Discrete-time Dynamic Optimization

Consider the problem:

$$\underset{u_t}{Max} \sum_{t=0}^{T} \beta^t r(x_t, u_t) = r(x_0, u_0) + \beta r(x_1, u_1) + \cdots + \beta^T r(x_T, u_T)$$

subject to: $x_{t+1} = g(x_t, u_t)$, x_0 given.

x_t: state variable.
u_t: control variable
β^t: discount factor
$r(x_t, u_t)$: single-period reward function
$g(x_t, u_t)$: state transition function

Classical Optimization Techniques

Example: The Ramsey Problem

$$\underset{c_t}{Max} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to: $k_{t+1} = f(k_t) - c_t$ and $k_{T+1} \geq 0$

Form the Lagrangian:

$$\mathcal{L} = u(c_0) + \cdots + \beta^t u(c_t) + \beta^{t+1} u(c_{t+1}) + \cdots + \beta^T u(c_T)$$
$$+ \lambda_0 [f(k_0) - c_0 - k_1] + \cdots + \lambda_t [f(k_t) - c_t - k_{t+1}] + \lambda_T k_{T+1}$$

First-order conditions include:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{\partial u}{\partial c_t} - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = \beta^{t+1} \frac{\partial u}{\partial c_{t+1}} - \lambda_{t+1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t + \lambda_{t+1} f'(k_{t+1}) = 0$$
$$\lambda_T \geq 0, \quad k_{T+1} \geq 0, \quad \lambda_T k_{T+1} = 0$$

Sustituting for the λ terms, we obtain:

$$1 = \beta \frac{u'(c_{t+1})}{u'(c_t)} f'(k_{t+1})$$

Euler Equation: difference equation system in c_t, k_t
Initial condition: k_0.
Final value condition: $k_{T+1} = 0$

Infinite Horizon Case

Now suppose that $T \rightarrow +\infty$
If a steady-state exists, then $c_{t+1} = c_t$, and so:

$$\lim_{t\to\infty} k(t) = \bar{k}$$
$$\lim_{t\to\infty} c(t) = \bar{c} = f(\bar{k})$$
$$\beta f'(\bar{k}) = 1$$

Backwards Induction with Finite Horizon

In the finite-horizon problem, we may try another approach.
Problem:

$$\underset{c_t}{Max} \sum_{t=0}^{T} \beta^t \ln(c_t)$$

subject to: $k_{t+1} = k_t^\alpha - c_t$ and $k_{T+1} \geq 0$

Suppose we start by solving the problem at time T.

$$\underset{c_T}{Max} \ln(c_T)$$

subject to: $k_{T+1} = k_T^\alpha - c_T \geq 0$

First we trivially obtain: $c_T^* = k_T^\alpha$.
Question: What is the value of setting $c_T^* = k_T^\alpha$?
Denote this value as $V_0 = \ln(k_T^\alpha) = \alpha \ln k_T$
The zero subscript indicates zero remaining periods.
Next we pose:

$$\underset{c_{T-1}}{Max} [\ln c_{T-1} + \beta V_0], \text{ subject to:} k_T = k_{T-1}^\alpha - c_{T-1}$$

or: $$\underset{c_{T-1}}{Max} [\ln c_{T-1} + \beta\alpha \ln k_T], \text{ subject to:} k_T = k_{T-1}^\alpha - c_{T-1}$$

or: $$\underset{c_{T-1}}{Max} [\ln c_{T-1} + \beta\alpha \ln(k_{T-1}^\alpha - c_{T-1})]$$

