701 Final
Autumn, 2010

1.) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a.) Eigenvalues solve $|A - \lambda I| = 0$, or

$$\begin{vmatrix} \frac{3}{4} - \lambda & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} - \lambda \end{vmatrix} = -\frac{3}{8} - \frac{3}{4}\lambda + \frac{1}{2}\lambda + \lambda^2 + \frac{1}{4} = 0$$

$$\lambda = \frac{\frac{1}{4} \pm \sqrt{9/16}}{2} \Rightarrow \begin{matrix} \lambda_1 = \frac{1}{2} \\ \lambda_2 = -\frac{1}{4} \end{matrix}$$

b.) v_1 solves $[A - \lambda_1 I] v_1 = 0$

$$\begin{bmatrix} \frac{3}{4} - \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} - \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $v_{11} = 1$ then $(\frac{1}{4})(1) - \frac{1}{2} v_{22} = 0 \Rightarrow v_{22} = \frac{1}{2}$

Therefore $v_1 = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ or any multiple thereof.

v_2 solves $[A - \lambda_2 I] v_2 = 0$

$$\begin{bmatrix} \frac{3}{4} - (-\frac{1}{4}) & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} - (-\frac{1}{4}) \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $v_{21} = 1$ then $(1)(1) - \frac{1}{2} v_{22} = 0 \Rightarrow v_{22} = 2$

Therefore $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any multiple thereof.

c.) The y variables satisfy $y = M^{-1}x$, where

$$M = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}$$

First calculate M^{-1}

$$M^{-1} = \frac{\text{Adjoint}(M)'}{|M|} = \frac{\begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}}{3/2} = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Therefore $y_1 = \frac{4}{3}x_1 - \frac{2}{3}x_2$

$$y_2 = -\frac{1}{3}x_1 + \frac{2}{3}x_2$$

d.) $x(0) = \begin{bmatrix} 1 \\ 1\frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 7/4 \end{bmatrix}$

$$y(0) = M^{-1}x(0) = \begin{bmatrix} 4/3 & -2/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 7/4 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$$

e.) $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_1(0)e^{\lambda_1 t} \\ y_2(0)e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \cdot e^{\frac{1}{2}t} \\ \frac{5}{6} \cdot e^{-\frac{1}{4}t} \end{bmatrix}$

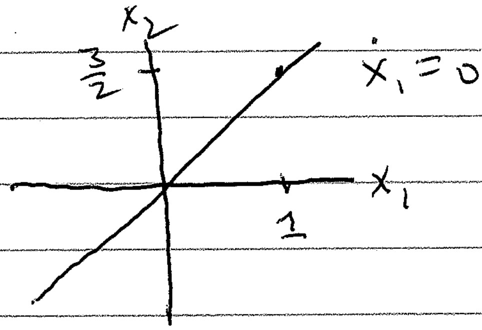
Then $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = M \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{6}e^{1/2 t} \\ 5/6 \, e^{-\frac{1}{4}t} \end{bmatrix}$

$$x_1(t) = \frac{1}{6}e^{1/2 t} + \frac{5}{6}e^{-\frac{1}{4}t}$$

$$x_2(t) = \frac{1}{12}e^{1/2 t} + \frac{5}{3}e^{-\frac{1}{4}t}$$

f.) $\dot{x}_1 = \frac{3}{4}x_1 - \frac{1}{2}x_2$

$\dot{x}_1 = 0 \Leftrightarrow \frac{3}{4}x_1 - \frac{1}{2}x_2 = 0 \Rightarrow x_2 = \frac{3}{2}x_1$

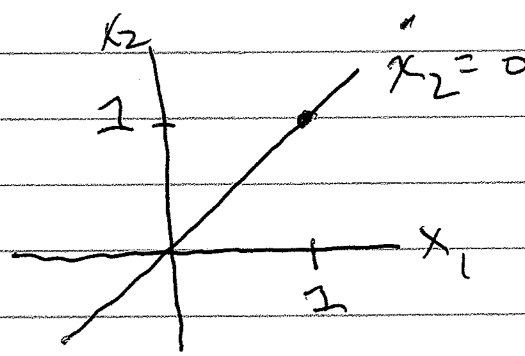


g.) $\frac{\partial \dot{x}_1}{\partial x_1} = \frac{3}{4} > 0$

To the right of $\dot{x}_1 = 0$, x_1 is increasing.

