \frac{2}{k} (\theta y' - y - \frac{1}{2}y^2) \\
   y(\hat{\theta}) = k/4 \\
   y'(\hat{\theta}) = \sqrt{3k}/2 \\
   y'(1) = 1
\end{cases}$$

(47)

$^{24}$We need $y''(\theta) \geq 0$ to ensure $q'(\theta) \geq 0$. 

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Proposition 3 For \( k \in (0, 4/3) \), there is a unique symmetric equilibrium in the duopoly model, which exhibits perfect sorting and is given by

\[
y(\theta) = \begin{cases} 
\frac{3}{4}(\theta - \theta^D)^2 & : \theta^D \leq \theta \leq \hat{\theta}^D \\
y^*(\theta) & : \hat{\theta}^D \leq \theta \leq 1
\end{cases}
\]

where \((\hat{\theta}^D, y^*(\theta))\) is the unique solution to the system (47), and \(\theta^* = \hat{\theta}^D - \sqrt{k/3}\).

For \( k \geq 4/3 \),

\[
y(\theta) = \frac{3}{4} \left( \theta - \frac{1}{3} \right)^2, \theta \in \left[ \frac{1}{3}, 1 \right].
\]

Proof. See Appendix. ■

In the proof we show that given \( k \in (0, 4/3) \) the solution to the differential equation system (47) exists and is unique. Moreover, \( y^*(\theta) \) is strictly convex. (47) is not a standard ordinary differential equation (ODE) system partly due to the fact that the boundary conditions involve an endogenously determined endpoint \((\hat{\theta})\). Thus no existing ODE theorem can be directly applied to show the existence and uniqueness. The proof is somewhat tedious and hence relegated to the appendix.

It is apparent from the proof that there is no closed-form solution for \( y^*(\theta) \), the equilibrium rent provision schedule for \( \theta \in [\hat{\theta}^D, 1] \). So the schedule of \( y^*(\theta) \) can only be obtained from numerical computations.\(^25\)

As in the monopoly model, the equilibrium menu of contracts exhibits perfect sorting. Thus \( \theta^* \) also becomes a measure of contracts offered by each firm. Let \( q_D(\cdot) \) and \( q_M(\cdot) \) be the equilibrium quality provision schedules in the duopoly model and monopoly model, respectively. Despite the absence of the closed-form solution in the duopoly model, we are able to rank \( \theta^D \) and \( \theta^M \), and the schedules \( q_D(\cdot) \) and \( q_M(\cdot) \) unambiguously:

**Proposition 4** Given \( k \in (0, 4/3) \), \( \theta^* < \theta^* \), and \( q_D(\theta) > q_M(\theta) \) for \( \theta \in [\theta^D, 1] \), which implies that compared to the monopoly benchmark, in duopoly equilibrium each firm offers more contracts

\(^{25}\)Note that the structure of the differential equation system (47) renders a straightforward algorithm to compute the equilibrium schedule. First, given any \( k \in (0, 4/3) \) we choose an initial value \( \hat{\theta} \) in \((0, 1)\). Then a schedule of \( y(\cdot) \) is plotted according to the initial conditions \((y(\hat{\theta}) \text{ and } y'(\hat{\theta}))\) and the differential equation. Given this plotted schedule we can verify the last boundary condition \( y'(1) = 1 \). If this condition holds within some pre-determined tolerance level, then the plotted schedule is what we want; otherwise we go back to the first step and adjust the initial value for \( \hat{\theta} \). This procedure repeats itself until \( \hat{\theta} \) converges. The converged \( \hat{\theta} \) along with the associated schedule \( y(\theta) \) results in the computed equilibrium schedule for \( \theta \in [\hat{\theta}, 1] \).
targeting more types of consumers, quality distortions are smaller, and the market coverage is larger.

**Proof.** See Appendix. ■

Proposition 4 is shown by comparing the two differential equation systems under two market structures. Figure 3 is an illustration for the comparisons.

![Figure 3: Duopoly vs. Monopoly](image)

To see the intuition behind this comparison result, let’s start by assuming that in the duopoly case each firm offers the optimal symmetric menu of contracts as offered in the monopoly case. As a result the partial coverage and full coverage ranges are the same under both market structures. Note that in the full coverage range \( \theta \in [\hat{\theta}^M, 1] \), the market share effect is absent under monopoly since the market is fully covered and the “competition” between the two brands is internalized by the monopolist; however, under duopoly the market share effect is present since each firm (brand) tries to steal the other firm’s market share. Thus the market share effect is stronger under duopoly, and each firm (brand) has an incentive to increase the rent provision. Therefore moving from monopoly to duopoly, \( \theta^{*D} < \theta^{*M} \), and \( q_D(\theta) > q_M(\theta) \) (by the screening condition (5)). Another way to see this is that competition under duopoly increases rent provisions to higher-type consumers (served in the full coverage range), which relaxes the screening condition in the vertical dimension: under duopoly firms would worry less about providing additional (informational) rent for the higher-type consumers, as the higher-type consumers are going to enjoy higher rent anyway due to competition. Consequently those consumers not served under monopoly may be served under duopoly, and quality distortions
become smaller. These two effects also jointly imply a larger market coverage under duopoly.

Our result has subtle implications about the distribution of consumers covered over the range of quality provisions. First note that in our analysis, the range of quality provisions is endogenously determined in equilibrium, which is from 0 to 1 in both the monopoly and duopoly cases. Our comparison indicates that moving from monopoly to duopoly denser contracts are offered covering more consumer types over the higher quality range. This is illustrated by figure 4 below, where the equilibrium schedules of quality provisions \( q_M(\cdot) \) and \( q_D(\cdot) \) are plotted against the consumer types under both market structures:

![Figure 4: Consumer Coverage over Quality Range](image)

By Proposition 4, duopoly induces smaller quality distortions, hence \( q_D(\theta) \geq q_M(\theta) \) (the equality only holds at \( \theta = 1 \) where both offer the efficient quality provision \( q = 1 \)). To see the specific effect of changing market structure on the number of contracts offered, we can compare the difference in consumer coverages (in the vertical type dimension) in both market structures for any given range of quality provisions; equivalently we can compare the difference of the intervals on the \( \theta \)-axis projected from any quality interval. To this end, first note that in terms of quality provisions, both the monopoly and the duopoly share the same coverage ranges: it is easily verified that \( q_M(\hat{\theta}^M) = q_D(\hat{\theta}^D) = \sqrt{3k}/2 \). Let \( \hat{q} = \sqrt{3k}/2 \), then as depicted in figure 4, \([0, \hat{q}]\) is the common partial coverage range and \([\hat{q}, 1]\) is the common full coverage range.