Proceed as:

$$\frac{d[\ln c_{T-1} + \beta\alpha \ln(k_{T-1}^\alpha - c_{T-1})]}{dc_{T-1}} = 0$$

$$\frac{1}{c_{T-1}} - \frac{\alpha\beta}{k_{T-1}^\alpha - c_{T-1}} = 0$$

$$k_{T-1}^\alpha - c_{T-1} = \alpha\beta c_{T-1}$$

$$c_{T-1}^* = \frac{k_{T-1}^\alpha}{(1+\alpha\beta)}$$

$$\begin{aligned} k_T^* &= k_{T-1}^{\alpha} - c_{T-1}^* \\ &= k_{T-1}^{\alpha} - \frac{k_{T-1}^{\alpha}}{(1+\alpha\beta)} \\ &= \left[1 - \frac{1}{1+\alpha\beta}\right] k_{T-1}^{\alpha} \\ &= \frac{\alpha\beta}{1+\alpha\beta} k_{T-1}^{\alpha} \end{aligned}$$

We now derive V_1:

$$\begin{aligned} V_1 &= \ln c_{T-1}^* + \alpha\beta \ln k_T^* \\ &= \ln\left(\frac{k_{T-1}^{\alpha}}{(1+\alpha\beta)}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta} k_{T-1}^{\alpha}\right) \\ &= \alpha \ln k_{T-1} - \ln(1+\alpha\beta) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + \alpha^2\beta \ln k_{T-1} \end{aligned}$$

Simplfying:

$$V_1 = \alpha(1+\alpha\beta)\ln k_{T-1} - \ln(1+\alpha\beta) + \alpha\beta \ln\left(\frac{\alpha\beta}{(1+\alpha\beta)}\right)$$

This is of the form:

$$V_1 = \alpha(1+\alpha\beta)\ln k_{T-1} + \Omega_{T-1}$$

Next we pose:

$$\underset{c_{T-2}}{Max}\ [\ln c_{T-2} + \beta V_1], \text{ subject to:} k_{T-1} = k_{T-2}^{\alpha} - c_{T-2}$$

Rewrite as:

$$\underset{c_{T-2}}{Max}\ [\ln c_{T-2} + \beta(\alpha(1+\alpha\beta)\ln(k_{T-2}^{\alpha} - c_{T-2}) + \Omega_{T-1})]$$

First-order Condition:

$$\frac{d[\quad]}{dc_{T-2}} = \frac{1}{c_{T-2}} - \frac{\alpha\beta(1+\alpha\beta)}{k_{T-2}^{\alpha} - c_{T-2}} = 0$$

Solution is of the form:

$$\begin{aligned} c_{T-2}^* &= \frac{k_{T-2}^{\alpha}}{(1+\alpha\beta+(\alpha\beta)^2)} \\ k_{T-1}^* &= \frac{\alpha\beta+(\alpha\beta)^2}{(1+\alpha\beta+(\alpha\beta)^2)} k_{T-2}^{\alpha} \end{aligned}$$

Next:

$$\begin{aligned} V_2 &= \ln c_{T-2}^* + \beta V_1 \\ &= \ln c_{T-2}^* + \beta[\alpha(1+\alpha\beta)\ln k_{T-1}^* + \Omega_{T-1}] \end{aligned}$$

Collecting terms, we find that:

$$V_2 = \alpha(1+\alpha\beta+(\alpha\beta)^2)\ln k_{T-2} + \Omega_{T-2}$$

A pattern is emerging in which:

$$V_{T-t} = \left(\alpha \sum_{i=0}^{T-t} (\alpha\beta)^i\right) \ln k_t + \Omega_{T-t}$$

$$c_t^* = \frac{k_t^\alpha}{\sum_{i=0}^{T-t} (\alpha\beta)^i}$$

We have solved for a series of policy functions, c_t^*.
As a byproduct, we have also solved for a series of value functions, V_{T-t}.
Now consider the infinite horizon problem:

$$\underset{c_t}{Max} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to: $k_{t+1} = f(k_t) - c_t$ and $\lim_{t\to+\infty} k_t \geq 0$

We conjecture that the solution is given by:

$$c_t^* = \lim_{t\to+\infty} \frac{k_t^\alpha}{\sum_{i=0}^{T-t} (\alpha\beta)^i} = (1 - \alpha\beta) k_t^\alpha$$

$$V = \lim_{t\to+\infty} \left(\alpha \sum_{i=0}^{T-t} (\alpha\beta)^i + \Omega_{T-t}\right)$$

$$= \frac{\alpha}{(1-\alpha\beta)} \ln k_t + \text{constant}$$

Infinite Horizon Problems: A Recursive Approach

$$\underset{u_t}{Max} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

subject to: $x_{t+1} = g(x_t, u_t)$, x_0 given.