h.) $\dot{x}_2 = \frac{1}{2}x_1 - \frac{1}{2}x_2$

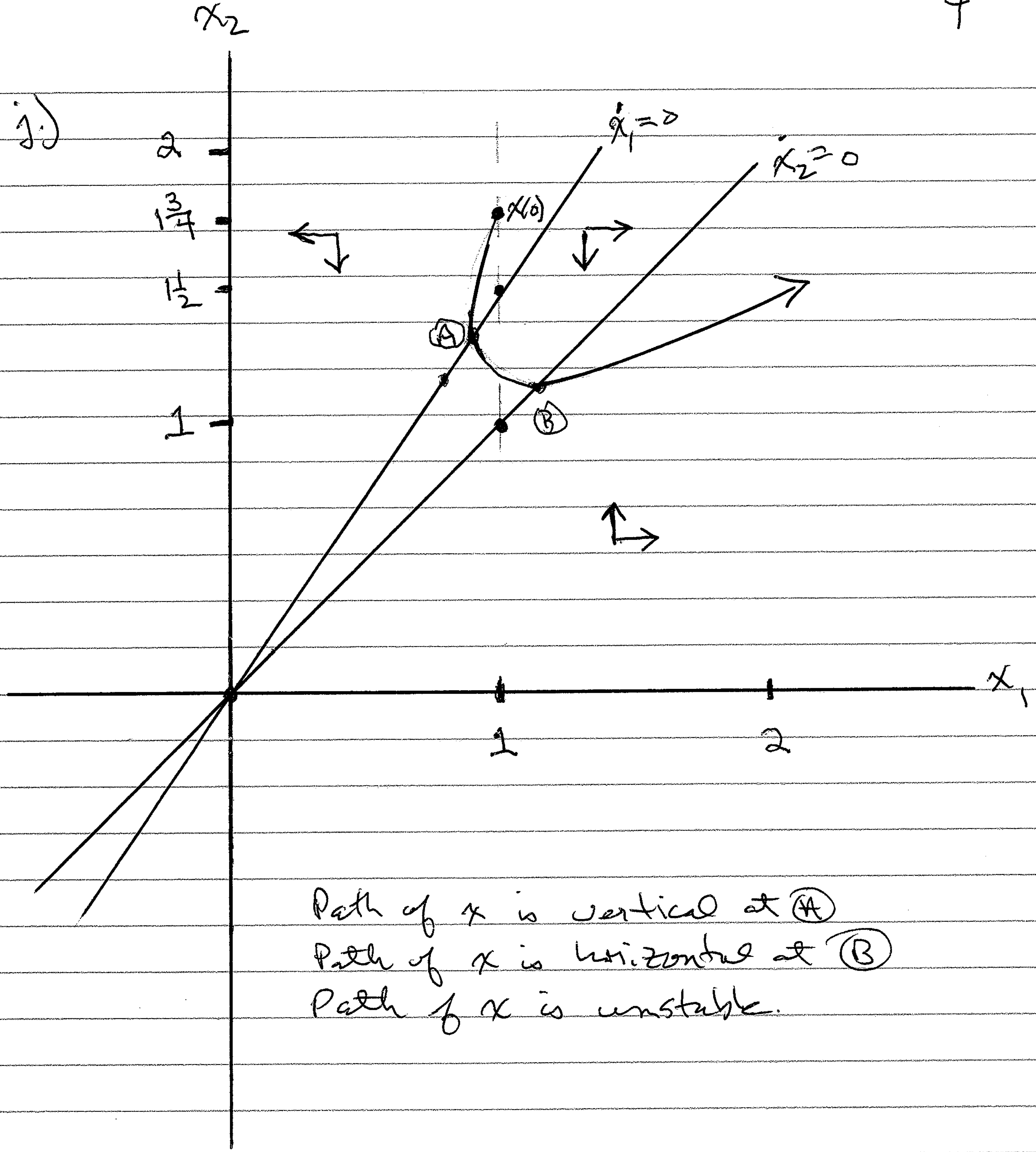
$\dot{x}_2 = 0 \Leftrightarrow \frac{1}{2}x_1 - \frac{1}{2}x_2 = 0 \Rightarrow x_2 = x_1$



i.) $\frac{\partial \dot{x}_2}{\partial x_2} = -\frac{1}{2} < 0$

Above $\dot{x}_2 = 0$, x_2 is decreasing.

j)



Path of x is vertical at A
Path of x is horizontal at B
Path of x is unstable.

k.) The only stable paths follow the stable eigenvector.