As can be seen from the figure, the effects of changing market structure on the consumer coverage are different in these two ranges. In the range of \([0, \hat{q}]\), the quality provision schedules in both cases
are parallel to each other. As a result, given any quality interval within $[0, \hat{q}]$, the projected interval moves to the left as we move from monopoly to duopoly. Note that although the total measure of consumer coverage (the length of the projected interval) remains the same, the composition of the consumer types changes: the coverage shifts to the lower types. In the range of $[\hat{q}, 1]$, the slope of the quality schedule is steeper in the monopoly. As a result, the projected interval of consumer coverage from any given quality interval moves to the left and becomes larger as we move from monopoly to duopoly: as illustrated, given any quality interval, say $[q_1, q_2]$, the projected intervals are $[\theta^M_1, \theta^M_2]$ in the monopoly case and $[\theta^D_1, \theta^D_2]$ in the duopoly case. Thus for this range, not only the composition of consumer coverage changes, the total measure of contracts offered also increases: the competition leads to denser contracts offered for any quality interval over this range.

More formally, the total measure of consumer types served per unit of quality range can be regarded as a measure of contract density. In our continuous type model, the inverse of the slope of the quality provision schedule, $d\theta/dq = 1/q'(\theta)$, evaluated at point $(\theta, q)$, is the contract density at quality $q$. From figure 4, it is apparent that over the lower quality range $([0, \hat{q}])$, $q_M'(\theta) = q_D'(\theta)$, so moving from monopoly to duopoly does not alter the contract density (though the composition of consumer coverage is shifted to lower types); while over the higher quality range $([\hat{q}, 1])$, $q_M'(\theta) > q_D'(\theta)$, thus moving from monopoly to duopoly leads to denser contracts offered (more consumer types being served).

In a very different model, Johnson and Myatt (2003) show that an incumbent may respond to entry by either expanding (fighting brand) or contracting (pruning) the product line (the range of qualities). In their model, introducing competition only has an effect on the lower end of quality range, while in our model, moving from monopoly to duopoly unambiguously leads to denser contracts offered in the higher end of the quality range. This difference is obviously due to differences in modeling and assumptions between our approach and theirs. For example, they assume that the entrant cannot produce in some high quality range where the incumbent is able to produce. Thus as entry occurs the incumbent may respond by ceding the market of low quality products to the entrant while focusing on production in the high quality range (pruning). On the other hand, in our model firms are assumed to be technologically equal and the competition for higher type consumers is most intense. As a result, competition has the most effect over the higher end of the quality range.26

26 This is the case in many market settings. For example, for car models in the relatively higher quality range, more features or options are offered for consumers to choose from, while for car models in the lower quality range, usually only standard features or options are available.
As in the monopoly case, we are also interested in how a change in $k$ affects the number of contracts offered by each firm. For convenience of comparison, we show the schedules of both $\theta^*D$ and $\theta^*M$ against $k$ in Figure 5 below, where the schedule of $\theta^*D$ is plotted from numerical computation.

![Figure 5: Comparison of Participation Thresholds](image)

As can be seen from the figure, $\theta^*M$ is always decreasing as $k$ increases. But for the duopoly model, there is a cutoff $k^*$ such that for $k \in (0, k^*)$, $\theta^*D$ is increasing in $k$; and for $k \in (k^*, 4/3)$, $\theta^*D$ is decreasing in $k$ (for $k \geq 4/3$, $\theta^*D = \theta^*M = 1/3$ is independent of $k$). Our computation shows that the turning point $k^*$ is approximately .91. Note that the decreasing trend of $\theta^*D$ in the range of $(k^*, 4/3)$ is not quantitatively significant; in this range of $k$, $\theta^*D$ is in the range of $[0.33, 0.35]$. On the other hand, the increasing trend of $\theta^*D$ in the range of $(0, k^*)$ is quantitatively significant; when $k = k^*$, $\theta^*D$ equals to 0.35, while as $k$ converges to 0, $\theta^*D$ converges to 0 as well. The following comparative statics result is obtained from numerical computations:27

**Proposition 5** In the duopoly case, when $k \in (0, k^*)$, as $k$ decreases each firm offers more contracts targeting more consumer types, and quality distortions become smaller; when $k \in (k^*, 4/3)$, as $k$ decreases each firm offers fewer contracts targeting fewer consumer types, and the effect on quality distortions is not uniform: there is a cutoff type, say $\theta$, such that when $\theta \in [0, \tilde{\theta})$, quality distortions become bigger, while when $\theta \in (\tilde{\theta}, 1)$, quality distortions become smaller; when $k \geq 4/3$, both firms are local monopolists hence $k$ does not affect the number of contracts or quality distortions.

27The MATLAB code for all the computations in this paper is available upon request.
Thus the effects of changing $k$ on $\theta^*$ and quality distortions in the duopoly case are dramatically different from those in the monopoly benchmark. Again the intuitions spelled out previously continue to help, with the details being a bit more subtle. In the monopoly case $k$ is interpreted as the degree of horizontal differentiation. In addition to this interpretation, in the duopoly case $k$ can also be interpreted as the degree of competition between two firms; The lower the $k$, the less the horizontal differentiation between two brands, and as a result, the more fierce the competition between two firms.\(^{28}\)

A decrease in $k$ while holding $y(\cdot)$ fixed leads to an increase in the market share in Phase I (the local monopoly range). Following the intuition suggested for Proposition 2, each firm would then have incentive to decrease the rent provision in this range, which can be achieved by raising $\theta^*$ or lowering $q(\cdot)$. However the effect on Phase II (the competition range) is different. As $k$ decreases, the competition becomes more intense. As a result, the impact of the market share effect on firms’ profit becomes relatively more important than that of the marginal effect on firms’ profit (which is further reinforced by a decrease in $\hat{\theta}$), therefore each firm would have incentive to raise rent provisions, which can be achieved by lowering $\theta^*$ or raising $q(\cdot)$. So the effects on $\theta^*$ and $q(\cdot)$ of decreasing $k$ in two phases work in exactly the opposite directions. The net effect depends on which effect dominates.\(^{29}\) When $k \in (0, k^*)$, i.e., when the initial competition between two firms is not too weak, the competition range is more important relative to the local monopoly range,\(^{30}\) thus the effect in the competition range dominates and each firm offers more contracts and quality distortions reduce in equilibrium. On the other hand, when $k \in (k^*, 4/3)$, i.e., when the initial competition between two firms is weak, the local monopoly range is relatively more important,\(^{31}\) thus the effect in the local monopoly range dominates and each firm offers fewer contracts, though the effect on quality distortions is not uniform: as $k$ decreases, there is a cutoff type, say $\tilde{\theta}$, such that when $\theta \in [0, \tilde{\theta})$, $q(\cdot)$ moves downward, while when $\theta \in (\tilde{\theta}, 1)$, $q(\cdot)$ moves slightly upward. This non-uniform effect actually makes perfect sense. When $k \in (k^*, 4/3)$, the competition is weak so the movement of the quality schedule should follow the pattern in the monopoly case. This explains why as $k$ decreases

\(^{28}\)In the limit as $k \to 0$, there is no horizontal differentiation between two brands, and the intensity of competition between two firms becomes maximal.