The objective may also be written out as:

$$\underset{u_t}{Max} \left[r(x_0, u_0) + \beta r(x_1, u_1) + \beta^2 r(x_2, u_2) + \cdots\right]$$

Next consider the problem:

$$\underset{u_t}{Max} \left[r(x_1, u_1) + \beta r(x_2, u_2) + \beta^2 r(x_3, u_3) + \cdots\right]$$

$$\text{subject to: } x_{t+1} = g(x_t, u_t),\ x_1 \text{ given.}$$

Now suppose that we know the solution to the second problem.
Solution consists of $\{u_1^*, u_2^*, \cdots, x_2^*, x_3^*, \cdots\}$.
Now define:

$$V(x_1) = [r(x_1, u_1^*) + \beta r(x_2^*, u_2^*) + \beta^2 r(x_3^*, u_3^*) + \cdots]$$

Conjecture that $V(\bullet)$ is time-invariant.
Next return to the original problem:

$$\underset{u_t}{Max}\ [r(x_0, u_0) + \beta r(x_1, u_1) + \beta^2 r(x_2, u_2) + \cdots]$$

This problem is characterized by time-consistency.
We can reoptimize at any time.
The remaining solution is unchanged.
The problem can be restated as:

$$\underset{u_0}{Max}\ [r(x_0, u_0) + \beta r(x_1, u_1^*|x_1) + \beta^2 r(x_2^*|x_1, u_2^*|x_1) + \cdots]$$

$$= \underset{u_0}{Max}\ [r(x_0, u_0) + \beta V(x_1)] \text{ where } x_1 = g(x_0, u_0)$$

Define the function, $h(x_0)$, as the solution, $u^*(x_0)$.
The functions V and h are related through the Bellman equation:

$$V(x) = \max_u\ \{r(x, u) + \beta V[g(x, u)]\}$$

$$= \max_u\ \{r(x, u) + \beta V[\tilde{x}]\}, \text{ where } \tilde{x} = g(x, u)$$

Solving the "max" part of the RHS requires that we set:

$$u = h(x)$$

where $h(x)$ solves the FOC of the maximization problem.
The Bellman equation is a functional equation.
The solution consists of the functions V and h.
For well-behaved problems:

1. Solution consists of a unique concave value function V.
2. Solution for V is the limit as $j \to \infty$ of:

$$V_{j+1}(x) = \max_u\ \{r(x, u) + \beta V_j(\tilde{x})\}$$

$$\text{subject to : } \tilde{x} = g(x, u),\ x \text{ given.}$$

3. The time-invariant policy function, $u_t = h(x_t)$, is unique.
4. Off corners, limiting value function is differentiable with:

$$V'(x) = \frac{\partial r}{\partial x}[x, h(x)] + \beta \frac{\partial g[x, u]}{\partial x} V'(\tilde{x})$$

where $\tilde{x} = g[x, h(x)]$

In the Bellman equation, replace u with the optimal value $h(x)$ to obtain:

$$V(x) = r[x, h(x)] + \beta V\{g[x, h(x)]\}$$

Differentiate both sides with respect to x:

$$\begin{aligned} V'(x) &= \frac{\partial r}{\partial x}[x, h(x)] + \frac{\partial r[x,u]}{\partial u}\frac{dh}{dx} \\ &+ \beta V' \bullet \left(\frac{\partial g[x,u]}{\partial x} + \frac{\partial g[x,u]}{\partial u}\frac{dh}{dx} \right) \end{aligned}$$