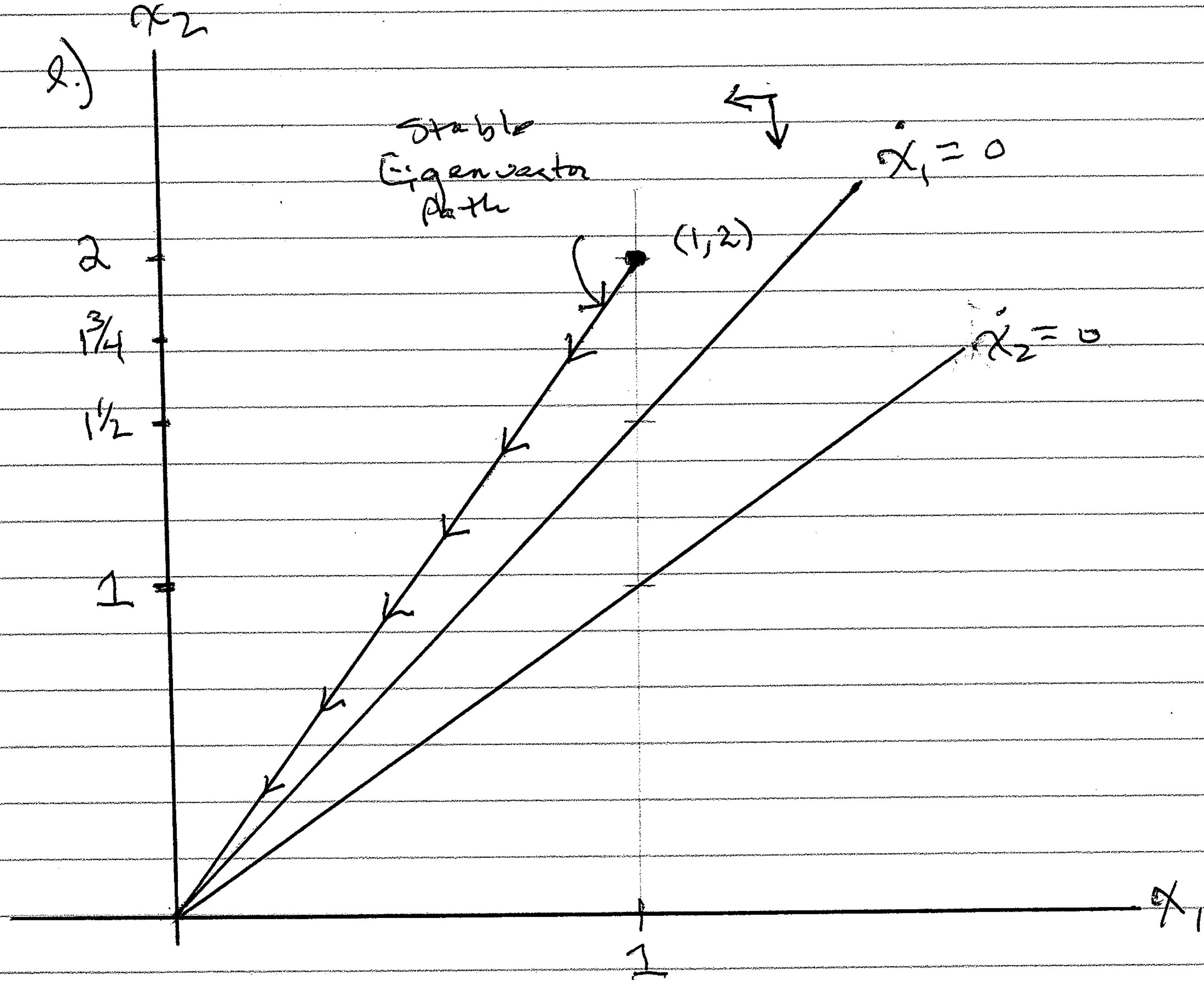
The stable eigenvalue is $\lambda_2 = -\frac{1}{4}$

The stable eigenvector is $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Along this path $x_2 = 2x_1$

If $x_1(0) = 1$ then we require $x_2(0) = 2$

l.)