\(^{29}\)In terms of the rent provision schedule $y(\cdot)$, a decrease in $k$ tends to increase $y(\theta)$ in the competition range and decrease $y(\theta)$ in the local monopoly range. But $y(\cdot)$ has to be continuous at the junction of two ranges to satisfy the IC constraint.

\(^{30}\)In the limit as $k \to 0$, the local monopoly range disappears.

\(^{31}\)As $k \geq 4/3$, the competition range disappears and both firms behave as if they were local monopolists.
the quality schedule in lower type range moves downward while the schedule in higher type range remains almost unchanged – recall that in the monopoly case, as \( k \) decreases the schedule \( q(\cdot) \) in the partial coverage range moves downward (equation (28)) while it stays the same in the full coverage range (equation (31)).

Again our computations show that the effect of changing \( k \) on either \( \theta^*_{\text{M}} \) or quality distortions over the range \( k > k^* \) is not quantitatively significant. However, it is qualitatively important as it provides a “continuity” for our intuitions to work when moving from monopoly to duopoly.

From Figure 5, it is apparent that \( \theta^*_{\text{M}} - \theta^*_{\text{D}} \) is decreasing in \( k \). Thus one potentially testable empirical implication is that the impact of market structure on the number of contracts offered depends on the degree of horizontal differentiation: the smaller the \( k \), the bigger the impact of introducing competition on the number of contracts offered.

6 Extension to the \( n \)-Firm Model

In this section we extend our analysis to any arbitrary finite \( n \)-firm case. Specifically, in the horizontal dimension there are \( n \) brands owned and operated by \( n \) distinct firms \( (n \geq 2) \), the locations of which evenly split the unit circle; and each firm offers vertically differentiated products. Each firm’s objective is to maximize the profit from its own brand, given other firms’ menus of contracts. Again we look for symmetric Bertrand-Nash equilibria in which each firm offers the same menu of contracts.

An \( n \)-tuple \( (y^*(\theta), \ldots, y^*(\theta)) \) constitutes a symmetric equilibrium if, given that all other firms offer \( y^*(\theta) \) for \( \theta \in [\theta^*, 1] \), each firm’s best response is also to choose \( y_i(\theta) = y^*(\theta), \theta \in [\theta^*, 1] \).

Given that all firms other than \( i \) offer the schedule \( y^*(\theta), \theta \in [\theta^*, 1] \), it can be easily verified that firm \( i \)’s relaxed program (by ignoring the constraint of the monotonicity of \( q_i(\cdot) \)) is as follows:

\[
\begin{align*}
\max & \quad \int_{\theta_i^*}^{\tilde{\theta}} \left[ \theta q_i(\theta) - y_i(\theta) - c(q_i(\theta)) \right] \frac{y_i(\theta)}{k} d\theta \\
& \quad + \int_{\tilde{\theta}}^{1} \left[ \theta q_i(\theta) - y_i(\theta) - c(q_i(\theta)) \right] \cdot \left[ \frac{1}{2n} + \frac{1}{2k}(y_i(\theta) - y^*(\theta)) \right] d\theta \\
\text{s.t.} & \quad y_i(\theta) = q_i(\theta) \\
& \quad y_i(\theta^*_i) = 0, \quad \theta^*_i \text{ free} \\
& \quad y_i(\tilde{\theta}) = \frac{k}{n} - y^*(\tilde{\theta}), \quad \tilde{\theta} \text{ free} \\
& \quad y_i(1) \text{ free}
\end{align*}
\]

\[32 \text{ As a direct consequence each firm is effectively competing with two adjacent firms.}\]
Applying Lemma 2, firm \(i\)'s optimal rent provision \(y^*(\theta)\) in the local monopoly range \((\theta < \hat{\theta})\) is the same as that in the duopoly model which is independent of \(n\). The optimal rent provision in the competition range \((\theta > \hat{\theta})\) and the optimal switching point \(\hat{\theta}\) are characterized by the following system:

\[
\begin{align*}
  y'' &= 2 - \frac{n}{2}(\theta y' - y - \frac{1}{2}y^2) \\
  y(\hat{\theta}) &= \frac{k}{2n} \\
  y'(\hat{\theta}) &= \sqrt{\frac{3k}{2n}} \\
  y'(1) &= 1
\end{align*}
\]

If we define \(k' = k/n\) as the normalized degree of horizontal differentiation, then by inspection, in terms of \(k'\) the differential equation system (48) is exactly the same as the differential equation system (47) in the duopoly case (where \(k' = k/2\)). This implies that the analysis of the \(n\)-firm case can be translated into the analysis of the duopoly case through normalizing \(k\) by \(n\), and in terms of \(k'\) the solution to the \(n\)-firm model is the same as the solution to the duopoly model. Thus all the results from the duopoly model carry over to the \(n\)-firm competitive model. In particular, the \(n\)-firm competitive model has a unique symmetric equilibrium, and such equilibrium exhibits perfect sorting, hence the participation threshold \(\theta^*\) becomes a measure for the number of contracts offered by each firm.\(^{33}\) Moreover, the effect of an increase in \(n\) (while holding \(k\) fixed) on the equilibrium is exactly the same as the effect of a decrease in \(k\) on the duopoly equilibrium. To re-state the results in the duopoly case in terms of \(k'\), let’s define \(k^* = k/k' \approx .455\). Then as \(k'\) increases, for \(k' < k^*\), \(\theta^*\) increases and \(q(\cdot)\) decreases, for \(k^* < k' < 2/3\), \(\theta^*\) decreases while \(q(\cdot)\) increases for lower types but decreases for higher types, and for \(k' \geq 2/3\), both \(\theta^*\) and \(q(\cdot)\) are independent of \(k'\). Translating this into \(n\)-firm case, we have the following result:

**Proposition 6** Fix \(k > 0\) and define \(n^* = k/k^*\). When \(n > n^*\), an increase in \(n\) leads to more contracts offered by each firm and smaller quality distortions; when \(n \in (1.5k, n^*)\), an increase in \(n\) leads to fewer contracts offered by each firm, and larger quality distortions for lower types and smaller quality distortions for higher types; when \(n \leq 1.5k\), each firm is a local monopolist, hence the number of contracts and quality distortions are independent of \(n\).\(^{33}\)

---

\(^{33}\)In Gal-Or’s (1983) quantity-setting model, symmetric Cournot equilibria may exist when the number of firms is small, but may fail to exist as the number of firms becomes larger. In contrast, in our model the symmetric Bertrand-Nash equilibrium always exists and is unique.
Proposition 6 thus implies that the effect of increasing competition on the number of contracts offered or quality distortions depends on the initial state of competition, and that effect is not monotonic. This result is consistent with a recent empirical study of cellular phone markets by Seim and Viard (2004), which we briefly discuss below.