The terms multiplying $\frac{dh}{dx}$ must be zero by the FOC.
If $\frac{\partial g}{\partial x} = 0$, then we obtain the simpler form:

$$V'(x) = \frac{\partial r}{\partial x}[x, h(x)]$$

Guess and Verify

Must be lucky or faced with standard problem type.
Consider the deterministic Ramsey Problem:

$$\underset{c_t}{Max} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to $k_{t+1} = k_t^\alpha - c_t$ and $\lim_{t\to\infty} k_t \geq 0$

Guess $V(k_t) = \Omega + B \ln k_t$
If correct we need to choose c_t to maximize:

$$\begin{aligned} V(k_t) &= \underset{c_t}{Max}\ [\ln c_t + \beta(\Omega + B\ln k_{t+1})] \\ &= \underset{c_t}{Max}\ [\ln c_t + \beta\Omega + \beta B \ln(k_t^\alpha - c_t)] \end{aligned}$$

First-order Condition:

$$\frac{1}{c_t} - \frac{\beta B}{(k_t^\alpha - c_t)} = 0$$

Solving:

$$c_t^* = \frac{k_t^\alpha}{1 + \beta B}$$

Then:

$$\begin{aligned} k_{t+1}^* &= k_t^\alpha - c_t^* = k_t^\alpha - \frac{k_t^\alpha}{1+\beta B} \\ &= \left(1 - \frac{1}{1+\beta B}\right) k_t^\alpha \\ &= \left(\frac{\beta B}{1+\beta B}\right) k_t^\alpha \end{aligned}$$

Now plug into Bellman equation:

$$\begin{aligned} V(k_t) &= \ln c_t^* + \beta\Omega + \beta B \ln(k_{t+1}^*) \\ &= \ln\left(\frac{k_t^\alpha}{1+\beta B}\right) + \beta\Omega + \beta B \ln\left(\frac{\beta B}{1+\beta B} k_t^\alpha\right) \\ &= \beta\Omega + \beta B \ln \beta B + (1+\beta B)\ln\left(\frac{k_t^\alpha}{1+\beta B}\right) \\ &= \beta\Omega + \beta B \ln \beta B - (1+\beta B)\ln(1+\beta B) + (1+\beta B)\alpha \ln k_t \end{aligned}$$

Therefore:

$$B = \alpha(1+\beta B) \Rightarrow B = \frac{\alpha}{1-\alpha\beta}$$

$$\Omega = \beta\Omega + \beta B \ln \beta B - (1+\beta B)\ln(1+\beta B)$$

$$\Omega = \frac{1}{1-\beta}\left\{\frac{\alpha\beta}{1-\alpha\beta}\ln\left(\frac{\alpha\beta}{1-\alpha\beta}\right) + \frac{\ln(1-\alpha\beta)}{(1-\alpha\beta)}\right\}$$

$$c_t^* = \frac{k_t^\alpha}{1+\beta B} = (1-\alpha\beta)k_t^\alpha$$

Howard's Policy Improvement Algorithm

This methodology iterates on the h function.
In particular consider the case of a linear h
We generate a sequence, $\{h_0, h_1, \cdots\}$
Hope to converge.
Ramsey Example:

$$\underset{c_t}{Max} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to $k_{t+1} = k_t^\alpha - c_t$ and $\lim_{t\to\infty} k_t \geq 0$

Start with $u_t = k_{t+1} = h_0 \bullet k_t^\alpha \Rightarrow c_t = (1-h_0)k_t^\alpha$.
Next form:

$$\begin{aligned}
J_0(k_0) &= \sum_{t=0}^{\infty} \beta^t \ln[(1-h_0)k_t^\alpha] \\
&= \sum_{t=0}^{\infty} \beta^t \{\ln(1-h_0) + \alpha \ln k_0\} \\
&= \frac{\ln(1-h_0)}{1-\beta} + \sum_{t=0}^{\infty} \alpha\beta^t \ln k_t \\
&= \frac{\ln(1-h_0)}{1-\beta} + \alpha[\ln k_0 + \beta \ln k_1 + \beta^2 \ln k_2 + \cdots]
\end{aligned}$$