2. Max $\int_0^T p(t)\, u(t)\, e^{-\rho t}\, dt$

a.) Max $\int_0^T p(t)\, u(t)\, e^{-\rho t}\, dt = \int_0^T u^{-\alpha} \cdot u\, e^{-\rho t}\, dt$

subject to $\dot{y} = -u$

$H = u^{(1-\alpha)} e^{-\rho t} - \lambda u$

b.) $\frac{\partial H}{\partial u} = (1-\alpha) u^{-\alpha} e^{-\rho t} - \lambda = 0$

$\Rightarrow \lambda = (1-\alpha) u^{-\alpha} e^{-\rho t}$

$\dot{\lambda} = -\frac{\partial H}{\partial y} = 0 \Rightarrow \dot{\lambda} = 0 \Rightarrow \lambda$ is time-constant.

c.) Denote the optimal value of λ as λ^*

$\lambda^* = (1-\alpha) u^{-\alpha} e^{-\rho t}$

Rearrange to obtain

$u(t) = \left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha} e^{-\frac{\rho}{\alpha} t}$

But $\dot{y} = -u = -\left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha} e^{-\rho/\alpha\, t}$

$y(t) = -\int^t \left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha} e^{-\frac{\rho}{\alpha} s}\, ds$

$$y(t) = \left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha}\left[\frac{\alpha}{\rho}e^{-\frac{\rho}{\alpha}\rho}\right]^t, \text{ or}$$

$$y(t) = c + \left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha}\frac{\alpha}{\rho}e^{-\frac{\rho}{\alpha}t}$$

where c is a yet-to-be-determined constant.

d.) For given T, $y(0)$ and $y(T)$:

$$y(0) = c + \frac{\alpha}{\rho}\left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha}$$

$$y(T) = c + \frac{\alpha}{\rho}\left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha} e^{-\frac{\rho}{\alpha}T}$$

Subtracting $y(0) - y(T) = \frac{\alpha}{\rho}\left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha}\left(1 - e^{-\frac{\rho}{\alpha}T}\right)$

along with $c = y(0) - \frac{\alpha}{\rho}\left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha}$

Rearranging $\frac{\alpha}{\rho}\left[\frac{(1-\alpha)}{\lambda^*}\right]^{1/\alpha} = \frac{y(0) - y(T)}{1 - e^{-\frac{\rho}{\alpha}T}}$

Finally:

$$y(t) = y(0) - \left[\frac{y(0) - y(T)}{1 - e^{-\frac{\rho}{\alpha}T}}\right] + \left[\frac{y(0) - y(T)}{1 - e^{-\frac{\rho}{\alpha}T}}\right]e^{-\frac{\rho}{\alpha}t}$$

e.) The complementary slackness condition would be

$$y(T) \geq 0 \quad \lambda(T) \geq 0 \quad y(T)\lambda(T) = 0$$

If $\lambda(T) = 0$ then $\lambda(t) = 0 \;\; \forall t$

The F.O.C. then requires $0 = (1-\alpha)u^{-\alpha}e^{-\rho t}$
This requires that $u(t) \to +\infty$ which sends $y(t)$ negative

f) As one possibility, we could add a Kuhn-Tucker term to the Hamiltonian
This would produce

$$\mathcal{L} = u^{(1-\alpha)}e^{-\rho t} - \lambda u + \theta(u_{max} - u)$$

Now we get

$$\frac{\partial \mathcal{L}}{\partial u} = (1-\alpha)u^{-\alpha}e^{-\rho t} - \lambda - \theta = 0$$

We still require $\lambda(t) = \lambda^* \;\; \forall t$
and

$$u_{max} - u \geq 0 \quad \theta \geq 0 \quad \theta \cdot (u_{max} - u) = 0 \;\; \forall t$$

If $\theta = 0 \;\; \forall t$ then $u \leq u_{max}$ never binds
Otherwise $\theta > 0$ and $\frac{\partial H}{\partial u} > 0$ for some t.

The interior (unconstrained) soln has $\dot{u} < 0$
Therefore if the constraint ever binds, it binds for small t.

3.) $\underset{c(t)}{\text{Max}} \int_0^\infty c(t) e^{-\rho t} dt$

s.t. $\dot{a}(t) = r a(t) - c(t)$

$c_{min} \le c(t) \le c_{max}$

a.) $\dot{a} = r a - c_{max}$

Complimentary Soln $y_c = \alpha e^{rt}$

Particular Soln $\dot{a} = 0 \Rightarrow r a = c_{max} \Rightarrow a(t) = \frac{c_{max}}{r}$

Full Soln $a(t) = \frac{c_{max}}{r} + \alpha e^{rt}$ for arbitrary α

b.) Same procedure as part (a):

$a(t) = \frac{c_{min}}{r} + \alpha e^{rt}$

c.) $\hat{H} = c(t) + q(t)[r a(t) - c(t)]$

d.) $\frac{\partial \hat{H}}{\partial c} = 1 - q(t)$ If $q(t) \neq 1$, no adjustment in c can make $\frac{\partial \hat{H}}{\partial q} = 0$

Therefore we are likely to have c at a corner, either $c = c_{min}$ or $c = c_{max}$.