Nonlinear pricing is a standard practice for cellular firms to sort consumers with respect to their different usage. Most wireless service is sold under “three-part tariffs”: monthly fee, peak minutes, and off-peak minutes. Seim and Viard (2004) study the effect of entry on the tariff choices (the number of contracts) of incumbent cellular firms. Before 1996, most geographic cellular market areas (CMA’s) had a duopoly market structure, with two firms operating the wireless service in each CMA. After the FCC auctioned off the PCS spectrum, PCS entrants began to enter the cellular markets. Due to some exogenous reasons, by 1998 there were significant variations in the amount of entry by PCS providers across cellular markets.34

Utilizing this heterogeneity across different cellular markets, Seim and Viard test the relationship between the number of calling plans offered by the incumbent providers and the number of entrants. They found that, generally speaking, incumbents introduce more calling plans in markets with more entrants. Moreover, they found that the relationship is not monotonic: if only one entrant enters, the incumbent duopolists actually reduce the number of calling plans; if more than one entrant enters, the number of calling plans offered by the incumbents increases as the number of entrants increases. They also show that this relationship cannot be explained by demographic heterogeneity or cost differences across markets. This non-monotonic relationship is consistent with the predictions in our Proposition 6.

7 Conclusion

Our paper is the first in the nonlinear pricing literature to consider a market with both horizontally and vertically differentiated products while allowing for (1) competition between firms, and (2) incomplete market coverage in both horizontal and vertical dimensions. Since the set of consumers served is endogenously determined, our setup enables us to study the effects of the horizontal differentiation (competition) on the number of contracts offered by each firm and quality distortions.

Specifically, using a two-phase optimal control technique, we characterize the unique symmetric

34This was due to two factors. First, some licenses were undeveloped because the winning bidders went bankrupt. Second, different cellular markets require different amounts of time to build a sufficiently large network of wireless infrastructure.
optimal menu of contracts in the monopoly benchmark and the unique symmetric equilibrium menu of contracts in the duopoly model. We show that when moving from monopoly to duopoly each brand offers more contracts targeting more (vertical) types of consumers, and the quality distortions become smaller. We then extend our analysis to the arbitrary $n$-firm case, and show that the major insights obtained from the base model continue to hold. In particular, we show that as long as the competition among firms is not too weak, further increasing competition (i.e., increasing the number of firms) leads to more contracts offered by each firm and smaller quality distortions.

Our results have empirical relevance regarding how competition affects the variety of goods, services, or contracts offered by firms targeting different types of consumers. The predictions of our model are largely consistent with some existing empirical evidence.

One restriction in this paper is that we assume a uniform distribution of consumers’ vertical types. While we maintain this assumption for ease of analysis, we believe that it is not crucial for our main results to hold. The reason is as follows. If we work with some other distributions instead of the uniform distribution, we may end up with partial pooling in equilibrium. However, in that case Lemma 1 and hence the screening condition (5) still hold.\textsuperscript{35} Thus the same insight regarding the interplay between horizontal differentiation and screening in the vertical dimension continues to apply; for example, in the case that the initial competition is not too weak, as competition increases, the IC constraint relaxes and we conjecture that the range of partial pooling shrinks and the participation threshold moves downward, which in turn implies that more contracts will be offered targeting more consumer types. It would be worthwhile to work out a detailed model to confirm this insight, which is left for future research.

\textsuperscript{35}Note that the equilibrium rent provision formula (5) holds as long as $q'(\cdot) > 0$, which encompasses the case of partial pooling.
Appendix

Proof of Lemma 2:

The idea is to work with a comparison path to construct the first variation of the functional (10). This variation is then used to derive the desired necessary conditions for optimality.

We look for functions \( q^*(\theta), y^*(\theta) \) and \( \theta_0, \theta_1 \) to maximize (10) subject to (11)-(14). Suppose that \( F_1, F_2, f_1, f_2 \) are continuously differentiable in all arguments. First let’s fix the notation:

- \( q^* \): the optimal control function over \( \theta_0 \leq \theta \leq \theta_2 \) with \( \theta_1 \) being the switching point moving from \( F_1 \) to \( F_2 \).
- \( y^* \): the corresponding optimal state variable satisfying (11)-(14).
- \( J^* \): the value achieved in (10) when evaluated along \( q^*, y^* \).
- \( \delta \theta_0 \) and \( \delta \theta_1 \): small changes in \( \theta_0 \) and \( \theta_1 \), respectively.
- \( q \): a feasible control over \( \theta_0 + \delta \theta_0 \leq \theta \leq \theta_2 \) with switching point at \( \theta_1 + \delta \theta_1 \).
- \( \delta q \): \( q^* - q \).
- \( y \): the feasible state which corresponds to \( q \).
- \( J \): the value of (10) when evaluated along the comparison path.

Finally, let \( \lambda(\theta) \) be the multiplier associated with the state variable. Then

\[
J - J^* = \int_{\theta_0 + \delta \theta_0}^{\theta_1 + \delta \theta_1} F_1(\theta, y, q) \, d\theta + \int_{\theta_1 + \delta \theta_1}^{\theta_2} F_2(\theta, y, q) \, d\theta - \left[ \int_{\theta_0}^{\theta_1} F_1(\theta, y^*, q^*) \, d\theta + \int_{\theta_1}^{\theta_2} F_2(\theta, y^*, q^*) \, d\theta \right]
\]

\[
= \int_{\theta_0 + \delta \theta_0}^{\theta_1 + \delta \theta_1} \left[ F_1(\theta, y, q) + \lambda f_1(\theta, y, q) - \lambda y^* \right] \, d\theta + \int_{\theta_1 + \delta \theta_1}^{\theta_2} \left[ F_2(\theta, y, q) + \lambda f_2(\theta, y, q) - \lambda y^* \right] \, d\theta
\]

\[
- \left[ \int_{\theta_0}^{\theta_1} \left[ F_1(\theta, y^*, q^*) + \lambda f_1(\theta, y^*, q^*) - \lambda y^* \right] \, d\theta + \int_{\theta_1}^{\theta_2} \left[ F_2(\theta, y^*, q^*) + \lambda f_2(\theta, y^*, q^*) - \lambda y^* \right] \, d\theta \right].
\]

Let \( h(\theta) = y(\theta) - y^*(\theta) \), and define \( \delta J \) as the first variation of \( J - J^* \). Following the derivations paralleling those in Amit (the proof of Theorem 1), we have

\[
\delta J = \lambda(\theta_0) h(\theta_0) - F_1^*(\theta_0) \delta \theta_0 + F_1^*(\theta_1) \delta \theta_1 - F_2^*(\theta_1) \delta \theta_1
\]

\[
+ \int_{\theta_0}^{\theta_1} \left[ \left( \frac{\partial F_1^*}{\partial y} + \lambda \frac{\partial f_1^*}{\partial y} + \lambda' \right) h + \left( \frac{\partial F_1^*}{\partial q} + \lambda \frac{\partial f_1^*}{\partial q} \right) \delta q \right] \, d\theta
\]

\[
+ \int_{\theta_1}^{\theta_2} \left[ \left( \frac{\partial F_2^*}{\partial y} + \lambda \frac{\partial f_2^*}{\partial y} + \lambda' \right) h + \left( \frac{\partial F_2^*}{\partial q} + \lambda \frac{\partial f_2^*}{\partial q} \right) \delta q \right] \, d\theta
\]

\[
- \lambda(\theta_1^-) h(\theta_1^-) - \lambda(\theta_1^+) h(\theta_1^+) - \lambda(\theta_2) h(\theta_2).
\]