But recall that:

$$k_{t+1} = h_0 \bullet k_t^\alpha \Rightarrow \ln k_{t+1} = \ln(h_0) + \alpha \ln k_t.$$

Therefore:

$$\begin{aligned}
\ln k_1 &= \ln(h_0) + \alpha \ln k_0 \\
\ln k_2 &= \ln h_0 + \alpha \ln k_1 \\
&= \ln h_0 + \alpha[\ln(h_0) + \alpha \ln k_0] \\
&= (1+\alpha)\ln h_0 + \alpha^2 \ln k_0
\end{aligned}$$

Pattern becomes:

$$\ln k_t = [1 + \alpha + \alpha^2 + \cdots + \alpha^{t-1}] \ln h_0 + \alpha^t \ln k_0$$

Now recall:

$$
\begin{aligned}
J_0(k_0) &= \frac{\ln(1-h_0)}{1-\beta} + \alpha[\ln k_0 + \beta \ln k_1 + \beta^2 \ln k_2 + \cdots] \\
&= \frac{\ln(1-h_0)}{1-\beta} + \alpha \ln k_0 \\
&\quad + \alpha\beta[\ln(h_0) + \alpha \ln k_0] \\
&\quad + \alpha\beta^2[(1+\alpha)\ln h_0 + \alpha^2 \ln k_0] \\
&\quad + \alpha\beta^3[(1+\alpha+\alpha^2)\ln h_0 + \alpha^3 \ln k_0] + \cdots
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
J_0(k_0) &= \text{constant} + \alpha \ln k_0 + \alpha^2\beta \ln k_0 + \alpha^3\beta^2 \ln k_0 + \cdots \\
&= \text{constant} + \alpha[1 + \alpha\beta + (\alpha\beta)^2 + \cdots] \ln k_0 \\
&= \text{constant} + \frac{\alpha}{(1-\alpha\beta)} \ln k_0
\end{aligned}
$$

Now denote the hypothetical value function as J_0.
Bellman Equation is given by:

$$
\begin{aligned}
V(k_t) &= \ln(k_t^\alpha - k_{t+1}) + \beta J_0(k_{t+1}) \\
&= \ln(k_t^\alpha - k_{t+1}) + \beta \times \text{constant} + \frac{\alpha\beta}{(1-\alpha\beta)} \ln k_{t+1}
\end{aligned}
$$

Maximize with respect to k_{t+1}
First-order condition:

$$
\frac{-1}{k_t^\alpha - k_{t+1}} + \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{k_{t+1}} = 0
$$

$$
\frac{1}{k_t^\alpha - k_{t+1}} = \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{k_{t+1}}
$$

Solve for $k_{t+1}^* = \alpha\beta k_t \Rightarrow h_1 = \alpha\beta$ and $c_t = (1-h_1)k_t$
Iteration works in one step.

Value Function Iteration

We may also iterate on the value function. We start with an arbitrarily selected candidate value function. As one possibility, choose:

$$
V(k_t) = v_0(k_t) = 0 \forall k_t
$$

Now consider the problem:

$$
\underset{u_t}{Max}\ \{u(c_t) + \beta v_0(k_t)\} = \{\ln(k_t^\alpha - u_t) + \beta \bullet 0\}
$$

Trivially,

$$
c_t^* = k_t^\alpha \text{ and } u_t^* = k_{t+1}^* = 0
$$

The new value function becomes:

$$v_1(k_t) = \ln(k_t^\alpha) + \beta \bullet 0$$
$$= \alpha \ln k_t$$

New problem:

$$\underset{u_t}{Max} \; \{u(c_t) + \beta v_1(k_t)\} = \{\ln(k_t^\alpha - u_t) + \alpha\beta \ln u_t\}$$