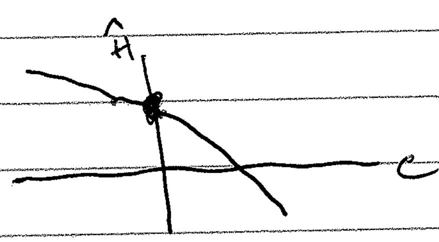
e.) $\dot{q} = -\frac{\partial \hat{H}}{\partial a} + \rho q = -[rq] + \rho q$

$\dot{q} = (\rho - r) q$ and so

$q(t) = q(0) e^{(\rho - r)t}$

For $q(0) > 0$ q decreases over time

f.) The corner solution $c = c_{min}$ occurs when $\partial \hat{H} / \partial c < 0$. In this case the individual would prefer to decrease c below c_{min}



$\partial \hat{H} / \partial c < 0$ when $q > 1$
Because q decreases over time, this case is likely for small t.

g.) The corner soln $c = c_{max}$ occurs when $\frac{\partial \hat{H}}{\partial c} > 0$
This requires $q < 1$. Because q decreases over time, this is likely to occur for large t.

h) The state variable is $a(t)$. The state variable must be continuous. For $t < \hat{t}$, the path of a follows

$$a(t) = \frac{c_{min}}{r} + \alpha e^{rt}$$

But $a(0)$ is given so

$$a(0) = \frac{c_{min}}{r} + \alpha \Rightarrow \alpha = a(0) - \frac{c_{min}}{r}$$

Therefore for $0 \leq t \leq \hat{t}$:

$$a(t) = \frac{c_{min}}{r} + \left[a(0) - \frac{c_{min}}{r}\right] e^{rt}$$

For $t > \hat{t}$ the path of a follows

$$a(t) = \frac{c_{max}}{r} + \alpha e^{rt}$$

But we have already argued that $c = c_{max}$ for <u>all</u> $t > \hat{t}$ and so $\alpha = 0$

For continuity, we require

$$\frac{c_{max}}{r} = \frac{c_{min}}{r} + \left[a(0) - \frac{c_{min}}{r}\right] e^{r\hat{t}}$$

Rearrange: $e^{r\hat{t}} = \dfrac{\left[\frac{c_{max}}{r} - \frac{c_{min}}{r}\right]}{\left[a(0) - \frac{c_{min}}{r}\right]}$

i.) We know that $q(t) = q(0) e^{(\rho - r)t}$

This time function is continuous. As we know that q is above 1 for $t < \hat{t}$ and q is below 1 for $t > \hat{t}$ it must be true that

$$q(\hat{t}) = 1$$

This also allows us to prove that

$$q(\hat{t}) = q(0) e^{(\rho - r)\hat{t}} \quad \text{and so}$$

$$q(0) = e^{-(\rho - r)\hat{t}}$$

j.) We would use the Lagrangian:

$$\mathcal{L} = c(t) + q(t)\left[ra(t) - c(t)\right] + \theta_1(t)\left[c(t) - c_{min}\right] + \theta_2(t)\left[c_{max} - c(t)\right]$$

FOCs $\frac{\partial \mathcal{L}}{\partial c} = 1 - q + \theta_1 - \theta_2 = 0$

$\dot{q} = -\frac{\partial \mathcal{L}}{\partial a} + \rho q \Rightarrow q(t) = q(0) e^{(\rho - r)t}$ same as before

When $c = c_{max}$ $\theta_1 > 0$ and $\theta_2 = 0$

When $c = c_{min}$ $\theta_1 = 0$ and $\theta_2 > 0$

For small t $\theta_1 > 0$ and so

$$1 - q(0)e^{(\rho - r)t} + \theta_1 = 0 \Rightarrow q > 1$$

For large t $\theta_2 > 0$ and so

$$1 - q(0)e^{(\rho - r)t} - \theta_2 = 0 \Rightarrow q < 1$$