(49)
Define the following:

\[
\begin{align*}
\delta y_0 &= y(\theta_0 + \delta \theta_0) - y^*(\theta_0) \\
\delta y_1 &= y(\theta_1 + \delta \theta_1) - y^*(\theta_1) \\
\delta y_2 &= y(\theta_2 + \delta \theta_2) - y^*(\theta_2).
\end{align*}
\]

Then since

\[
\begin{align*}
y(\theta_0 + \delta \theta_0) &\approx y(\theta_0) + y^*(\theta_0)\delta \theta_0 \\
y(\theta_1 + \delta \theta_1) &\approx y(\theta_1) + y^*(\theta_1^-)\delta \theta_1 \text{ if } \delta \theta_1 < 0 \\
y(\theta_1 + \delta \theta_1) &\approx y(\theta_1) + y^*(\theta_1^+)\delta \theta_1 \text{ if } \delta \theta_1 > 0.
\end{align*}
\]

We have

\[
\begin{align*}
\delta y_0 &\approx y^*(\theta_0)\delta \theta_0 + h(\theta_0) \\
\delta y_1 &\approx y^*(\theta_1^-)\delta \theta_1 + h(\theta_1^-) \text{ if } \delta \theta_1 < 0 \\
\delta y_1 &\approx y^*(\theta_1^+)\delta \theta_1 + h(\theta_1^+) \text{ if } \delta \theta_1 > 0.
\end{align*}
\]

Therefore,

\[
\begin{align*}
h(\theta_0) &\approx \delta y_0 - y^*(\theta_0)\delta \theta_0 = -y^*(\theta_0)\delta \theta_0 \quad (50) \\
h(\theta_1^-) &\approx \delta y_1 - y^*(\theta_1^-)\delta \theta_1 = [R'(\theta_1^-) - y^*(\theta_1^-)]\delta \theta_1 \quad (51) \\
h(\theta_1^+) &\approx \delta y_1 - y^*(\theta_1^+)\delta \theta_1 = [R'(\theta_1^+) - y^*(\theta_1^+)]\delta \theta_1 \quad (52) \\
h(\theta_2) &= y(\theta_2) - y^*(\theta_2) = \delta y_2. \quad (53)
\end{align*}
\]

Substituting (50)-(53) into (49), we have

\[
\delta J = -[F_1^*(\theta_0) + \lambda(\theta_0)y^*(\theta_0)]\delta \theta_0 \\
+ \int_{\theta_0}^{\theta_1} \left[ \left( \frac{\partial F_1^*}{\partial y} + \lambda \frac{\partial f_1^*}{\partial y} + \lambda' \right) h + \left( \frac{\partial F_1^*}{\partial q} + \lambda \frac{\partial f_1^*}{\partial q} \right) dq \right] d\theta \\
+ \int_{\theta_1}^{\theta_2} \left[ \left( \frac{\partial F_2^*}{\partial y} + \lambda \frac{\partial f_2^*}{\partial y} + \lambda' \right) h + \left( \frac{\partial F_2^*}{\partial q} + \lambda \frac{\partial f_2^*}{\partial q} \right) dq \right] d\theta \\
+ [F_1^*(\theta_1) + \lambda(\theta_1^-)y^*(\theta_1^-) - \lambda(\theta_1^-)R'(\theta_1^-)] \\
- (F_2^*(\theta_1) + \lambda(\theta_1^+)y^*(\theta_1^+) - \lambda(\theta_1^+)R'(\theta_1^+))]\delta \theta_1 - \lambda(\theta_2)\delta y_2. \quad (54)
\]
(54) represents the first variation based on which the necessary conditions for the maximization problem (10)-(14) can be derived. Let

\[-\lambda' = \frac{\partial F_1^*}{\partial y} + \lambda \frac{\partial f_1^*}{\partial y} = \frac{\partial H_1}{\partial y} \quad \text{for } \theta_0 \leq \theta < \theta_1\]  

\[-\lambda' = \frac{\partial F_2^*}{\partial y} + \lambda \frac{\partial f_2^*}{\partial y} = \frac{\partial H_2}{\partial y} \quad \text{for } \theta_1 < \theta \leq \theta_2.\]  

(55)  

(56)

It is possible that \(\delta \theta_0 = \delta \theta_1 = \delta y_2 = 0\). Then (54) reduces to

\[\delta J = \int_{\theta_0}^{\theta_1} \left( \frac{\partial F_1^*}{\partial q} + \lambda \frac{\partial f_1^*}{\partial q} \right) \delta q d\theta + \int_{\theta_1}^{\theta_2} \left( \frac{\partial F_2^*}{\partial q} + \lambda \frac{\partial f_2^*}{\partial q} \right) \delta q d\theta.\]

From this we conclude that for optimality, the following conditions must hold:

\[\frac{\partial F_1^*}{\partial q} + \lambda \frac{\partial f_1^*}{\partial q} = \frac{\partial H_1}{\partial q} = 0, \quad \text{for } \theta_0 \leq \theta \leq \theta_1\]  

\[\frac{\partial F_2^*}{\partial q} + \lambda \frac{\partial f_2^*}{\partial q} = \frac{\partial H_2}{\partial q} = 0, \quad \text{for } \theta_1 \leq \theta \leq \theta_2.\]  

(57)  

(58)

Substituting (55)-(58) into (54), we have

\[\delta J = -[F_1^*(\theta_0) + \lambda(\theta_0)y'\theta_0)]\delta \theta_0 - \lambda(\theta_2)\delta y_2 + \theta_1^*(\theta_1)\lambda(\theta_1)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^*\]

\[= \left( [H_1(\theta_1) - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^*] \right) \delta \theta_1.\]  

(59)

If \(\delta \theta_1\) is freely variable, which occurs when the optimal solution involves \(\theta_0 < \theta_1 < \theta_2\), then nonpositivity of (59) is ensured only if

\[H_1(\theta_1) - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^*.\]

If feasible modifications are \(\delta \theta_1 \geq 0\), which occurs when the optimal solution involves \(\theta_0 = \theta_1 < \theta_2\), then nonpositivity of (59) is ensured only if

\[H_1(\theta_1) - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^* - \lambda(\theta_1^*)\theta_1^*.\]  

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If feasible modifications are \( \delta \theta_1 \leq 0 \), which occurs when the optimal solution involves \( \theta_0 < \theta_1 = \theta_2 \), then nonpositivity of (59) is ensured only if
\[
H_1(\theta_1) - \lambda(\theta_1^-) R'(\theta_1^-) \geq H_2(\theta_1) - \lambda(\theta_1^+) R'(\theta_1^+).
\]

We have thus derived all the necessary conditions in Lemma 2. ■

Proof of Proposition 3:

Following the derivations preceding to the Proposition, the proof will be completed by showing that \( \forall k \in (0, 4/3] \), there is a unique \( \hat{\theta} \in (0, 1] \) and a unique \( y(\theta) \) defined over \( [\hat{\theta}, 1] \) satisfying the differential equation system (47). Moreover, the solution of \( y(\theta) \) is strictly convex.