First-order condition:

$$0 = \frac{-1}{k_t^\alpha - u_t} + \frac{\alpha\beta}{u_t}$$
$$u_t = \alpha\beta k_t^\alpha - \alpha\beta u_t$$

Therefore:

$$u_t^* = k_{t+1}^* = \frac{\alpha\beta}{(1+\alpha\beta)} k_t^\alpha \Rightarrow c_t^* = k_t^\alpha - u_t^*$$
$$c_t^* = \left(1 - \frac{\alpha\beta}{1+\alpha\beta}\right) k_t^\alpha = \frac{k_t^\alpha}{(1+\alpha\beta)}$$

The new value function becomes:

$$v_2(k_t) = \ln\left(\frac{k_t^\alpha}{(1+\alpha\beta)}\right) + \beta \bullet \alpha \ln k_{t+1}^*$$
$$= \ln\left(\frac{k_t^\alpha}{(1+\alpha\beta)}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{(1+\alpha\beta)} k_t^\alpha\right)$$
$$= \ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha \ln k_t + \alpha\beta \ln\left(\frac{\alpha\beta}{(1+\alpha\beta)}\right) + \alpha^2\beta \ln k_t$$

Collecting terms:

$$v_2(k_t) = \ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{(1+\alpha\beta)}\right) + \alpha(1+\alpha\beta) \ln k_t$$

Next consider:

$$\underset{u_t}{Max} \; \left\{\ln(k_t^\alpha - u_t) + \beta \bullet \left[\ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{(1+\alpha\beta)}\right) + \alpha(1+\alpha\beta) \ln u_t\right]\right\}$$

First-order condition:

$$0 = \frac{-1}{k_t^\alpha - u_t} + \alpha\beta(1+\alpha\beta)\frac{1}{u_t}$$
$$u_t = \alpha\beta(1+\alpha\beta)k_t^\alpha - \alpha\beta(1+\alpha\beta)u_t$$
$$u_t^* = k_{t+1}^* = \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta(1+\alpha\beta)} k_t^\alpha$$

We can also solve for:

$$\begin{aligned} c_t^* &= k_t^\alpha - u_t^* \\ &= k_t^\alpha \left[\frac{1 + \alpha\beta(1+\alpha\beta) - \alpha\beta(1+\alpha\beta)}{1+\alpha\beta(1+\alpha\beta)} \right] \\ &= \frac{1}{1+\alpha\beta(1+\alpha\beta)} k_t^\alpha \\ &= \frac{1}{1+\alpha\beta+(\alpha\beta)^2} k_t^\alpha \end{aligned}$$

A pattern is emerging in which, in the limit:

$$\begin{aligned} c_t^*(1+\alpha\beta+(\alpha\beta)^2+\cdots) &= k_t^\alpha \\ \frac{c_t^*}{(1-\alpha\beta)} &= k_t^\alpha \end{aligned}$$

Therefore:

$$c_t^* = (1-\alpha\beta)k_t^\alpha$$

Stochastic Problems

Consider the problem:

$$\underset{u_t}{Max}\ E_0\left(\sum_{t=0}^{\infty} \beta^t r(x_t, u_t)\right)$$

subject to: $x_{t+1} = g(x_t, u_t, \epsilon_{t+1})$,

where: ϵ_t is iid, with Prob$\{\epsilon_t \leq e\} = F(e), \forall t$

At each t, x_t and ϵ_t are known.
But u_t must be chosen before the realization of ϵ_{t+1}
Solution to the stochastic problem is a contingency plan.
Choice of u_t should not be made until the realization of ϵ_t.
Classical procedures no longer apply.
The Classical solution jointly determines all of the u_t at $t = 0$.
Not very helpful for deciding on a contingency plan.
We first consider:

$$\underset{u_t}{Max}\ E_1[r(x_1,u_1) + \beta r(x_2,u_2) + \cdots]$$

Conjecture that the solution is $u_t = h(x_t)$.
Solution is an autonomous contingency rule.
If $h(\bullet)$ is known, we may compute:

$$E_1 V(x_1) = E_1[r(x_1, h(x_1)) + \beta r(x_2, h(x_2)) + \cdots]$$

with $x_{t+1} = g(x_t, h(x_t), \epsilon_{t+1})$

We again call $V(\bullet)$ the value function.
Next, we reconsider:

$$\underset{u_t}{Max}\ E_0\left(\sum_{t=0}^{\infty}\beta^t r(x_t,u_t)\right)$$

Right-hand side may be rewritten as:

$$= E_0[r(x_0,u_0) + \beta r(x_1,u_1) + \beta^2 r(x_2,u_2) + \cdots]$$
$$= E_0\{r(x_0,u_0) + \beta E_1[r(x_1,h(x_1)) + \beta r(x_2,h(x_2)) + \cdots]\}$$
$$= r(x_0,u_0) + \beta E_0\{E_1[r(x_1,h(x_1)) + \beta r(x_2,h(x_2)) + \cdots]\}$$
$$= r(x_0,u_0) + \beta E_0\{E_1[V(x_1)]\} = r(x_0,u_0) + \beta E_0[V(x_1)]$$

Original problem may therefore be written as:

$$\underset{u_0}{Max}\ [r(x_0,u_0) + \beta E_0 V(x_1)] \text{ where } x_1 = g(x_0,u_0,\epsilon_1)$$

The Bellman equation is given by:

$$V(x) = \underset{u}{Max}\ r(x,u) + \beta E[V(g(x,u,\epsilon))|x]$$

$$\text{where: } E[V(g(x,u,\epsilon))|x] = \int V(g(x,u,\epsilon))dF(\epsilon)$$

We may now iterate on:

$$V_{j+1}(x) = \underset{u}{Max}\ (r(x,u) + \beta E[V_j(g(x,u,\epsilon))|x])$$

Differentiating the RHS, we obtain:

$$\frac{\partial r(x,u)}{\partial u} + \beta E\left[\frac{\partial g}{\partial u}(x,u,\epsilon)V'(g(x,u,\epsilon))|x\right] = 0$$

The value function also satisfies:

$$V'(x) = \frac{\partial r(x,h(x))}{\partial x} + \beta E\left[\frac{\partial g}{\partial x}(x,h(x),\epsilon)V'(g(x,h(x),\epsilon))|x\right]$$

When $\frac{\partial g}{\partial x} = 0$, the formula for $V'(x)$ becomes:

$$V'(x) = \frac{\partial r(x,h(x))}{\partial x}$$

Combine $V'(x)$ and FOC to obtain the stochastic Euler equation:

$$\frac{\partial r(x_t,u_t)}{\partial u_t} + \beta E\left[\frac{\partial g(x_t,u_t,\epsilon_{t+1})}{\partial u_t}\frac{\partial r(x_{t+1},u_{t+1})}{\partial x_{t+1}}\middle| x_t\right] = 0$$

Stochastic Guess and Verify

Consider the stochastic version of the Ramsey problem:

$$\underset{c_t}{Max}\ E_0\left(\sum_{t=0}^{\infty}\beta^t \ln(c_t)\right)$$

subject to $k_{t+1} = \theta_t k_t^{\alpha} - c_t$ and $\lim_{t\to\infty} k_t \geq 0$

$$E_t[\ln(\theta_{t+1})] = 0$$

Guess that:

$$V(k_t,\theta_t) = \Omega + F\ln k_t + G\ln(\theta_t)$$

Recall that:

$$c_t = \theta_t k_t^{\alpha} - k_{t+1}$$

We may now write the maximization as:

$$\underset{k_{t+1}}{Max}\ \{\ln(\theta_t k_t^{\alpha} - k_{t+1}) + \beta E[\Omega + F\ln k_{t+1} + G\ln(\theta_{t+1})]\}$$