First letting \( z(\theta) = y(\theta) - \frac{1}{2} \theta^2 \), we have
\[
z''(\theta) = 1 + \frac{1}{k}(z'(\theta)^2 + 2z(\theta)).
\]
(60)

Let \( z'(\theta) = v(z(\theta)) \), then \( z''(\theta) = v'(z)z'(\theta) = vv'(z) \). (60) thus becomes:
\[
v \frac{dv}{dz} = 1 + \frac{1}{k}(v^2 + 2z).
\]
(61)

Substituting \( w(z) = v^2(z) \) into (61), we have
\[
w' - \frac{2}{k}w = 2 + \frac{4}{k}z.
\]
which leads to
\[
w(z) = ce^{2z/k} - 2z - 2k.
\]
where \( c \) is a parameter to be determined by the boundary conditions.

The system (47) can now be written in terms of function \( z(\theta) \) as follows:
\[
(z'(\theta))^2 = ce^{2z(\theta)/k} - 2z(\theta) - 2k
\]
\[
z(\hat{\theta}) = \frac{k}{4} - \frac{1}{2} \hat{\theta}^2 := \hat{z}
\]
\[
z'(\hat{\theta}) = \frac{\sqrt{3k}}{2} - \hat{\theta}
\]
(62)
\[
z'(1) = 0.
\]

Define \( \alpha \) such that \( c = kae^{-2z/k} \), and \( \delta \) such that \( \hat{\theta} = \frac{\sqrt{3k}}{2} \delta \) \( \hat{\theta} \in (0, 1] \) implies \( \delta \in (0, 2/\sqrt{3k}) \).

Also define
\[
u(\theta) = \frac{2}{k}(z(\theta) - \hat{z}).
\]
Then we have
\[ u'^2 = \frac{4}{k^2} z'^2 = \frac{4}{k^2} (kae^u - 2z - 2k) = \frac{4}{k} \left( \alpha e^u - u - \frac{2}{k} \hat{z} - 2 \right). \]

Letting \( f(u) = \alpha (e^u - 1) - u + \beta \), where \( \beta = \alpha - \frac{2}{k} \hat{z} - 2 \), then
\[ u'^2 = \frac{4}{k} f(u). \]

At \( \hat{\theta} \), \( u(\hat{\theta}) = 0 \), \( u'(\hat{\theta}) = \sqrt{3/k(1 - \delta)} \), hence
\[ \beta = \frac{k}{4} u''(\hat{\theta}) = \frac{3}{4} (1 - \delta)^2 \quad (63) \]
\[ \alpha = \beta + \frac{2}{k} \hat{z} + 2 = \frac{13}{4} - \frac{3}{2} \delta. \quad (64) \]

The system (62) can now be rewritten as follows:
\[ u'^2 = \frac{4}{k} f(u) \quad (65) \]
\[ u(\hat{\theta}) = 0 =: \hat{u} \quad (66) \]
\[ u'(\hat{\theta}) = \sqrt{3/k(1 - \delta)} := \hat{u}' \quad (67) \]
\[ u'(1) = 0 := u'_1 \quad (68) \]

where
\[ f(u) = \alpha (e^u - 1) - u + \beta = \left( \frac{13}{4} - \frac{3}{2} \delta \right) (e^u - 1) - u + \frac{3}{4} (1 - \delta)^2. \quad (69) \]

For notational convenience let \( u_1 := u(1) \). Then \( u'_1 = 0 \Rightarrow f(u_1) = 0 \).

First, from (67)-(68) it can be verified that \( \hat{\theta} = 1 \Rightarrow k = 4/3 \). So for \( k \in (0, 4/3) \) we must have \( \hat{\theta} < 1 \), or \( \delta < 2/\sqrt{3k} \).

The rest of the proof is completed in 6 steps:

1. Show that (65) implies \( u' = -\frac{2}{\sqrt{k}} \sqrt{f(u)} \) and \( \delta \geq 1 \).

Suppose not, then \( u' = \frac{2}{\sqrt{k}} \sqrt{f(u)} \geq 0 \). By (67) \( \delta \leq 1 \), and \( \alpha \geq 7/4 \), which implies \( f'(u) = \alpha e^u - 1 \geq 7/4 - 1 > 0 \) for all \( u \geq 0 \). But then \( f(u_1) > f(\hat{u}) = f(0) = \beta \geq 0 \), a contradiction. Therefore we must have \( u' = -\frac{2}{\sqrt{k}} \sqrt{f(u)} \leq 0 \) and hence \( \delta \geq 1 \). Since \( u \) is decreasing, we have \( u_1 \leq \hat{u} = 0 \). It can be verified that for \( k \in (0, 4/3) \), \( u_1 = \hat{u} = 0 \) is impossible.\(^{36} \)

Hence \( u_1 < \hat{u} = 0 \) for \( k \in (0, 4/3) \), and \( f(u) \geq 0 \) on \( [u_1, 0] \).

\(^{36} u_1 = \hat{u} \Rightarrow u = 0 \), which implies \( z = \hat{z} \) and \( \hat{\theta} = \sqrt{\frac{3k}{4}} \). Therefore \( y(\theta) = \frac{1}{4} \theta^2 + \hat{z} = \frac{1}{4} \theta^2 + \frac{k}{4} - \frac{1}{2} \hat{\theta}^2 = \frac{1}{2} \theta^2 - \frac{k}{4} \). But then \( y(\theta) \) does not satisfy the differentiation equation in system (47), a contradiction.
2. Show that in the solution to system (65)-(68), \( \alpha > 0 \), which implies that the original solution \( y(\cdot) \) is strictly convex.

Suppose not, i.e., suppose \( \alpha \leq 0 \). Then \( f'(u) = \alpha e^u - 1 < 0 \), which implies that \( f(\hat{u}) < f(u_1) = 0 \). But \( f(\hat{u}) = \alpha(e^{\hat{u}} - 1) - \hat{u} + \beta = \beta \geq 0 \), contradiction. So \( \alpha > 0 \).

Since
\[
y'' = 1 + z'' = \frac{1}{k} e^{2z/k} = \frac{\alpha e^{2(z^2 - \hat{z})/k}}{k},
\]
\( \alpha > 0 \) (or \( \delta < 13/6 \)) implies that the original solution \( y(\cdot) \) must be strictly convex.

3. Show that given \( \delta \) (or \( \hat{\theta} \)), the solution of \( u(\cdot) \) (and hence \( y(\cdot) \)) exists and is unique.

Since \( f''(u) = \alpha e^u > 0 \), \( f \) is strictly convex (with \( f(\pm \infty) = \infty \)). Hence \( f(u) > 0 \) on \((u_1, 0] \).

\[
f'(u) = \alpha e^u - 1 = 0 \Rightarrow u_{\min} = -\ln \alpha.
\]

Let \( A(\delta) = \min f(u) = f(-\ln \alpha) = \ln \alpha + \frac{3}{4}(\delta^2 - 2) \). Since \( f(u_1) = 0 \), we must have \( A(\delta) \leq 0 \).

We next show that \( A(\delta) < 0 \). Suppose not, then \( u_1 = u_{\min} = -\ln \alpha < 0 \), which implies
\[
f(u) \approx a(u - u_1)^2 \text{ near } u_1, \text{ where } a \text{ is a positive real number.}
\]

\[
\int_{u_1}^{0} \frac{du}{\sqrt{f(u)}} = -\frac{2}{\sqrt{k}} \int_{1}^{\hat{\theta}} d\theta = \frac{2}{\sqrt{k}}(1 - \hat{\theta}) < \infty.
\]  \((70)\)

But on the other hand,
\[
\int_{u_1}^{0} \frac{du}{\sqrt{f(u)}} = -\frac{1}{\sqrt{a}} \int_{u_1}^{\hat{u}} \frac{du}{u - u_1} = \infty,
\]
a contradiction.

Therefore \( A(\delta) < 0 \) and hence in the neighborhood of \( u_1 \), \( f(u) = O(u - u_1) \).

Define
\[
\Phi(u) = \int_{0}^{u} \frac{dv}{\sqrt{f(v)}} = \int_{\hat{\theta}}^{\theta} -\frac{2}{\sqrt{k}} ds = -\frac{2}{\sqrt{k}}(\theta - \hat{\theta}).
\]

Note that \( \Phi(u) \) is well defined for any \( u \in [u_1, 0] \), as \( f(u) = O(u - u_1) \) near \( u_1 \) (which implies \( |\int_{0}^{u_1} \frac{dv}{\sqrt{f(v)}}| < \infty \)).

Since \( \Phi(u) \) is a strictly increasing function over \([u_1, 0] \), inverting we have
\[
u(\theta) = \Phi^{-1}\left(-\frac{2}{\sqrt{k}}(\theta - \hat{\theta})\right) \text{ for } \theta \in [\hat{\theta}, 1].
\]  \((71)\)

Thus given \( \hat{\theta} \), \( u(\cdot) \) (and hence \( y(\cdot) \)) is uniquely determined by \((71) \). It remains to show that \( \hat{\theta} \) (or \( \delta \)) exists and is unique.
4. Show that in the solution \( \delta \in [1, \min\{\delta_0, 2/\sqrt{3k}\}) \) (where \( \delta_0 \) is defined below).

Since \(- \ln \alpha = u_{\min} < u_1 < 0\), we have \( \alpha > 1 \) or \( \delta < \frac{3}{2} \). We thus have \( \delta \in [1, \frac{3}{2}] \) (from step 1). It is straightforward to verify that \( A(\delta) \) is strictly increasing over the interval \([1, \frac{3}{2}]\) and there is a unique \( \delta_0 \in [1, \frac{3}{2}] \) such that \( A(\delta_0) = 0 \). Since \( A(\delta) < 0 \), we thus have \( \delta \in [1, \delta_0) \). Combining this with \( \delta < 2/\sqrt{3k} \), in the solution to the system (65)-(68) we must have \( \delta \in [1, \min\{\delta_0, 2/\sqrt{3k}\}) \).

By (70) we have \( \int_{u_1}^0 \frac{du}{\sqrt{f(u)}} = \frac{2}{\sqrt{k}} (1 - \hat{\theta}) = \frac{2}{\sqrt{k}} - \sqrt{3} \delta \).

Define

\[
\xi(\delta) = \sqrt{3} \delta + \int_{u_1}^0 \frac{du}{\sqrt{f(u)}}. \tag{72}
\]

5. Show that given any \( k \in (0, \frac{4}{3}) \), there is a \( \delta \in (1, \min\{\delta_0, 2/\sqrt{3k}\}) \) satisfying \( \xi(\delta) = \frac{2}{\sqrt{k}} \).

First \( f - \beta + u + \alpha = \alpha e^u \) implies \( (f - \beta + u + \alpha)' = f - \beta + u + \alpha \). That is, \( f' - f - u = \text{constant} = f'(u_1) - f(u_1) - u_1 = f'(u_1) - u_1 \).

Hence \( f' - f - (u - u_1) = f'(u_1) > 0 \), which leads to

\[
f'(u_1) \int_{u_1}^0 \frac{1}{\sqrt{f}} du = \int_{u_1}^0 \frac{f'}{\sqrt{f}} du - \int_{u_1}^0 \sqrt{f} du = \int_{u_1}^0 \frac{u - u_1}{\sqrt{f}} du. \tag{73}
\]

Define \( \xi_1(\delta) = \int_{u_1}^0 \sqrt{f} du \), and \( \xi_2(\delta) = \int_{u_1}^0 \frac{u - u_1}{\sqrt{f}} du \). Note that \( f'(u_1) = \alpha e^{u_1} - 1 > 0 \), and \( \int_{u_1}^0 \frac{f'}{\sqrt{f}} du = 2\sqrt{f(0)} = 2\sqrt{3} = \sqrt{3}(\delta - 1) \). Therefore by (73) we have

\[
\xi(\delta) = \frac{1}{\alpha e^{u_1} - 1} \left[ \sqrt{3}(\delta - 1) - \xi_1(\delta) - \xi_2(\delta) \right] + \sqrt{3} \delta. \tag{74}
\]

Since \( u_1(\delta) \) is continuous in \( \delta \), both \( \xi_1(\delta) \) and \( \xi_2(\delta) \) are also continuous in \( \delta \). Therefore, \( \xi(\delta) \) is continuous in \( \delta \).

First, consider \( \delta \to 1^+ \). It is easily verified that \( \beta \to 0^+ \), \( \alpha \to (\frac{4}{3})^- \). Hence \( f(u) \to g(u) = \frac{1}{4} (e^u - 1) - u \), and \( u_1(\delta) \to 0^- \).

By (74), \( \xi(\delta) < \frac{1}{\alpha e^{u_1} - 1} \sqrt{3}(\delta - 1) + \sqrt{3} \delta \to \sqrt{3} \). Since \( \sqrt{3} < 2/\sqrt{k} \), we have \( \xi(\delta) < 2/\sqrt{k} \) for \( \delta \) sufficiently close to \( 1^+ \).