First-order condition:

$$\begin{aligned} 0 &= \frac{\partial}{\partial k_{t+1}}\{\ln(\theta_t k_t^{\alpha} - k_{t+1}) + \beta[\Omega + F\ln k_{t+1}]\} \\ &= \frac{-1}{\theta_t k_t^{\alpha} - k_{t+1}} + \frac{\beta F}{k_{t+1}} \\ k_{t+1} &= \beta F[\theta_t k_t^{\alpha} - k_{t+1}] \end{aligned}$$

Rearrange to obtain:

$$k_{t+1}^* = \frac{\beta F}{1+\beta F}\theta_t k_t^{\alpha}$$

Next recall that:

$$\begin{aligned} c_t^* &= \theta_t k_t^{\alpha} - k_{t+1}^* \\ &= \theta_t k_t^{\alpha} - \frac{\beta F}{1+\beta F}\theta_t k_t^{\alpha} \end{aligned}$$

Collecting terms:

$$c_t^* = \left[1 - \frac{\beta F}{1+\beta F}\right]\theta_t k_t^{\alpha} = \frac{\theta_t k_t^{\alpha}}{1+\beta F}$$

We next evaluate the Bellman equation at c_t^* and k_{t+1}^* to obtain:

$$\begin{aligned} V(k_t,\theta_t) &= \max\{\ln c_t + \beta E_t V(k_{t+1},\theta_{t+1})\} \\ &= \ln c_t^* + \beta E_t[\Omega + F\ln k_{t+1}^* + G\ln(\theta_{t+1})] \\ &= \ln\left[\frac{\theta_t k_t^{\alpha}}{1+\beta F}\right] + \beta\Omega + \beta F\ln\left[\frac{\beta F}{1+\beta F}\theta_t k_t^{\alpha}\right] + \beta G \bullet 0 \end{aligned}$$

Therefore:

$$\begin{aligned} V(k_t,\theta_t) = {} & \ln(\theta_t) + \alpha\ln(k_t) - \ln(1+\beta F) \\ & + \beta\Omega + \beta F\ln\left(\frac{\beta F}{1+\beta F}\right) + \beta F\ln(\theta_t) + \alpha\beta F\ln(k_t) \end{aligned}$$

Expand to:

$$V(k_t,\theta_t) = [\beta\Omega - \ln(1+\beta F) + \beta F\ln(\beta F) - \beta F\ln(1+\beta F)] + (1+\beta F)\ln(\theta_t) + \alpha(1+\beta F)\ln(k_t)$$

Now match coefficients:

$$\Omega = \frac{\beta F\ln(\beta F) - (1+\beta F)\ln(1+\beta F)}{1-\beta}$$

$$F = \alpha(1+\beta F) \Rightarrow F = \frac{\alpha}{1-\alpha\beta}$$

$$G = 1+\beta F = \frac{1}{1-\alpha\beta}$$

Finally:

$$k_{t+1} = \frac{\beta F}{1+\beta F}\theta_t k_t^{\alpha} = \alpha\beta\theta_t k_t^{\alpha}$$

$$c_t = \theta_t k_t^{\alpha} - k_{t+1} = (1-\alpha\beta)\theta_t k_t^{\alpha}$$

$$\Omega = \frac{1}{1-\beta}\left\{\frac{\alpha\beta}{(1-\alpha\beta)}\ln(\alpha\beta) + \frac{\ln(1-\alpha\beta)}{(1-\alpha\beta)}\right\}$$

and:

$$V(k_t,\theta_t) = \Omega + \frac{\alpha\ln k_t + \ln(\theta_t)}{(1-\alpha\beta)}$$