Second, consider \( \delta \to b = \min\{\delta_0, 2/\sqrt{3k}\} \) from the left. We discuss the following two cases:

Case 1: \( \delta_0 > 2/\sqrt{3k} \). Then when \( \delta \to b^- = (2/\sqrt{3k})^- \), \( \xi(\delta) > \sqrt{3} \delta = \sqrt{3} b = 2/\sqrt{k} \) (the inequality is due to (72)).
Case 2: \( \delta_0 \leq 2/\sqrt{3k} \). For \( \delta \to b^- = \delta_0^- \), \( A(\delta) = f_{\text{min}} \to 0^- \) (since \( A(\delta_0) = 0 \)). So \( u_1 \to (-\ln \alpha)^+ \), and by (72), \( \xi(\delta) \to \infty \). So when \( \delta \to b^- \), \( \xi(\delta) > 2/\sqrt{k} \).

By the mean-value theorem, there exists \( \delta \in (1, \min\{\delta_0, 2/\sqrt{3k}\}) \) such that \( \xi'(\delta) = \frac{2}{\sqrt{k}} \).

6. Show that the solution from step 5 is unique.

We have

\[
    f(u_1) = \alpha(e^{u_1} - 1) - u_1 + \beta = 0. \tag{75}
\]

Differentiating (75) with respect to \( \delta \), we have

\[
    -\frac{3}{2}(e^{u_1} - 1) + \frac{3}{2}(\delta - 1) + (\alpha e^{u_1} - 1)u_1' = 0.
\]

which gives

\[
    u_1' = \frac{3}{2}(e^{u_1} - \delta) \quad \text{and} \quad \frac{3}{2}(\delta - 1) + (\alpha e^{u_1} - 1)u_1' = 0.
\]

By (74),

\[
    \xi_1' = \frac{d\xi_1}{d\delta} = \frac{d\xi_1}{du_1} \frac{du_1}{d\delta} = 0 \cdot \frac{du_1}{d\delta} = 0
\]

\[
    \xi_2' = \frac{d\xi_2}{du_1} u_1' = 0 + \int_{u_1}^0 \frac{-1}{\sqrt{f}} du_1' = -(\xi - \sqrt{3} \xi_2' u_1').
\]

So

\[
    \xi' = \sqrt{3} + \frac{1}{\eta}(\sqrt{3} - \xi_1 - \xi_2) - \frac{1}{\eta^2} \left(\alpha e^{u_1} u_1' - \frac{3}{2} e^{u_1}\right) \left[\sqrt{3}(\delta - 1) - \xi_1 - \xi_2\right]
\]

\[
    = \sqrt{3} + \frac{1}{\eta}[\sqrt{3} + (\xi_1 - \sqrt{3} \delta) u_1'] + \frac{e^{u_1}(\alpha e^{u_1} - 1)}{\eta}(\xi - \sqrt{3} \delta)
\]

\[
    = \sqrt{3} + \frac{1}{\eta}\sqrt{3} + \frac{1}{\eta}(\xi - \sqrt{3} \delta) \left[u_1'(1 - \alpha e^{u_1}) + \frac{3}{2} e^{u_1}\right]
\]

\[
    = \sqrt{3} + \frac{1}{\eta}\sqrt{3} + \frac{1}{\eta}(\xi - \sqrt{3} \delta) \frac{3}{2} \delta
\]

\[
    > 0 \quad \text{(since} \quad \xi - \sqrt{3} \delta = \int_{u_1}^0 \frac{du}{\sqrt{f(u)}} > 0 \).}
\]

Therefore \( \xi(\delta) \) is strictly increasing in \( \delta \in (1, \min\{\delta_0, 2/\sqrt{3k}\}) \), which implies that there is a unique \( \delta \) satisfying \( \xi(\delta) = \frac{2}{\sqrt{k}} \).
Proof of Proposition 4:

Suppose \( \theta^* \leq \theta^* \). Since \( \hat{\theta}^D - \hat{\theta}^M = \hat{\theta}^M - \hat{\theta}^M = \sqrt{\frac{k}{3}} \), \( \hat{\theta}^M \leq \hat{\theta}^D \). By (28) and (40) we have

\[
q_M(\hat{\theta}^M) = q_D(\hat{\theta}^D) = \sqrt{3k}/2
\]

From (31),

\[
q_M'(\theta) = 2 \quad \text{for} \quad \theta \in [\hat{\theta}^M, \hat{\theta}^D]
\]

\[
\Rightarrow q_M(\hat{\theta}^D) \geq q_D(\hat{\theta}^D)
\]

(76)

From (41),

\[
q_D'(\theta) = 2 - \frac{2}{k} \left[ \theta y' - y - \frac{1}{2} y'^2 \right]
\]

In equilibrium, a firm’s profit from a type \( \theta \) consumer is positive for \( \theta > \theta^* \), i.e. \( \theta y' - y - \frac{1}{2} y'^2 > 0 \) hence

\[
q_M'(\theta) = 2 > q_D'(\theta) \quad \text{for} \quad \theta \in [\hat{\theta}^D, 1]
\]

(77)

Note that \( k \in \left(0, \frac{4}{3}\right) \) implies \( \hat{\theta}^D < 1 \). Combining this with (76) and (77), we have \( q_M(1) > q_D(1) \), which contradicts the fact that \( q_M(1) = q_D(1) = 1 \). Therefore \( \theta^M > \theta^D \) in equilibrium.

To show that \( q_D(\theta) > q_M(\theta) \), we consider the following cases:

For \( \theta \in [\theta^*, \hat{\theta}^D] \), by (28) and (40) we have \( q_D(\theta) > q_M(\theta) \) as \( \theta^* < \theta^M \).

For \( \theta \in (\hat{\theta}^D, \hat{\theta}^M] \), \( q_D(\theta) > q_M(\theta) = \sqrt{3k}/2 \), and \( q_M(\theta) \leq q_M(\hat{\theta}^M) = \sqrt{3k}/2 \). Hence \( q_D(\theta) > q_M(\theta) \).

For \( \theta \in (\hat{\theta}^M, 1) \), \( q_D'(\theta) < 2 = q_M'(\theta) \) and \( q_M(1) = q_D(1) \) implies that \( q_D(\theta) > q_M(\theta) \).

To sum up, \( q_D(\theta) > q_M(\theta) \), hence \( \theta - q_D(\theta) < \theta - q_M(\theta) \) for \( \theta \in [\theta^D, 1) \), which implies that quality distortion is smaller in the duopoly case.

That the market coverage is larger in the duopoly case is a direct consequence from (1) \( \hat{\theta}^D < \hat{\theta}^M \), and (2) \( y_D(\theta) > y_M(\theta) \). This is visualized by Figure 3.
References


